Fuzzy Space-Time Fractional Telegraph Equations

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Abstract - *In this paper, the Laplace Variational Iteration Method (LVIM)and the Caputo fuzzy fractional derivatives applied to find the exact fuzzy solution of the fuzzy space-time fractional telegraph equations. We present a new concept of solutions, we investigate the problem of finding new solutions. An example is given for which the new solutions are found.*

Keywords - *Laplace Variational Iteration Method, Caputo fuzzy fractional.*

I. INTRODUCTION

The theory of fractional calculus, which deals with the investigation and applications of derivatives and integrals of arbitrary order has a long history. The theory of fractional calculus developed mainly as a pure theoretical field of mathematics, in the last decades it has been used in various fields as rheology, viscoelasticity, electrochemistry, diffusion processes, etc. The telegraph equations have a wide variety of application in physics and engineering. The applications arise, for example, in the propagation of electrical signals and optimization of guided communication systems [4, 8, 9]. It is recently shown by [2] that a quaternionic momentum eigenvalue produces a telegraph equation. This equation is found to describe the propagation of a quantum particle. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning. Therefore, assume space-time telegraph models have imprecise parameters, since fuzzy sets theory is a powerful tool for modeling imprecise and processing vague in mathematical models, hence, the our idea is solving space-time telegraph equations with fuzzy parameters via the new concept of solution and, utilizing the Caputo-type fuzzy fractional derivatives [14] and Laplace Variational Iteration Method(LVIM).

The (LVIM)[1]combined the Laplace transform and variational iteration method [15, 3, 10], has been introduced by many authors in solving various types of problems. and apply it to space-time one dimensional fractional telegraph equations in a half-space domain (signaling problem). This approach enables us to overcome the difficulties that arise in finding the general Lagrange multiplier.

The paper is organized as follows: in Section 2, we call some fundamental results on fuzzy numbers. In Sections 3, illustrates the construction of Laplace variational technique In Section 4, we study the fuzzy space-time telegraph models using the concept of caputo fuzzy fractional derivative and present a new concept of solution In Section 5, we present an example to illustrate our method.

II. PRELIMINARIES

We place a bar over a capital letter to denote a fuzzy number of $\mathbb{R}^n$ So, $\bar{A}$, $\bar{K}$, $\bar{\gamma}$, $\bar{\beta}$ etc. all represent fuzzy numbers of $\mathbb{R}^n$ for some n. We write $\mu_{\bar{A}}(t)$, a number in $[0,1]$, for the membership function of $\bar{A}$ evaluated at $t \in \mathbb{R}^n$. An α −cut of $\bar{A}$, is always a closed and bounded interval that written $\bar{A}[\alpha]$, is defined as $\{t|\mu_{\bar{A}}(t) \geq \alpha\}$for $0 < \alpha < 1$ We separately specify $\bar{A}[0]$, as the closure of the union of all the $\bar{A}[\alpha]$, for $0 < \alpha \leq 1$.

Definition1 : [7] Let $\mathbb{R}_{\mathcal{F}} = \{\bar{A}|\ \bar{A} : \mathbb{R} \to [0,1]\}$, satisfies (1)-(4) :

1. $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, $\bar{A}$ is normal.
2. $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, $\bar{A}$ is a fuzzy convex set.
3. $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, $\bar{A}$ is upper semi-continuous on $\mathbb{R}$.
4. $\bar{A}[0]$ is a compact set.

Then $\mathbb{R}_{\mathcal{F}}$ is called fuzzy number space and $\forall \bar{A} \in \mathbb{R}_{\mathcal{F}}$, $\bar{A}$ is called a fuzzy number.

Definition2 : [7,13]We represent an arbitrary fuzzy number by an ordered pair of functions $\bar{A}[\alpha] = [A_1(\alpha), A_2(\alpha)]$, $\alpha \in [0,1]$, which satisfy the following requirements :

1. $A_1(\alpha)$ is a nondecreasing function over $[0,1]$,
2. $A_2(\alpha)$ is a nonincreasing function on $[0,1]$
3. $A_1(\alpha)$ and $A_2(\alpha)$ are bounded left continuous on $(0,1]$, and right continuous at $\alpha = 0$, and

4. $A_1(\alpha) \leq A_2(\alpha)$, for $0 \leq \alpha \leq 1$.

