On Some Topological Indices of Tensor Product Graphs

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Abstract - *A topological Index of a graph is a real numbers associated with the graph that is invariant under its automorphisms. The elementary topological indices are numbers of vertices and edges of a finite graph. The well-known one is Wiener Index due to Wiener [4]. It has wide applicability in molecular chemistry.*

The concept of Geometric – Arithmetic degree Index of a graph H, denoted by G/H degree (H), is introduced by Mogharrab and Fath – Tabar [2] as follows.

Key words: *Wiener Index, Geometric Arithmetic degree, Squared Geometric Arithmetic degree*.

I. INTRODUCTION

In this paper, we consider topological indices of standard graphs and that of tensor product graphs.

A topological index of a graph is a real number associated with the graph that is invariant under its automorphism.

The elementary topological indices are the number of vertices/edges of a finite graph. The well known one is Wiener index due to Wiener [5]. It has wide applicability in molecular chemistry.

II. PRELIMINARIES

We now present the necessary definitions and observations needed for the development of further ones in the succeeding sections.

The concept of Geometric-Arthimetic degree (deg) index of a graph H, denoted by G/A deg(H), introduced by Mogharrab and Fath-Taber [2] is the following:

Definition 1.1: Let H be a non-empty, connected, simple and finite graph. Then

$$G/A-\deg(\mathrm{H})=\sum_{uv\in E(H)}\frac{G.M \text{ of } \{\mathrm{d(u),d(v)}\}}{A.M \text{ of } \{\mathrm{d(u),d(v)}\}}$$

, where u, v$\in$ V(H) (vertex set H), E(H) is the edge set of H and d_H is the degree function of H. (when there is only one graph under consideration we omit and write V(H), E(H), and d_H as V, E and d respectively and the edge with ends u & v is denoted by uv.

Similarly, we can define the squared Geometric-Arthimetic degree index of H as

$$G^2/A-\deg(\mathrm{H})=\sum_{uv\in E(H)}\frac{(G.M)^2 \text{ of } \{\mathrm{d(u),d(v)}\}}{A.M \text{ of } \{\mathrm{d(u),d(v)}\}}.$$

The Geometric-twiced Arthimetic degree index of H denoted as

$$G/2A-\deg(\mathrm{H})=\sum_{uv\in E(H)}\frac{(G.M)\ \mathrm{of}\ \{\mathrm{d(u),d(v)}\}}{2(A.M)\ \mathrm{of}\ \{\mathrm{d(u),d(v)}\}}$$

and the squared Geometric-twice Arthimetic degree index of H, denoted as

$$G^2/2A-\deg\ (\mathrm{H})=\sum_{uv\in E(H)}\frac{(G.M)^2\ \mathrm{of}\ \{\mathrm{d(u),d(v)}\}}{2(A.M)\ \mathrm{of}\ \{\mathrm{d(u),d(v)}\}}..$$

Observations 1.2:

(a) If H is a regular graph then d(u) = d(v) for all u, v $\in$ V and so G.M of {d(u), d(v)} = A.M of {d(u), d(v)} for all u, v $\in$ V and hence

$$G/A-\deg(H)=\sum_{uv\in E}1$$

$$=|\mathrm{E}|$$

$$=\text{The number of edges of H, if H is a finite graph}.$$

(b)

$$(i)\, G/2A-\deg(H)=\frac{1}{2}G/A-\deg(H)\ \text{and}$$

$$(ii)\ G^2/2A-\deg(H)=\frac{1}{2}G^2/A-\deg(H).$$

(c) There is no interest in either empty or infinite or disconnected graph. So by a graph we mean a non-empty, finite, simple and connected one.

In the following section, we consider these indices for standard graphs.

§ 2. Basic results concerning standard graphs

Theorem 2.1: For the complete graph K_n (n being any integer $\geq$ 2),

(i) $G/A-\deg(K_n) = n(n-1)/2$ and
(ii) $G^2/A-\deg(K_n) = n(n-1)^2/2$.

Proof: We know that K_n has n vertices and any two vertices in K_n are adjacent. So it is a (n – 1)-regular graph with n(n-1)/2 edges.

Hence G/A – degree $(K_n) = |E(K_n)| = n(n-1)/2$.

