# Binomial Transforms of Generalized Fibonacci-Like Sequences 

Yasemin Taşyurdu<br>Department of Mathematics, Faculty of Science and Art, University of Erzincan Binali Yıldırım, Erzincan, Turkey


#### Abstract

In this study, we apply binomial transforms to generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences. We obtain the Binet formulas and generating functions of these transforms. Also, we introduce an infinite triangle consist of the terms of generalized Fibonacci-Like sequences and their binomial transforms.


Keywords Fibonacci sequence, Lucas sequences, Binomial Transforms, Binet Formulo.

## I. Introduction

Special number sequences such as Fibonacci, Lucas, Pell and Jacobsthal have been studied by mathematicians for their intrinsic theory and applications. The well known Fibonacci $\left\{F_{n}\right\}$ and Lucas $\left\{L_{n}\right\}$ sequences are defined by recurrence relations

$$
\begin{array}{ll}
F_{n}=F_{n-1}+F_{n-2}, & F_{0}=0, F_{1}=1 \\
L_{n}=L_{n-1}+L_{n-2}, & L_{0}=2, L_{1}=1
\end{array}
$$

respectively, for $n \geq 2$ [5].
On the other hand, there are so many studies known as Fibonacci-Like sequences in the literature that concern about these special number sequences such as Fibonacci and Lucas sequences.

Some generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences can be generalized as follows:

Definition 1. [3] Generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ associated with Fibonacci and Lucas sequences is defined by recurrence relation

$$
\begin{equation*}
B_{n}=B_{n-1}+B_{n-2}, \quad n \geq 2 \tag{1.1}
\end{equation*}
$$

with initial conditions $B_{0}=2 s$ and $B_{1}=s+1, s$ being a fixed positive integer.
Definition 2. [4] Generalized Fibonacci-Like sequence $\left\{D_{n}\right\}$ associated with Fibonacci and Lucas sequences is defined by recurrence relation

$$
\begin{equation*}
D_{n}=D_{n-1}+D_{n-2}, \quad n \geq 2 \tag{1.2}
\end{equation*}
$$

with initial conditions $D_{0}=2$ and $D_{1}=1+m, m$ being a fixed positive integer.
The relation between Fibonacci sequence and generalized Fibonacci-Like sequences can be written as

$$
\begin{aligned}
& B_{n}=F_{n}+s L_{n} \\
& D_{n}=m F_{n}+L_{n}
\end{aligned}
$$

respectively, where $s, m$ being fixed positive integers. A few terms of these sequences are

$$
\begin{aligned}
& \left\{B_{n}\right\}=\{2 s, 1+s, 1+3 s, 2+4 s, 3+7 s, \ldots\} \\
& \left\{D_{n}\right\}=\{2,1+m, 3+m, 4+2 m, 7+3 m, \ldots\} .
\end{aligned}
$$

The corresponding characteristic equation of equations (1.1) and (1.2) is

$$
x^{2}-x-1=0
$$

and its roots are $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$. Then the Binet formulas for generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ are given respectively by

$$
\begin{aligned}
& B_{n}=\frac{\alpha^{n}-\beta^{n}}{\sqrt{5}}+s\left(\alpha^{n}+\beta^{n}\right) \\
& D_{n}=m\left(\frac{\alpha^{n}-\beta^{n}}{\sqrt{5}}\right)+\left(\alpha^{n}+\beta^{n}\right)
\end{aligned}
$$

Also, the roots $\alpha$ and $\beta$ verify the relations such as $\alpha+\beta=1, \alpha-\beta=\sqrt{5}, \alpha \beta=-1$.

In addition, some matrices based transforms can be introduced for a given sequence. Binomial transform is one of these transforms and there are also other ones such as rising and falling binomial transforms ([6]). Many researchers studied on these transforms to these number sequences. Falcon and Plaza applied binomials transforms to the $k$-Fibonacci sequences ([2]). Bhadouria et al. investigated binomial transforms of $k$-Lucas sequences ([1]).