It should be noted that for $a \leq b \leq c$, $a, b, c \in \mathbb{R}$ a triangular fuzzy number $\bar{A}$ =(a,b,c) is given such that $a_1(\alpha)$ = b-(1- α)(b-a) and $a_1(\alpha)$ =b+(1- α)(c-b) are the endpoint of the α-cut set, for all $0 \leq \alpha \leq 1$. In this paper we use triangular fuzzy numbers. For arbitrary fuzzy numbers $\bar{A}\ [\alpha] = [a_1(\alpha), a_2(\alpha)]$ and $\bar{B}\ [\alpha] = [b_1(\alpha), b_2(\alpha)]$ we have algebraic operations as follows :

1. $(\bar{A} + \bar{B})\ [\alpha] = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]$
2. $(\bar{A} - \bar{B})\ [\alpha] = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)]$
3. $k\ \bar{A}\ [\alpha] = \begin{cases} [k\ a_1(\alpha), k\ a_2(\alpha)], & \text{k} \geq 0 \\ [k\ a_2(\alpha), k\ a_1(\alpha)], & \text{k} < 0 \end{cases}$

Definition3: Let $\bar{A}$, $\bar{B} \in \mathbb{R}_{\mathcal{F}}$. If there exists $\bar{C} \in \mathbb{R}_{\mathcal{F}}$. such that $\bar{A} = \bar{B} + \bar{C}$, then $\bar{C}$ is called the Hukuhara difference (H-difference) of $\bar{A}$, $\bar{B}$, and it is denoted by
$\bar{A} \ominus \bar{B} \neq \bar{A} + (-1)\,\bar{B}$.

Definition4: Let $F{:}\,T \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function. One says, F is (1) -differentiable at $x_0 \in T$ if there exists an element $F'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that for all $h > 0$ sufficiently near to 0, there exist

$$F(x_0 + h) \ominus F(x_0), F(x_0) \ominus F(x_0 - h)$$

$$\lim_{h\to 0^+} \frac{F(x_0 + h) \ominus F(x_0)}{h} = \lim_{h\to 0^+} \frac{F(x_0) \ominus F(x_0 - h)}{h} = F'(x_0) \qquad (1)$$

F is (2) -differentiable at $x_0 \in T$ if there exists an element $F'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that for all $h < 0$ sufficiently near to 0, there exist

$$F(x_0 + h) \ominus F(x_0), F(x_0) \ominus F(x_0 - h)$$

$$\lim_{h\to 0^-} \frac{F(x_0 + h) \ominus F(x_0)}{h} = \lim_{h\to 0^-} \frac{F(x_0) \ominus F(x_0 - h)}{h} = F'(x_0) \qquad (2)$$

Definition5: The Caputo fractional derivative of order $\beta > 0$ of a crisp continuous function $f(x), x > 0$ is defined by [11]

$$D_x^\beta f(x) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} f^{(n)}(\text{t})\text{dt}, & \text{n} - 1 < \beta \leq n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x), & \beta = n \in \mathbb{N} \end{cases} \qquad (3)$$

where D_x^β is called the Caputo derivative operator.

From Definition **5** the following result is obtained:

$$D_x^\beta t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} t^{\beta-\gamma}, & \text{n} - 1 < \beta \leq n,\ \gamma > n-1,\ \gamma \in \mathbb{R} \\ 0, & \text{n} - 1 < \beta \leq n,\ \gamma \leq n-1. \end{cases} \qquad (4)$$

Definition6: The Laplace transform of fractional order derivative, is defined by [11]

$$\mathcal{L}\left[D_x^\beta f(x)\right] = s^\beta \mathcal{L}[f(x)] - \sum_{k=0}^{n-1} s^{\beta-k-1} [f^{(k)}(x)]_{x=0}, \quad \text{n} - 1 < \beta \leq n,\ n \in \mathbb{N} \qquad (5)$$

Definition7: The Mittag-Leffler function with two parameters is defined by [16,11]

$$E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \gamma)}, \beta, \gamma, z \in \mathbb{C},\ \ Re(\beta) > 0, Re(\gamma) > 0$$

It follows Definition **7** that
$E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$, and $E_{2,3}(z^2) = \frac{1}{z^2}[\cosh(z) - 1]$.