By definition, $G^2/A-\deg(K_n) = 2\sum_{e=uv\in E(K_n)}\frac{d(u)d(v)}{d(u)+d(v)}$

$$=2\sum_{e\in E(K_n)}\frac{(n-1)^2}{2(n-1)}$$

$$=(n-1)|E(K_n)|$$

$$=n(n-1)^2/2.$$

Theorem 2.2: For the cycle C_n (n being any integer $\geq$ 3),

(i) $G/A - \deg(C_n) = n$ and
(ii) $G^2/A - \deg(C_n) = 2n$.

Proof. We know that C_n has n vertices and is a 2-regular graph; hence it has n edges. So, by the observation

(i) $G/A - \deg(C_n) = =|E(C_n)| = n$ and

(ii) $G^2/A - \deg(C_n) = 2\sum_{e=uv\in E(C_n)} \frac{d(u)d(v)}{d(u)+d(v)} == 2\sum_{e\in E(C_n)} \frac{4}{4} = 2|E(C_n)| = 2n.$

Theorem 2.3: For the path P_n, (n being any integer ≥ 2),

(i)

$$G/A - \deg(P_n) = \begin{cases} 1 \text{ if } n=2, \\ (n-3) + \frac{4\sqrt{2}}{3} \text{if } n \geq 3. \end{cases}$$

(ii)

$$G^2/A - \deg(P_n) = \begin{cases} 1 \text{ if } n=2, \\ 2(n-3) + \frac{8}{3} \text{if } n \geq 3. \end{cases}$$

Proof. Since $P_2 = K_2$, we have $G/A - \deg(P_2) = 1$.

Let n be any integer ≥ 3 and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$.

Now $E(P_n) = \{ u_i u_{i+1} : i = 1, 2, \ldots,(n-1)\}$,

$d(u_1) = d(u_n) = 1$ and $d(u_i) =2$, $i = 2, \ldots, (n-1)$.]

So when n=3,

$$G/A - \deg(P_3) = 2\left\{ \frac{\sqrt{d(u_1)d(u_2)}}{d(u_1)+d(u_2)} + \frac{\sqrt{d(u_2)d(u_3)}}{d(u_2)+d(u_3)} \right\}$$

$$= 2\left\{ \frac{\sqrt{1.2}}{1+2} + \frac{\sqrt{2.1}}{2+1} \right\}$$

$$= \frac{4\sqrt{2}}{3}$$

$$= (3-3) + \frac{4\sqrt{2}}{3}.$$

Let n be any integer ≥ 4, Now,

$$G/A - \deg(P_n) = 2\sum_{i=1}^{n-1} \frac{\sqrt{d(u_i)d(u_{i+1})}}{d(u_i)+d(u_{i+1})}$$

$$=2\left\{\frac{\sqrt{d(u_1)d(u_2)}}{d(u_1)+d(u_2)}+\sum_{i=2}^{n-2}\frac{\sqrt{d(u_i)d(u_{i+1})}}{d(u_i)+d(u_{i+1})}+\frac{\sqrt{d(u_{n-1})d(u_n)}}{d(u_{n-1})+d(u_n)}\right\}$$

$$=2\left\{\frac{\sqrt{1.2}}{1+2}+\sum_{i=2}^{n-2}\frac{\sqrt{2.2}}{2+2}+\frac{\sqrt{2.1}}{2+1}\right\}$$

$$=2\left\{\frac{\sqrt{2}}{3}+\frac{1}{2}(n-2-2+1)+\frac{\sqrt{2}}{3}\right\}$$

$$=2\left\{\frac{2\sqrt{2}}{3}+\frac{(n-3)}{2}\right\}$$

$$=(n-3)+\frac{4\sqrt{2}}{3}.$$

Now let us consider G^2/A – deg (P_n).

G^2/A – deg (P_2) = G^2/A – deg (K_2)

$$=\frac{2(2-1)^2}{2}=1.$$

When n = 3, G^2/A – degree (P_3) $= 2\left\{\frac{d(u_1)d(u_2)}{d(u_1)+d(u_2)}+\frac{d(u_2)d(u_3)}{d(u_2)+d(u_3)}\right\}=2\left\{\frac{1.2}{1+2}+\frac{2.1}{2+1}\right\}$