In [7], the binomial transform $B$ of the integer sequence $A=\left\{a_{0}, a_{1}, a_{2}, \ldots\right\}$, which is denoted by $B(A)=\left\{b_{n}\right\}$ and defined by

$$
b_{n}=\sum_{i=0}^{n}\binom{n}{i} a_{i}
$$

In this paper, we define binomial transforms of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ associated with Fibonacci and Lucas sequences. The generating functions and Binet formulas given $n$-th general term of these transforms are found by using their recurrence relations. Finally, we get an infinite triangle consist of the terms of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ and their binomial transforms.

## II. Main Results

## A. Binomial Transforms of Generalized Fibonacci-Like Sequences

In the section, we will mainly present binomial transforms of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ associated with Fibonacci and Lucas sequences.

Definition 3. The binomial transforms of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ are defined as follows:
i. The binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ is

$$
\begin{equation*}
p_{n}=\sum_{i=0}^{n}\binom{n}{i} B_{i} \tag{2.1}
\end{equation*}
$$

ii. The binomial transform $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{D_{n}\right\}$ is

$$
\begin{equation*}
r_{n}=\sum_{i=0}^{n}\binom{n}{i} D_{i} \tag{2.2}
\end{equation*}
$$

From equations (2.1) and (2.2), for $n \geq 0$, we can write a few terms of binomial transforms $P=\left\{p_{n}\right\}$ and $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ as follows:

$$
\begin{aligned}
& p_{0}=B_{0}=2 s \\
& p_{1}=B_{0}+B_{1}=1+3 s=B_{2}, \\
& p_{2}=B_{0}+2 B_{1}+B_{2}=3+7 s=B_{4} \\
& \quad \vdots \\
& p_{n}=B_{2 n}
\end{aligned}
$$

and

$$
\begin{aligned}
& r_{0}=D_{0}=2 \\
& r_{1}=D_{0}+D_{1}=3+m=D_{2} \\
& r_{2}=D_{0}+2 D_{1}+D_{2}=7+3 m=D_{4} \\
& \quad \vdots \\
& r_{n}=D_{2 n}
\end{aligned}
$$

The following Lemma is the key in the proof of the main theorem.
Lemma 1. i. The binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ verifies the relation

$$
\begin{equation*}
p_{n+1}=\sum_{i=0}^{n}\binom{n}{i}\left(B_{i}+B_{i+1}\right) \tag{2.3}
\end{equation*}
$$

ii. The binomial transform $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{D_{n}\right\}$ verifies the relation

$$
\begin{equation*}
r_{n+1}=\sum_{i=0}^{n}\binom{n}{i}\left(D_{i}+D_{i+1}\right) \tag{2.4}
\end{equation*}
$$

Proof: Firstly we will just prove to (i), since prove to (ii) can be obtained in the same manner with (i).
i. By using equation (2.1), we obtain

$$
\begin{aligned}
p_{n+1} & =\sum_{i=1}^{n+1}\binom{n+1}{i} B_{i}+B_{0} \\
& =\sum_{i=1}^{n+1}\binom{n}{i} B_{i}+\sum_{i=1}^{n+1}\binom{n}{i-1} B_{i}+B_{0} \\
& =\sum_{i=0}^{n}\binom{n}{i} B_{i}+\sum_{i=0}^{n}\binom{n}{i} B_{i+1} \\
& =\sum_{i=0}^{n}\binom{n}{i}\left(B_{i}+B_{i+1}\right)
\end{aligned}
$$

where $\binom{n}{n+1}=0$ and the well known binomial equation $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$. This is desired result.
From equations (2.3) and (2.4) by using the equations (2.1) and (2.2), we can obtain

$$
p_{n+1}=p_{n}+\sum_{i=0}^{n}\binom{n}{i} B_{i+1}
$$

and

$$
r_{n+1}=r_{n}+\sum_{i=0}^{n}\binom{n}{i} D_{i+1}
$$

Now we present the main theorem of this paper.
Theorem 1. i. Recurrence relation of binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ is
$p_{n+1}=3 p_{n}-p_{n-1}$ for $n \geq 1$
with initial conditions $p_{0}=2 s$ and $p_{1}=1+3 s$.
ii. Recurrence relation of binomial transform $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{D_{n}\right\}$ is