III.LAPLACE VARIATIONAL ITERATION METHOD (LVIM)

Consider the following general multiterms fractional telegraph equation:

$$\begin{cases} D_x^{\beta}\,\mathrm{U(t,x)} + \mathrm{f(t,x)} = \mathrm{a_1 D_t^{\delta}\,U(t,x)} + \mathrm{a_2 D_t^{\gamma}\,U(t,x)} + \mathrm{a_3\,U(t,x)} \\ \qquad \mathrm{U(0,t) = h(t), \quad D_x U(0,t) = g(t)} \end{cases} \tag{6}$$

where $1 < \beta, \delta \le 2, 0 < \gamma \le 1,\ t, x \ge 0$ and $\mathrm{a_1, a_2, a_3}$ are constants.

The concept of the technique (LVIM) [1] is illustrated in the following context. By applying Laplace transform with respect to x, on both sides of (6) we get

$$s^{\beta}\tilde{u}(s,t) - s^{\beta-1}u(0,t) - s^{\beta-2}u_x(0,t) = \mathcal{L}\left[\mathrm{a_1}\frac{\partial^{\delta}}{\partial t^{\delta}}u(x,t) + \mathrm{a_2}\frac{\partial^{\gamma}}{\partial t^{\gamma}}u(x,t) + \mathrm{a_3 u}(x,t) - \mathrm{f}(x,t)\right] \quad (7)$$

$$u(s,t) = \frac{1}{s}h(t) + \frac{1}{s^2}g(t) - \frac{1}{s^{\beta}}\mathcal{L}[\mathrm{f}(x,t)] + \frac{1}{s^{\beta}}\,\mathcal{L}\left[\mathrm{a_1}\frac{\partial^{\delta}}{\partial t^{\delta}}u(x,t) + \mathrm{a_2}\frac{\partial^{\gamma}}{\partial t^{\gamma}}u(x,t) + \mathrm{a_3 u}(x,t)\right] \quad (8)$$

By taking the inverse Laplace transform to (7) we have

$$u(x,t) = h(t) + xg(t) - \mathcal{L}^{-1}[\frac{1}{s^{\beta}}\mathcal{L}[\mathrm{f}(x,t)]] + \mathcal{L}^{-1}[\frac{1}{s^{\beta}}\,\mathcal{L}\left[\mathrm{a_1}\frac{\partial^{\delta}}{\partial t^{\delta}}u(x,t) + \mathrm{a_2}\frac{\partial^{\gamma}}{\partial t^{\gamma}}u(x,t) + \mathrm{a_3 u}(x,t)\right]\,] \quad (9)$$

Now the fractional derivative of order β with respect to x is removed, and the dependent variable $u(x,t)$ in the left hand side of (9) became free of derivatives. Next step, we differentiate (7) with respect to get

$$\frac{\partial}{\partial x}u(x,t) = g(t) - \frac{\partial}{\partial x}[\mathcal{L}^{-1}[\frac{1}{s^{\beta}}\mathcal{L}[\mathrm{f}(x,t)]]] + \frac{\partial}{\partial x}[\mathcal{L}^{-1}[\frac{1}{s^{\beta}}\,\mathcal{L}\left[\mathrm{a_1}\frac{\partial^{\delta}}{\partial t^{\delta}}u(x,t) + \mathrm{a_2}\frac{\partial^{\gamma}}{\partial t^{\gamma}}u(x,t) + \mathrm{a_3 u}(x,t)\right]]] \quad (10)$$

The above step has been taken to enable us to construct the correction functional for (9) to be