$=8/3 = 2(3-3)+8/3.$

Let $n \geq 4$; now

G^2/A – degree (P_n) =

$$2\sum_{i=1}^{n-1}\frac{d(u_i)d(u_{i+1})}{d(u_i)+d(u_{i+1})}$$

$$=2\left\{\frac{d(u_1)d(u_2)}{d(u_1)+d(u_2)}+\sum_{i=2}^{n-2}\frac{d(u_i)d(u_{i+1})}{d(u_i)+d(u_{i+1})}+\frac{d(u_{n-1})d(u_n)}{d(u_{n-1})+d(u_n)}\right\}$$

$$=2\left\{\frac{1.2}{1+2}+\sum_{i=2}^{n-2}\frac{2.2}{2+2}+\frac{2.1}{2+1}\right\}$$

$$=2\left\{\frac{4}{3}+((n-2)-2+1)\right\}$$

$$=2\left\{\frac{4}{3}+(n-3)\right\}$$

$$= 2(n-3) + \frac{8}{3}$$

Theorem 2.4: For the complete bipartite graph $K_{m,n}$ (m, n being any positive integers)

(i) $G/A - \deg(K_{m,n}) = \frac{2(mn)^{3/2}}{m+n}$ and

(ii) $G^2/A - \deg(K_{m,n}) = \frac{2(mn)^2}{m+n}$

Proof. We know that if {X, Y} is a bipartition of $K_{m,n}$ with |X| = m, and |Y| = n, then for any edge e=uv of $K_{m,n}$ with $u \in X$ and $v \in Y \Rightarrow$ deg(u) = n and deg(v) = m.

$K_{m,n}$ has (m+n) vertices and mn edges. Now, by definition

$$G/A - \deg(K_{m,n}) = 2 \sum_{e=uv \in E} \frac{\sqrt{nm}}{n+m}$$

$$= 2 \frac{\sqrt{mn}}{m+n} |E(K_{m,n})|$$

$$= 2 \frac{(mn)^{3/2}}{m+n}.$$

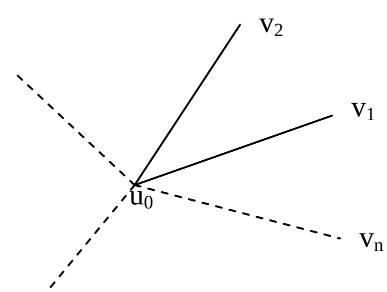
(Observe that $K_{1,1} = K_2$ and $G/A - \deg(K_{1,1}) = 1$).

By definition, $G^2/A - \deg(K_{m,n}) = 2 \frac{nm}{n+m} |E(K_{m,n})|$

$$= 2 \frac{(mn)^2}{m+n}$$

(Observe that $G^2/A - \deg(K_{1,1}) = 1$).

Theorem 2.5. For the star graph $S_{1,n}$, where n is any integer ≥ 3.



(i) $G/A - \deg(S_{1,n}) = 2 \frac{n^{3/2}}{(n+1)}$.

(ii) $G^2/A - \deg(S_{1,n}) = 2 \frac{n^2}{(n+1)}$

(Observe that $S_{1,1} = K_2 = P_2$ and $S_{1,2} = P_3 = K_{1,2}$. So we consider the case when $n \geq 3$).

Proof: Let $V(S_{1,n}) = \{u_0, v_1, v_2, \ldots, v_n\}$,

where $\deg(u_0) = n$ and $\deg v_j = 1$ for j=1,2,…n.

By definition,

$$G/A - deg(S_{1,n}) = 2\sum_{j=1}^{n} \frac{\sqrt{deg(u_0)deg(v_j)}}{(deg(u_0)+deg(v_j))}$$

$$= 2\sum_{j=1}^{n} \frac{\sqrt{n(1)}}{(n+1)}$$

$$= 2\frac{\sqrt{n}}{(n+1)}\sum_{j=1}^{n} 1$$

$$= 2\frac{\sqrt{n}}{(n+1)} n$$

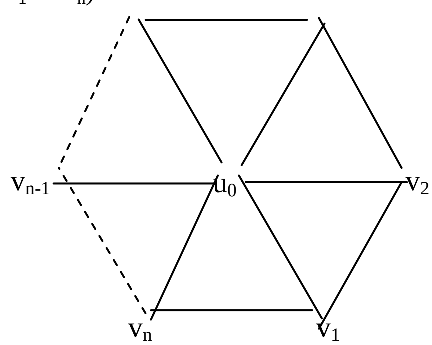
$$= 2\frac{n^{3/2}}{(n+1)}.$$

By definition, $G^2/A - deg(S_{1,n}) = 2\sum_{j=1}^{n} \frac{deg(u_0)deg(v_j)}{(deg(u_0)+deg(v_j))} = 2\sum_{j=1}^{n} \frac{(n)(1)}{(n+1)} = 2\frac{n}{(n+1)}\sum_{j=1}^{n} 1 = \frac{2n^2}{(n+1)}.$

Theorem 2.6: For the wheel graph $K_1 \vee C_n$ (n being any integer ≥ 3), $G/A - deg(K_1 \vee C_n) =$

$(1 + \frac{2\sqrt{3n}}{n+3} n)$.