$$
\begin{equation*}
r_{n+1}=3 r_{n}-r_{n-1} \quad \text { for } \quad n \geq 1 \tag{2.6}
\end{equation*}
$$

with initial conditions $r_{0}=2$ and $r_{1}=3+m$.
Proof. Firstly we will just prove to (i), since prove to (ii) can be obtained in the same manner with (i).
i. From Lemma 1 and equation (1.1) and (2.1), we obtain

$$
\begin{aligned}
p_{n+1} & =\sum_{i=0}^{n}\binom{n}{i}\left(B_{i}+B_{i+1}\right) \\
& =B_{0}+B_{1}+\sum_{i=1}^{n}\binom{n}{i}\left(B_{i}+B_{i}+B_{i-1}\right) \\
& =2 \sum_{i=1}^{n}\binom{n}{i} B_{i}+\sum_{i=1}^{n}\binom{n}{i} B_{i-1}+1+3 s
\end{aligned}
$$

where $B_{0}=2 s, B_{1}=1+s$. By using equation (2.1), we obtain

$$
\begin{equation*}
p_{n+1}=2 p_{n}+\sum_{i=1}^{n}\binom{n}{i} B_{i-1}+1+3 s \tag{2.7}
\end{equation*}
$$

From we have equation (2.7) by using that $\binom{n-1}{n}=0$, we get

$$
\begin{aligned}
p_{n} & =2 p_{n-1}+\sum_{i=1}^{n-1}\binom{n-1}{i} B_{i-1}+1+3 s \\
& =p_{n-1}+\sum_{i=0}^{n-1}\binom{n-1}{i} B_{i}+\sum_{i=1}^{n-1}\binom{n-1}{i} B_{i-1}+1+3 s \\
& =p_{n-1}+\left[\sum_{i=1}^{n}\binom{n-1}{i-1} B_{i-1}+\sum_{i=1}^{n}\binom{n-1}{i} B_{i-1}\right]+1+3 s \\
& =p_{n-1}+\sum_{i=1}^{n}\left[\binom{n-1}{i-1}+\binom{n-1}{i}\right] B_{i-1}+1+3 s .
\end{aligned}
$$

So, we have

$$
\begin{equation*}
p_{n}=p_{n-1}+\sum_{i=1}^{n}\binom{n}{i} B_{i-1}+1+3 s \tag{2.8}
\end{equation*}
$$

where the well known binomial equation $\binom{n}{i}=\left(\begin{array}{c}i=1 \\ n-1 \\ i\end{array}\right)+\binom{n-1}{i-1}$. From equations (2.7) and (2.8), we obtain

$$
\begin{aligned}
p_{n+1}-2 p_{n}-1-3 s & =p_{n}-p_{n-1}-1-3 s \\
p_{n+1} & =3 p_{n}-p_{n-1}
\end{aligned}
$$

which is desired result.

1) Binet formulas for the binomal transforms of generalized Fibonacci-Like sequences: In this section, we give the Binet formulas of the binomial transforms $P=\left\{p_{n}\right\}$ and $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$.

The characteristic equation of sequences $\left\{p_{n}\right\}$ and $\left\{r_{n}\right\}$ in equations (2.5) and (2.6) is

$$
x^{2}-3 x+1=0
$$

where $x_{1}$ and $x_{2}$ be roots of this equation. Also, the roots $x_{1}$ and $x_{2}$ verify the relations such as

$$
\begin{aligned}
x_{1} x_{2} & =1 \\
x_{1}+x_{2} & =3 \\
x_{1}-x_{2} & =\sqrt{5}
\end{aligned}
$$

