

On Generalized Fibonacci-Like Sequences by Hessenberg Matrices

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Abstract In this study, we consider Hessenberg matrices with applications to generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences. We then investigate the relationships between the Hessenberg matrices and generalized Fibonacci-Like sequences. We define some Hessenberg matrices and show that the determinants and permanents of these Hessenberg matrices are terms of generalized Fibonacci-Like sequences.

Keywords Fibonacci sequence, Lucas sequences, Hessenberg matrix.

I. INTRODUCTION

Sequences of integer numbers satisfying second order recurrence relations have been contributed over several years, with emphasis on studies of the well known Fibonacci sequences and then Lucas sequences. It has been obtained a lot of studies on Fibonacci and Lucas pattern based sequences which are known as Fibonacci-Like sequences. Some researchers defined various generalized Fibonacci-like sequences associated with Fibonacci and Lucas sequences and investigated the properties of the these sequences.

Some generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences can be generalized as follows:

Definition 1. [4] Generalized Fibonacci-Like sequence $\{B_n\}$ associated with Fibonacci and Lucas sequences is defined by recurrence relation

$$B_n = B_{n-1} + B_{n-2}, \quad n \geq 2 \quad (1.1)$$

with initial conditions $B_0 = 2s$ and $B_1 = s + 1$, s being a fixed positive integer.

Definition 2. [5] Generalized Fibonacci-Like sequence $\{D_n\}$ associated with Fibonacci and Lucas sequences is defined by recurrence relation

$$D_n = D_{n-1} + D_{n-2}, \quad n \geq 2 \quad (1.2)$$

with initial conditions $D_0 = 2$ and $D_1 = 1 + m$, m being a fixed positive integer.

The corresponding characteristic equation of equations (1.1) and (1.2) is

$$x^2 - x - 1 = 0$$

and its roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then the Binet's formulas for generalized Fibonacci-Like sequences $\{B_n\}$ and $\{D_n\}$ are given respectively by

$$B_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} + s(\alpha^n + \beta^n)$$

$$D_n = m \left(\frac{\alpha^n - \beta^n}{\sqrt{5}} \right) + (\alpha^n + \beta^n)$$

Also, the roots α and β verify the relations such as

$$\alpha + \beta = 1$$

$$\alpha - \beta = \sqrt{5}$$

$$\alpha\beta = -1.$$

The relation between Fibonacci and Lucas sequences and generalized Fibonacci-Like sequences can be written as

$$B_n = F_n + sL_n$$

$$D_n = mF_n + L_n$$

respectively, where s, m being fixed positive integers. A few terms of these sequences are

$$\{B_n\} = \{2s, 1 + s, 1 + 3s, 2 + 4s, 3 + 7s, \dots\}$$

$$\{D_n\} = \{2, 1 + m, 3 + m, 4 + 2m, 7 + 3m, \dots\}$$

There are so many articles in the literature that concern about these sequences of integer numbers contribute significantly to mathematics, especially to the field of matrix algebra. Many properties of these sequences of integer numbers are deduced directly from elementary matrix algebra. In matrix algebra, determinant and permanent are two important concepts. It is known that there are a lot of relationships between determinantal and permanent representations of matrices and well-known number sequences. Many researchers studied on determinantal and permanent representations of generalized Fibonacci-Like sequences and investigated the relationships between the Hessenberg matrices and these sequences of integer numbers ([3], [6], [7], [9], [10], [11]).

Let $A_n = [a_{ij}]$ be an $n \times n$ matrix and S_n is a symmetric group of permutations over the set $\{1, 2, \dots, n\}$. The determinant of A matrix defined by

$$\det A = \sum_{\alpha \in S_n} \text{sgn}(\alpha) \prod_{i=1}^n a_{i\alpha(i)}$$

where the sum ranges over all the permutations of the integers $1, 2, \dots, n$ ([9]). It can be denoted by $\text{sgn}(\alpha) = \pm 1$ the signature of α , equal to $+1$ if α is the product of an even number of transposition and -1 otherwise. The permanent of A matrix is defined by

$$\text{per } A = \sum_{\alpha \in S_n} \prod_{i=1}^n a_{i\alpha(i)}$$

where the summation extends over all permutations α of the symmetric group S_n ([7]).

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors r_1, r_2, \dots, r_m . We call A is contractible on column k , if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j . If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}^T]^T$ is called the contraction of A on row k relative to columns i and j .