(i) $G^2/A\text{-}deg(K_1 \vee C_n) = 9n(n+1)/(n+3)$.



Proof: Let the vertex set $V(K_1 \vee C_n) = \{u_0, u_1, u_2, \ldots, u_{n-1}, u_n\}$.

Now $E(K_1 \vee C_n) = \{u_0v_j: j=1, 2, \ldots, n\} \cup \{v_jv_{j+1}: j = 1, 2, \ldots, n\}$

(with the convention $v_{n+1} = v_1$). So

$d(u_0) = n$, $deg(v_j) = 3$ for $j = 1, 2, \ldots, n$

By definition, $G/A\text{-}deg(K_1 \vee C_n) = 2\sum_{j=1}^{n} \frac{\sqrt{d(u_0)d(v_j)}}{(d(u_0)+d(v_j))} + 2\sum_{j=1}^{n} \frac{\sqrt{d(v_j)d(v_{j+1})}}{(d(v_j)+d(v_{j+1}))}$

$$= 2\sum_{j=1}^{n} \frac{\sqrt{(n)(3)}}{(n+3)} + 2\sum_{j=1}^{n} \frac{\sqrt{(3)(3)}}{(3+3)}$$

$$= 2\frac{\sqrt{3n}}{(n+3)} n + n$$

$$= 2\ (\frac{\sqrt{3n}}{(n+3)} + 1)n$$

$$= \ (1 + 2\frac{\sqrt{3n}}{(n+3)})n\ .$$

$$G^2/A\text{-deg}(K_1 \vee C_n) = 2\sum_{j=1}^{n} \frac{\sqrt{d(u_0)\,d(v_j)}}{(d(u_0)+d(v_j))} + 2\sum_{j=1}^{n} \frac{\sqrt{d(v_j)\,d(v_{j+1})}}{(d(v_j)+d(v_{j+1}))}$$

$$= 2\sum_{j=1}^{n} \frac{\sqrt{(n)(3)}}{(n+3)} + 2\sum_{j=1}^{n} \frac{\sqrt{(3)(3)}}{(3+3)}$$

$$= 6\frac{n^2}{(n+3)} + 3n$$

$$= 3n\left(1 + \frac{2n}{n+3}\right) = 3n\left(\frac{3(n+1)}{(n+3)}\right) = \frac{9n(n+1)}{(n+3)}.$$

§3. Results related to Tensor Product Graphs

For the development of these results, we need the following:

Definition 3.1 [3]: H_1 and H_2 are disjoint graphs, then the tensor product of H_1 and H_2, denoted by $H_1 \wedge H_2$ is the graph whose vertex set is $V(H_1) \cup V(H_2)$ and the edge set being the set of all elements of the form $(u_1, v_1), (u_2, v_2)$, where $u_1u_2 \in V(H_1)$, $v_1, v_2 \in V(H_2)$, $u_1u_2 \in E(H_1)$ and $v_1v_2 \in E(H_2)$.

Result 3.2 [3]: H_1, H_2 be simple and connected graphs, then $H_1 \wedge H_2$ is connected iff either H_1 or H_2 contains an odd cycle.

Result 3.3 [1]: A non-empty connected graphs is Eulerian iff every vertex of the graph is of even degree.

Result 3.4 [1]: A simple graph is bipartite iff it contains no odd cycle.

Result 3.5 [4]: For integers m, $n \geq 2$, $K_m \wedge K_n$ is a simple, finite and (m-1)(n-1)-regular graph with mn vertices and ½mn(m – 1)(n – 1) edges. Further it is connected if one of m, n is ≥ 3.

Theorem 3.6: For integers m, $n \geq 2$, with $m+n \geq 5$,

(i) $G/A - \deg (K_m \wedge K_n) = ½\, mn(m-1)(n-1)$
(ii) $G^2/A - \deg (K_m \wedge K_n) = ½\, mn[(m-1)(n-1)]^2$.