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda\{\frac{\partial}{\partial\xi}u_n(x,t) - g(t) + \frac{\partial}{\partial\xi}[\mathcal{L}^{-1}[\frac{1}{s^{\beta}}\mathcal{L}[\mathrm{f}(x,t)]]] \\ - \frac{\partial}{\partial\xi}[\mathcal{L}^{-1}[\frac{1}{s^{\beta}}\,\mathcal{L}\left[\mathrm{a_1}\frac{\partial^{\delta}}{\partial t^{\delta}}u_n(x,t) + \mathrm{a_2}\frac{\partial^{\gamma}}{\partial t^{\gamma}}u_n(x,t) + \mathrm{a_3}u_n(x,t)\right]]]\}d\xi \quad (11) \end{aligned}$$

The general Lagrange multiplier for (11) can be identified optimally via variation theory to get $1 + \lambda|_{\xi=x}$=0, $\lambda'|_{\xi=x}$ =0 then, we obtain $\lambda = -1$

Substituting $\lambda = -1$ into (11) we get the iterative formula for $\mathrm{n} = 0, 1, 2, 3..$, as follows:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) - \int_0^x \{\frac{\partial}{\partial\xi}u_n(x,t) - g(t) + \frac{\partial}{\partial\xi}[\mathcal{L}^{-1}[\frac{1}{s^{\beta}}\mathcal{L}[\mathrm{f}(x,t)]]] \\ - \frac{\partial}{\partial\xi}[\mathcal{L}^{-1}[\frac{1}{s^{\beta}}\,\mathcal{L}\left[\mathrm{a_1}\frac{\partial^{\delta}}{\partial t^{\delta}}u_n(x,t) + \mathrm{a_2}\frac{\partial^{\gamma}}{\partial t^{\gamma}}u_n(x,t) + \mathrm{a_3}u_n(x,t)\right]]]\}d\xi \quad (12) \end{aligned}$$

Start with the initial iteration

$$u_0(x,t) = \mathrm{u(0,t)} = \mathrm{x}U_x(0,t) = \mathrm{h(t) + x\,g(t)}$$

The exact solution is given as a limit of the successive approximations $u_n(x,t)$, n=0,1,2… in other words, u(x,t)= $\lim_{n\to\infty} u_n(x,t)$

IV. FUZZY SPACE-TIME FRACTIONAL TELEGRAPH EQUATIONS

A. Caputo fuzzy fractional derivatives

Now we introduce definition and theorems for the order $1 < \beta < 2, \beta \ne 1$ derivative based on the selection of derivative type in each step of differentiation. (For $\beta = 1$ see [6] and $\beta = 2$ see[12,5]).

Theorem1: let $\mathrm{F} \in \mathrm{C}((0,\mathrm{a}), \mathbb{R}_{\mathcal{F}}) \cap L^1(0,\mathrm{a}), \mathbb{R}_{\mathcal{F}})$ be a fuzzy valued function and $[\mathrm{F(x)}] =^{\alpha} [f_1(\mathrm{x},\alpha), f_2(\mathrm{x},\alpha)]$ for $\alpha \in [0,1]$ and $x_0 \in (0,a)$ then for $0 < \beta < 1$.

(i) If F is (1)-Caputo fractional differentiable then $f_1(x,a)$ and $f_2(x,a)$, Caputo fractional differentiable and

$$[D_1^{\beta}F(x_0)]^{\alpha} = [D^{\beta}f_1(\mathrm{x_0},\alpha), D^{\beta}f_2(\mathrm{x_0},\alpha)]$$

(ii) If F is (2)-Caputo fractional differentiable then $f_1(x,a)$ and $f_2(x,a)$, Caputo fractional differentiable and

$$[D_2^{\beta}F(x_0)]^{\alpha} = [D^{\beta}f_2(\mathrm{x_0},\alpha), D^{\beta}f_1(\mathrm{x_0},\alpha)]$$

Where

$$D^{\beta}f_1(x_0,\alpha) = [\frac{1}{\Gamma(n-\beta)}\int_0^x (x-t)^{n-\beta-1}\,f_1^{(n)}(\mathrm{t},\alpha)\mathrm{dt}]_{x=x_0}$$

$$D^{\beta} f_2(x_0,\alpha) = [\frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} f_2^{(n)}(\mathrm{t},\alpha)\mathrm{dt}]_{x=x_0}$$

Proof. See [14].