Lemma 1. [1] Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A . Then,

$$\text{per } A = \text{per } B. \quad (1.3)$$

An $n \times n$ matrix $A_n = [a_{ij}]$ is called lower Hessenberg matrix if $a_{ij} = 0$ when $j - i > 1$, i.e.,

$$A_n = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix}.$$

Theorem 1. [2] Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $\det(A_0) = 1$. Then

$$\det(A_1) = a_{11}$$

and for $n \geq 2$

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} \left[(-1)^{n-r} a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} \det(A_{r-1}) \right]. \quad (1.4)$$

Theorem 2. [8] Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $\text{per}(A_0) = 1$. Then

$$\text{per}(A_1) = a_{11}$$

and for $n \geq 2$

$$\text{per}(A_n) = a_{n,n} \text{per}(A_{n-1}) + \sum_{r=1}^{n-1} \left[a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} \text{per}(A_{r-1}) \right]. \quad (1.5)$$

In this paper, we define four type lower Hessenberg matrices and show that the determinants and permanents of these type matrices are terms of generalized Fibonacci-Like sequences $\{B_n\}$ and $\{D_n\}$ associated with Fibonacci and Lucas sequences.

II. MAIN RESULTS

A. The Determinantal Representations of Generalized Fibonacci-Like Sequences

In this section, we define two type lower Hessenberg matrices and show that determinants of these Hessenberg matrices give terms of generalized Fibonacci-Like sequences $\{B_n\}$ and $\{D_n\}$.

Definition 3. The n -square Hessenberg matrix $U_n(s) = (u_{ij})$ is defined by

$$U_n(s) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2s & 1+s & \alpha(1+s) & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & \alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \beta & 1 & \alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \quad (2.1)$$

with $u_{i,i} = 1$, $u_{i,i+1} = \alpha$, $u_{i+1,i} = \beta$ for $3 \leq i \leq n$, $u_{11} = u_{12} = 1$, $u_{21} = -2s$, $u_{22} = 1+s$, $u_{23} = \alpha(1+s)$, $u_{32} = \beta$ and 0 otherwise.

Theorem 3. Let the matrix $U_n(s)$ be as in equation (2.1). Then for $n \geq 2$,

$$\det U_n(s) = B_n$$

where B_n is the n th term of generalized Fibonacci-Like sequence $\{B_n\}$.

Proof. We can use the mathematical induction on n to prove $\det U_n(s) = B_n$. Then,

$$n = 2, \quad \det U_2(s) = \begin{vmatrix} 1 & 1 \\ -2s & 1+s \end{vmatrix} = 1 + 3s = B_2$$

$$n = 3, \quad \det U_3(s) = \begin{vmatrix} 1 & 1 & 0 \\ -2s & 1+s & \alpha(1+s) \\ 0 & \beta & 1 \end{vmatrix} = 2 + 4s = B_3$$

$$n = 4, \quad \det U_4(s) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -2s & 1+s & \alpha(1+s) & 0 \\ 0 & \beta & 1 & \alpha \\ 0 & 0 & \beta & 1 \end{vmatrix} = 3 + 7s = B_4$$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$\det U_n(s) = B_n, \det U_{n-1}(s) = B_{n-1}, \dots$$

and we show that it is true for $n+1$. By our assumption and using equations (1.4) and (1.1), we have

$$\begin{aligned} \det U_{n+1}(s) &= u_{n+1,n+1} \det U_n(s) + \sum_{i=1}^n \left[(-1)^{n+1-r} u_{n+1,r} \prod_{j=r}^n u_{j,j+1} \det U_{r-1}(s) \right] \\ &= (1) \det U_n(s) + \sum_{i=1}^{n-1} \left[(-1)^{n+1-r} u_{n+1,r} \prod_{j=r}^n u_{j,j+1} \det U_{r-1}(s) \right] + (-1) u_{n+1,n} u_{n,n+1} \det U_{n-1}(s) \\ &= \det U_n(s) + [(-1)(\beta)(\alpha) \det U_{n-1}(s)] \\ &= \det U_n(s) + \det U_{n-1}(s) \\ &= B_n + B_{n-1} \\ &= B_{n+1} \end{aligned}$$

where $\alpha\beta = -1$. So the proof is completed. ■

Definition 4. The n -square Hessenberg matrix $V_n(m) = (v_{ij})$ is defined by

$$V_n(m) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1+m & \alpha(1+m) & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & \alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \beta & 1 & \alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \quad (2.2)$$

with $v_{i,i} = 1, v_{i,i+1} = \alpha, v_{i+1,i} = \beta$ for $3 \leq i \leq n, v_{11} = v_{12} = 1, v_{21} = -2, v_{22} = 1+m, v_{23} = \alpha(1+m), v_{32} = \beta$ and 0 otherwise.

Theorem 4. Let the matrix $V_n(m)$ be as in equation (2.2). Then for $n \geq 2$,

$$\det V_n(m) = D_n$$

where D_n is the n th term of generalized Fibonacci-Like sequence $\{D_n\}$.