where $x_{1}=\frac{3+\sqrt{5}}{2}$ and $x_{2}=\frac{3-\sqrt{5}}{2}$.
Theorem 2. i. The $n$. term of the binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ is given by

$$
\begin{equation*}
p_{n}=\frac{\left(1+s\left(3-2 x_{2}\right)\right) x_{1}^{n}-\left(1+s\left(3-2 x_{1}\right)\right) x_{2}^{n}}{\sqrt{5}} \tag{2.9}
\end{equation*}
$$

where $x_{1}=\frac{3+\sqrt{5}}{2}, x_{2}=\frac{3-\sqrt{5}}{2}$ and $s$ being fixed positive integer.
ii. The $n$. term of the binomial transform $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{D_{n}\right\}$ is given by

$$
\begin{equation*}
r_{n}=\frac{\left(3+m-2 x_{2}\right) x_{1}^{n}-\left(3+m-2 x_{1}\right) x_{2}^{n}}{\sqrt{5}} \tag{2.10}
\end{equation*}
$$

where $x_{1}=\frac{3+\sqrt{5}}{2}, x_{2}=\frac{3-\sqrt{5}}{2}$ and $m$ being fixed positive integer.
Proof. Firstly we will just prove to (i), since prove to (ii) can be obtained in the same manner with (i).
i. The characteristic equation of $p_{n+1}=3 p_{n}-p_{n-1}$ recurrence relation is $x^{2}-3 x+1=0$. The solutions of this equation are

$$
x_{1}=\frac{3+\sqrt{5}}{2} \text { and } x_{2}=\frac{3-\sqrt{5}}{2} \text {. }
$$

The general term of binomial transforms $P=\left\{p_{n}\right\}$ may be expressed in the forms

$$
p_{n}=U x_{1}^{n}+V x_{2}^{n}
$$

for some cofficients $U$ and $V$. Giving to $n$ the values $n=0$ and $n=1$, we obtain

$$
\begin{aligned}
2 s & =U+V \\
1+3 s & =U x_{1}+V x_{2}
\end{aligned}
$$

where $p_{0}=2 s$ and $p_{1}=1+3 s$ from Theorem 1 .

Then, we get

$$
U=\frac{1+s\left(3-2 x_{2}\right)}{\sqrt{5}} \text { and } V=-\frac{1+s\left(3-2 x_{1}\right)}{\sqrt{5}} \text {. }
$$

So, we take $U=\frac{s\left(3-2 x_{2}\right)+1}{\sqrt{5}}$ and $V=-\frac{1+s\left(3-2 x_{1}\right)}{\sqrt{5}}$ in form $p_{n}=U x_{1}^{n}+V x_{2}^{n}$ and obtain desired formula.
2) Generating function for the binomal transforms of generalized Fibonacci-Like sequences: In this section, we give the generating functions of the binomial transforms $P=\left\{p_{n}\right\}$ and $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$.

A generating function $g(x)$ is a formal power series

$$
g(x)=\sum_{n=0}^{\infty} a_{n} x^{n}
$$

whose coefficients give the sequence $\left\{a_{0}, a_{1}, a_{2}, \ldots\right\}$. Given a generating function is the analytic expression for the $n t h$ term in the corresponding series.

Theorem 3. i. The generating function of the binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ is

$$
\begin{equation*}
p_{n}(x)=\frac{2 s+(1-3 s) x}{1-3 x+x^{2}} \tag{2.11}
\end{equation*}
$$

where $s$ being fixed positive integer.
ii. The generating function of the binomial transform $R=\left\{r_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{D_{n}\right\}$ is

$$
\begin{equation*}
r_{n}(x)=\frac{2+(m-3) x}{1-3 x+x^{2}} \tag{2.12}
\end{equation*}
$$

where $m$ being fixed positive integer.
Prof. Firstly we will just prove to (i), since prove to (ii) can be obtained in the same manner with (i).
i. Let the terms of binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ are cofficient of a potential series centered at the origin and consider the corresponding analytic function $p_{n}(x)$ such that

$$
p_{n}(x)=p_{0}+p_{1} x+p_{2} x^{2}+\cdots
$$

Then we can write

$$
\begin{gathered}
3 p_{n}(x) x=3 p_{0} x+3 p_{1} x^{2}+3 p_{2} x^{3}+\cdots \\
p_{n}(x) x^{2}=p_{0} x^{2}+p_{1} x^{3}+p_{2} x^{4}+\cdots
\end{gathered}
$$