For a given fuzzy valued function $F(x)$, we have tow possibilities (as in definition 4) to obtain the Caputo fractional differentiable of F at x: $D_1^{\beta} F(x)$ and $D_2^{\beta} F(x)$ for $0 < \beta < 1$. Then for each of these two derivatives, we have again two possibilities for $1 < \beta < 2$ (see definition 3.1 [14]):

$D_1^{\beta}\mathrm{F(x)} \begin{smallmatrix} \nearrow D_{1,1}^{\beta}\mathrm{F(x)} \\ \searrow D_{2,1}^{\beta}\mathrm{F(x)} \end{smallmatrix}$ and for $D_2^{\beta}\mathrm{F(x)} \begin{smallmatrix} \nearrow D_{1,2}^{\beta}\mathrm{F(x)} \\ \searrow D_{2,2}^{\beta}\mathrm{F(x)} \end{smallmatrix}$

Definition8: Let $\mathrm{F} \in \mathrm{C}((0,\mathrm{a}),\mathbb{R}_{\mathcal{F}}) \cap L^1((0,\mathrm{a}),\mathbb{R}_{\mathcal{F}})$ and $n,m = \{1,2\}$. One says F is (n,m)-Caputo fractional differentiable at $x_0 \in (0,\mathrm{a})$ if $D_n^{\beta} F$ for $0 < \beta < 1$ exists on a neighborhood of x_0 as a fuzzy valued function and it is (m)-Caputo fractional differentiable at x_0 for $1 < \beta < 2$ are denoted by $D_{n,m}^{\beta} F(x_0)$ for $n,m = \{1,2\}$. Using Definition 4 we have:

Theorem2: Let $D_1^{\beta}\mathrm{F(x)}$ or $D_2^{\beta} F(x) \in \mathrm{C}((0,\mathrm{a}),\mathbb{R}_{\mathcal{F}})$ be a fuzzy valued function and $[\mathrm{F(x)}]^{\alpha} = [f_1(\mathrm{x},\alpha), f_2(\mathrm{x},\alpha)]$, for $\alpha \in [0,1]$ and $x_0 \in (0,a)$ then for $1 < \beta < 2$

(i) If $D_1^{\beta}\mathrm{F(x)}$ is (1)-Caputo fractional differentiable function then $D^{\beta} f_1(\alpha)$ and $D^{\beta} f_2(\alpha)$ Caputo fractional differentiable and

$[D_{1,1}^{\beta}\mathrm{F(x_0)}]^{\alpha} = [\,D^{\beta} f_1(\mathrm{x_0},\alpha),\ D^{\beta} f_2(\mathrm{x_0},\alpha)\,]$

(ii) If $D_1^{\beta}\mathrm{F(x)}$ is (2)-Caputo fractional differentiable function then $D^{\beta} f_1(\alpha)$ and $D^{\beta} f_2(\alpha)$ Caputo fractional differentiable and

$[D_{1,2}^{\beta}\mathrm{F(x_0)}]^{\alpha} = [\,D^{\beta} f_2(\mathrm{x_0},\alpha),\ D^{\beta} f_1(\mathrm{x_0},\alpha)\,]$

(iii) If $D_2^{\beta}\mathrm{F(x)}$ is (1)-Caputo fractional differentiable function then $D^{\beta} f_1(\alpha)$ and $D^{\beta} f_2(\alpha)$ Caputo fractional differentiable and

$[D_{2,1}^{\beta}\mathrm{F(x_0)}]^{\alpha} = [\,D^{\beta} f_2(\mathrm{x_0},\alpha),\ D^{\beta} f_1(\mathrm{x_0},\alpha)\,]$