Proof. We can use the mathematical induction on n to prove $\det V_n(m) = D_n$. Then,

$$n = 2, \quad \det V_2(m) = \begin{vmatrix} 1 & 1 \\ -2 & 1+m \end{vmatrix} = 3+m = D_2$$

$$n = 3, \quad \det V_3(m) = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1+m & \alpha(1+m) \\ 0 & \beta & 1 \end{vmatrix} = 4+2m = D_3$$

$$n = 4, \quad \det V_4(m) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -2 & 1+m & \alpha(1+m) & 0 \\ 0 & \beta & 1 & \alpha \\ 0 & 0 & \beta & 1 \end{vmatrix} = 7+3m = D_4$$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$\det V_n(m) = D_n, \det V_{n-1}(m) = D_{n-1}, \dots$$

and we show that it is true for $n+1$. By our assumption and using equation (1.4) and (1.2), we have

$$\begin{aligned} \det V_{n+1}(m) &= v_{n+1,n+1} \det V_n(m) + \sum_{i=1}^n \left[(-1)^{n+1-r} v_{n+1,r} \prod_{j=r}^n v_{j,j+1} \det V_{r-1}(m) \right] \\ &= (1) \det V_n(m) + \sum_{i=1}^{n-1} \left[(-1)^{n+1-r} v_{n+1,r} \prod_{j=r}^n v_{j,j+1} \det V_{r-1}(m) \right] + (-1) v_{n+1,n} v_{n,n+1} \det V_{n-1}(m) \\ &= \det V_n(m) + [(-1)(\beta)(\alpha) \det V_{n-1}(m)] \\ &= \det V_n(m) + \det V_{n-1}(m) \\ &= D_n + D_{n-1} \\ &= D_{n+1} \end{aligned}$$

where $\alpha\beta = -1$. So the proof is completed. ■

B. The Permanental Representations of Generalized Fibonacci-Like Sequences

In this section, we define two type lower Hessenberg matrices and show that permanents of these Hessenberg matrices give terms of generalized Fibonacci-Like sequences $\{B_n\}$ and $\{D_n\}$.

Definition 5. The n -square Hessenberg matrix $P_n(s) = p_{ij}$ is defined by

$$P_n(s) = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2s & 1+s & -\alpha(1+s) & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \quad (2.3)$$

with $p_{i,i} = 1$, $p_{i,i+1} = -\alpha$, $p_{i+1,i} = \beta$, for $3 \leq i \leq n$, $p_{11} = -p_{12} = 1$, $p_{21} = -2s$, $p_{22} = 1+s$, $p_{23} = -\alpha(s+1)$, $p_{32} = \beta$ and 0 otherwise.

Theorem 5. Let matrix $P_n(s)$ be as in equation (2.3). Then for $n \geq 2$,

$$\text{per} P_n(s) = \text{per} P_n^{(n-2)}(s) = B_n$$

where B_n is the n th term of generalized Fibonacci-Like sequence $\{B_n\}$.

Proof. By Definition 5, it can be contracted on first column. Let $P_n^r(s)$ be r th contraction of $P_n(s)$, $1 \leq r \leq n-2$. The matrix $P_n(s)$ can be contracted on first column, so that we get

$$\begin{aligned} P_n^1(s) &= \begin{pmatrix} 1+3s & -\alpha(1+s) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \\ &= \begin{pmatrix} B_2 & -\alpha B_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \end{aligned}$$

where $B_2 = 1+3s$ and $B_1 = 1+s$. Since $P_n^1(s)$ also can be contracted on first column,

$$\begin{aligned} P_n^2(s) &= \begin{pmatrix} 2+4s & -\alpha(1+3s) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \\ &= \begin{pmatrix} B_3 & -\alpha B_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \end{aligned}$$

where $B_3 = 2+4s$, $B_2 = 1+3s$ and $\alpha\beta = -1$. Continuing with this process, we have the r th contraction of the matrix $P_n(s)$ as

$$P_n^r(s) = \begin{pmatrix} B_{r+1} & -\alpha B_r & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix}$$

for $3 \leq r \leq n-4$. Hence

$$P_n^{n-3}(s) = \begin{pmatrix} B_{n-2} & -\alpha B_{n-3} & 0 \\ \beta & 1 & -\alpha \\ 0 & \beta & 1 \end{pmatrix}$$

which by contraction of $P_n^{n-3}(s)$ on first column, we obtain

$$P_n^{n-2}(s) = \begin{pmatrix} B_{n-2} - \alpha\beta B_{n-3} & -\alpha B_{n-2} \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} B_{n-2} + B_{n-3} & -\alpha B_{n-2} \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} B_{n-1} & -\alpha B_{n-2} \\ \beta & 1 \end{pmatrix}$$