Proof: $K_m \wedge K_n$ is a connected, regular graph. By the observation (1.2)(a) follows that

$G/A - \deg (K_m \wedge K_n) = |E(K_m \wedge K_n)|$

$= ½\, mn(m-1)(n-1)$ (By Result 3.5)

Let $w_1 = (u_1, v_1)$, $w_2 = (u_2, v_2) \in V(K_m \wedge K_n)$

Now, by definition,

$$G^2/A - \deg(K_m \wedge K_n) = 2 \sum_{w_1w_2 \in E(K_m \wedge K_n)} \frac{d(w_1)d(w_2)}{d(w_1)+d(w_2)}$$

$$= 2 \sum_{w_1w_2 \in E(K_m \wedge K_n)} \frac{\{(m-1)(n-1)\}^2}{2(m-1)(n-1)}$$

$$= (m-1)(n-1)\,|E(K_m \wedge K_n)|$$

$$= \tfrac{1}{2} mn\,[(m-1)(n-1)]^2.$$

Result 3.7 [4]: For integers m, $n \geq 3$, $C_m \wedge C_n$ is a simple 4-regular graph with mn vertices and 4mn edges. Further, it is connected iff atleast one of m, n is odd.

Theorem 3.8: For integers m, $n \geq 3$, when one of m, n is odd,

(i) $G/A - \deg(C_m \wedge C_n) = 2mn$ and
(ii) $G^2/A - \deg(C_m \wedge C_n) = 8mn$.

Proof: Since $C_m \wedge C_n$ is a 4-regular graph, it follows that

$$G/A - \deg(C_m \wedge C_n) = |E(C_m \wedge C_n)|$$

$$= 2mn.$$

Further, $$G^2/A - \deg(C_m \wedge C_n) = 2 \sum_{e \in E(C_m \wedge C_n)} \frac{(4)(4)}{4+4}$$

$$= 4|E(C_m \wedge C_n)|$$

$$= 8mn.$$

Result 3.9[4]: $K_2 \wedge C_n$ is a connected graph iff n is odd. Further it is isomorphic to C_{2n}.

Result 3.10[4]: $K_m \wedge C_n$ ($m > 2$, $n \geq 3$) is a simple, finite and $2(m-1)$-regular graph with mn vertices and $(m-1)(mn)$ edges.

Theorem 3.11: For the graph $K_2 \wedge C_n$ (n being an odd integer ≥ 3),

(i) $G/A - \deg(K_2 \wedge C_n) = 2n$ and
(ii) $G^2/A - \deg(K_2 \wedge C_n) = 4n$.

Proof: By Result 3.9, $K_2 \wedge C_n$ is isomorphic to C_{2n}.

Hence, by Th.2.2,

(i) $G/A - \deg(K_2 \wedge C_n) = 2n$ and
(ii) $G^2/A - \deg(K_2 \wedge C_n) = 4n$.

Theorem 3.12: For the graph $K_m \wedge C_n$ ($m, n \geq 3$)

(i) $G/A - \deg(K_m \wedge C_n) = (m-1)mn$ and
(ii) $G^2/A - \deg(K_m \wedge C_n) = 2(m-1)^2\, mn$.

Proof: By the result 3.10, $K_m \wedge C_n$ is a $2(m-1)$- regular graph with mn vertices and $(m-1)mn$ edges. So

$$G/A - \deg (K_m \wedge C_n) = 2 \sum_{e = w_1 w_2 \in E(K_m \wedge C_n)} \frac{\sqrt{d(w_1) d(w_2)}}{d(w_1) + d(w_2)}$$

$$= 2 \frac{2(m-1)}{4(m-1)} |E(K_m \wedge C_n)|$$

$$= |E(K_m \wedge C_n)|$$

$$= (m-1)mn.$$

$$G^2/A - \text{degree} (K_m \wedge C_n) = 2 \sum \frac{d(w_1) d(w_2)}{d(w_1) + d(w_2)}$$

$$= 2 \frac{4(m-1)^2}{4(m-1)} |E(K_m \wedge C_n)|$$

$$= 2(m-1)|E(K_m \wedge C_n)|$$

$$= 2(m-1)^2 mn.$$

III. CONCLUSIONS

In this paper, we obtain the Geometric-Arthimetic degree, squared Geometric-Arthimetic degree, and Geometric-twiced Arthimetic degree index of H and the basic results concerning to K_n, C_n, P_n, $K_{m,n}$, $S_{1,n}$ and $K_1 v C_n$ graphs. The results regarding tensor product of graphs are also discussed.

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