(iv) If $D_2^{\beta}\mathrm{F(x)}$ is (2)-Caputo fractional differentiable function then $D^{\beta} f_1(\alpha)$ and $D^{\beta} f_2(\alpha)$ Caputo fractional differentiable and

$[D_{2,2}^{\beta}\mathrm{F(x_0)}]^{\alpha} = [\,D^{\beta} f_1(\mathrm{x_0},\alpha),\ D^{\beta} f_2(\mathrm{x_0},\alpha)\,]$

Where

$$D^{\beta} f_1(x_0,\alpha) = [\frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} f_1^{(n)}(\mathrm{t},\alpha)\mathrm{dt}]_{x=x_0}$$

$$D^{\beta} f_2(x_0,\alpha) = [\frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} f_2^{(n)}(\mathrm{t},\alpha)\mathrm{dt}]_{x=x_0}$$

Proof. see [14].

IV.2 ***A new concept of solution for the Fuzzy Space-Time Fractional Telegraph Equations***

Consider the following fuzzy general multiterms fractional telegraph equation:

$$\begin{cases} D_x^{\beta}\mathrm{U(x,t)} + \mathrm{f(x,t)} = \mathrm{a_1 D_t^{\delta}\,U(x,t)} + \mathrm{a_2 D_t^{\gamma}\,U(x,t)} + \mathrm{a_3\,U(x,t)} \quad \mathrm{t} \in (0,\mathrm{a}), \qquad \mathrm{x} \in (0,\mathrm{b}) \\ \mathrm{U(0,t)} = \mathrm{h(t)}, \quad \mathrm{D_x U(0,t)} = \mathrm{g(t)} \end{cases} \tag{13}$$

Where $a_1, a_2, a_3 > 0$ and $h(t), g(t)$ and $f(t,x)$ are a continuous fuzzy functions.

Now we consider Equation (13) with generalized fractional differentiability and introduce a new class of solutions.

Definition9: Let $\lambda \in \{\gamma,\beta,\delta\}$ and $\mathrm{U(x,t)} \in \mathrm{C}(\mathrm{I} \times J, \mathbb{R}_{\mathcal{F}}) \cap L^1(\mathrm{I} \times J, \mathbb{R}_{\mathcal{F}})$ be a fuzzy valued function and $n,m \in \{1,2\}$. One says (n,m)-solution for problem (13) on $\mathrm{I} \times J$, if $D_n^{\lambda}\mathrm{U(x,t)}$ for $0 < \lambda < 1$ and $D_{n,m}^{\lambda}\mathrm{U(x,t)}$ for

$1 < \lambda < 2$ exist on $\mathrm{I} \times J$ and

$$\begin{cases} D_{n,m,(x)}^{\beta}\mathrm{U(x,t)} + \mathrm{f(x,t)} = \mathrm{a_1 D_{n,m,(t)}^{\delta}\,U(x,t)} + \mathrm{a_2 D_{n,(t)}^{\gamma}\,U(x,t)} + \mathrm{a_3\,U(x,t)} \quad \mathrm{t} \in \mathrm{I}, \qquad \mathrm{x} \in \mathrm{J} \\ \mathrm{U(0,t)} = \mathrm{h(t)}, \quad \mathrm{D_{n(x)} U(0,t)} = \mathrm{g(t)} \end{cases}$$

Let U be an, (n,m)-solution for (13). To find it, utilizing theorem (2) and considering the initial values, we can translate problem (13) to system space-time fractional telegraph equations hereafter, called corresponding (n,m)-system for problem (13).
Therefore, four space-time fractional telegraph equations systems are possible for problem (13), as follows:
(1,1)-system:

$$\begin{cases} D_x^{\beta}U_1(x,t,\alpha)+f_1(x,t,\alpha)=a_1D_t^{\delta}U_1(x,t,\alpha)+\ a_2D_t^{\gamma}U_1(x,t,\alpha)+a_3\,U_1(x,t,\alpha), \\ D_x^{\beta}U_2(x,t,\alpha)+f_2(x,t,\alpha)=a_1D_t^{\delta}U_2(x,t,\alpha)+\ a_2D_t^{\gamma}U_2(x,t,\alpha)+a_3\,U_2(x,t,\alpha), \\ U_1(0,t,\alpha)=h_1(t,\alpha),\ U_2(0,t,\alpha)=h_2(t,\alpha), \\ D_xU_1(0,t,\alpha)=g_1(t,\alpha),\ D_xU_2(0,t,\alpha)=g_2(t,\alpha) \end{cases} \quad (14)$$