by using equation (1.1). From the equation (1.3), we have

$$\text{per}P_n(s) = \text{per}P_n^{n-2}(s) = B_{n-1} - \alpha\beta B_{n-2} = B_{n-1} + B_{n-2} = B_n$$

where $\alpha\beta = -1$. So the proof is completed. ■

Definition 6. The n -square Hessenberg matrix $T_n(m) = (t_{ij})$ is defined by

$$T_n(m) = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1+m & -\alpha(1+m) & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \quad (2.4)$$

with $t_{i,i} = 1$, $t_{i,i+1} = -\alpha$, $t_{i+1,i} = \beta$ for $3 \leq i \leq n$, $t_{11} = -t_{12} = 1$, $t_{21} = -2$, $t_{22} = 1+m$, $t_{23} = -\alpha(1+m)$, $t_{32} = \beta$ and 0 otherwise.

Theorem 6. Let the matrix $T_n(m)$ be as in equation (2.4). Then for $n \geq 2$,

$$\text{per}T_n(m) = \text{per}T_n^{n-2}(m) = D_n$$

where D_n is the n th term of generalized Fibonacci-Like sequence $\{D_n\}$.

Proof. By Definition 6, it can be contracted on first column. Let $T_n^r(m)$ be r th contraction of $T_n(m)$, $1 \leq r \leq n-2$. The matrix $T_n(m)$ can be contracted on first column, so that we get

$$\begin{aligned} T_n^1(m) &= \begin{pmatrix} 3+m & -\alpha(1+m) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \\ &= \begin{pmatrix} D_2 & -\alpha D_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix} \end{aligned}$$

where $D_2 = 3+m$ and $D_1 = 1+m$. $T_n^1(m)$ also can be contracted on first column,

$$T_n^2(m) = \begin{pmatrix} 4+2m & -\alpha(3+m) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} D_3 & -\alpha D_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix}$$

where $D_3 = 4 + 2m$, $D_2 = 3 + m$ and $\alpha\beta = -1$. Continuing with this process, we have the r th contraction of the matrix $T_n(m)$ as

$$T_n^r(m) = \begin{pmatrix} D_{r+1} & -\alpha D_r & 0 & 0 & \cdots & 0 & 0 & 0 \\ \beta & 1 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & \beta & 1 & -\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \beta & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{pmatrix}$$

for $3 \leq r \leq n-4$. Hence

$$T_n^{n-3}(m) = \begin{pmatrix} D_{n-2} & -\alpha D_{n-3} & 0 \\ \beta & 1 & -\alpha \\ 0 & \beta & 1 \end{pmatrix}$$

which by contraction of $T_n^{n-3}(m)$ on first column, we obtain

$$T_n^{n-2}(m) = \begin{pmatrix} D_{n-2} - \alpha\beta D_{n-3} & -\alpha D_{n-2} \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} D_{n-2} + D_{n-3} & -\alpha D_{n-2} \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} D_{n-1} & -\alpha D_{n-2} \\ \beta & 1 \end{pmatrix}$$

by using equation (1.2). From the equation (1.3), we have

$$\text{per} T_n(m) = \text{per} T_n^{n-2}(m) = D_{n-1} - \alpha\beta D_{n-2} = D_{n-1} + D_{n-2} = D_n$$

where $\alpha\beta = -1$. So the proof is completed. ■

As the other way, equation (1.5) can be used for proofs of Theorem 5 and Theorem 6 too.

III. CONCLUSIONS

Sequences of integer numbers, such as the Fibonacci and Lucas sequences are well-known second order recurrence sequences in all of mathematics. Also, there are a lot of studies on its properties and applications to almost every field of science and art. In this paper we contribute for the study of generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences. We define some Hessenberg matrices and study on determinantal and permanental representations of generalized Fibonacci-Like sequences and investigate the relationships between the Hessenberg matrices and these sequences of integer numbers.

Several studies involving all types of Hessenberg matrices can easily be found in the literature. Here we have considered the four type lower Hessenberg matrices whose entries are terms of generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences. For these cases we have provided the determinants and permanents of these matrices.

In the future, we intend to discuss the determinants and permanents of Hessenberg type matrices associated with generalized Fibonacci-Like sequences associated with Pell, Pell-Lucas, Jacobsthal sequences.

ACKNOWLEDGMENT

The author declare that there is no conflict of interest regarding the publication of this paper and express their sincere thanks to the referee for his/her careful reading and suggestions that helped to improve this paper.

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