From the above last two equations, we obtain

$$
\begin{gathered}
p_{n}(x)-3 p_{n}(x) x+p_{n}(x) x^{2}=p_{0}+p_{1} x-3 p_{0} x \\
\left(1-3 x+x^{2}\right) p_{n}(x)=2 s+(1+3 s) x-(6 s) x
\end{gathered}
$$

where $p_{n+1}=3 p_{n}-p_{n-1}, \quad p_{0}=2 s$ and $p_{1}=1+3 s$ from equation (2.5). So, the generating function of the binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequence $\left\{B_{n}\right\}$ is

$$
p_{n}(x)=\frac{2 s+(1-3 s) x}{1-3 x+x^{2}}
$$

3) Triangles of the binomal transforms of generalized Fibonacci-Like sequences: In this section, we introduce an infinite triangle consist of the terms of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and $\left\{D_{n}\right\}$ and their binomial transforms $P=\left\{p_{n}\right\}$ and $R=\left\{r_{n}\right\}$. Let be $T$ an infinite triangle of numbers by using the following rule:
i. The left diagonal of the triangle consists of the terms of generalized Fibonacci-Like sequences,
ii. Any number off the left diagonal is the sum of the number to its left and number diagonally above it to the left.

For example, the triangle $T=\left\{t_{n}\right\}$ for generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and its binomial transform $P=\left\{p_{n}\right\}$ is following:

$$
\begin{aligned}
& 2 s \\
& 1+s \quad 1+3 s \\
& 1+3 s \quad 2+4 s \quad 3+7 s \\
& 2+4 s \quad 3+7 s \quad 5+11 s \quad 8+18 s \\
& 3+7 s \quad 5+11 s \quad 8+18 s \quad 13+29 s \quad 21+47 s
\end{aligned}
$$

Note that the antidiagonal sequences $\left\{t_{n}\right\}$ of this triangle verify the $t_{n+1}=t_{n}+t_{n-1}$ relation as generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ while the all diagonal sequences hold $t_{n+1}=3 t_{n}-t_{n-1}$ relation of binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$. That is the sequence on the left diagonal $\{2 s, 1+s, 1+3 s, 2+4 s, 3+7 s, \ldots\}$ is generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ and the sequence on the right diagonal $\{2 s, 1+3 s, 3+7 s, 8+18 s, 21+47 s, \ldots\}$ is the binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ which was defined in this study. So, every antidiagonal sequence $\left\{t_{n}\right\}$ of this triangle verify the relation as generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$ while the all diagonal sequences $\left\{t_{n}\right\}$ hold relation of the binomial transform $P=\left\{p_{n}\right\}$ of generalized Fibonacci-Like sequences $\left\{B_{n}\right\}$.

Similarly, we can obtain the triangle $T=\left\{t_{n}\right\}$ for generalized Fibonacci-Like sequences $\left\{D_{n}\right\}$ and its binomial transform $R=\left\{r_{n}\right\}$.

## III. Conclusions

There are a lot of studies on sequences of integer numbers such as the Fibonacci and Lucas sequences are well-known second order recurrence sequences in to almost every field of science and art. In this study, we present binomial transforms of generalized Fibonacci-Like sequences associated with Fibonacci and Lucas sequences. Also, the Binet formulas, generating functions of these transforms are obtained. Moreover, we introduce an infinite triangle consist of the terms of generalized Fibonacci-Like sequences and their binomial transforms.

In the future, we intend to discuss binomial transforms of generalized Fibonacci-Like sequences associated with Pell, Pell-Lucas, Jacobsthal sequences.

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