(1,2)-system:

$$\begin{cases} D_x^{\beta}U_2(x,t,\alpha)+f_1(x,t,\alpha)=a_1D_t^{\delta}U_2(x,t,\alpha)+\ a_2D_t^{\gamma}U_1(x,t,\alpha)+a_3\,U_1(x,t,\alpha), \\ D_x^{\beta}U_1(x,t,\alpha)+f_2(x,t,\alpha)=a_1D_t^{\delta}U_1(x,t,\alpha)+\ a_2D_t^{\gamma}U_2(x,t,\alpha)+a_3\,U_2(x,t,\alpha), \\ U_1(0,t,\alpha)=h_1(t,\alpha),\ U_2(0,t,\alpha)=h_2(t,\alpha), \\ D_xU_1(0,t,\alpha)=g_1(t,\alpha),\ D_xU_2(0,t,\alpha)=g_2(t,\alpha) \end{cases} \quad (15)$$

(2,1)-system:

$$\begin{cases} D_x^{\beta}U_2(x,t,\alpha)+f_1(x,t,\alpha)=a_1D_t^{\delta}U_2(x,t,\alpha)+\ a_2D_t^{\gamma}U_2(x,t,\alpha)+a_3\,U_1(x,t,\alpha), \\ D_x^{\beta}U_1(x,t,\alpha)+f_2(x,t,\alpha)=a_1D_t^{\delta}U_1(x,t,\alpha)+\ a_2D_t^{\gamma}U_1(x,t,\alpha)+a_3\,U_2(x,t,\alpha), \\ U_1(0,t,\alpha)=h_1(t,\alpha),\ U_2(0,t,\alpha)=h_2(t,\alpha), \\ D_xU_2(0,t,\alpha)=g_1(t,\alpha),\ D_xU_1(0,t,\alpha)=g_2(t,\alpha) \end{cases} \quad (16)$$

(2,2)-system:

$$\begin{cases} D_x^{\beta}U_1(x,t,\alpha)+f_1(x,t,\alpha)=a_1D_t^{\delta}U_1(x,t,\alpha)+\ a_2D_t^{\gamma}U_2(x,t,\alpha)+a_3\,U_1(x,t,\alpha), \\ D_x^{\beta}U_2(x,t,\alpha)+f_2(x,t,\alpha)=a_1D_t^{\delta}U_2(x,t,\alpha)+\ a_2D_t^{\gamma}U_1(x,t,\alpha)+a_3\,U_2(x,t,\alpha), \\ U_1(0,t,\alpha)=h_1(t,\alpha),\ U_2(0,t,\alpha)=h_2(t,\alpha), \\ D_xU_2(0,t,\alpha)=g_1(t,\alpha),\ D_xU_1(0,t,\alpha)=g_2(t,\alpha) \end{cases} \quad (17)$$

Our strategy of solving (13) is based on the selection of derivative type in the fuzzy space-time fractional telegraph equations. We first choose the type of solution and translate problem (13) to the corresponding system. Then, we solve the obtained space-time fractional telegraph equations by Laplace variational iteration method (LVIM). Finally we find such a domain in which the solution and its derivatives have valide level sets according to the type of Caputo fractional differentiable and using the representation Theorem [11] we can construct the solution of the fuzzy space-time fractional telegraph equations(13).

Theorem3: Let $n,m\in\{1,2\}$ and $U=\ [U_1,U_2]$, be an (n,m)-solution for problem (13) on $I\times J$. Then U_1 and U_2 solve the associated (n,m)-systems.
Proof.
Suppose U is the (n,m)-solution of problem(13). According to the definition9, then $D_n^{\lambda}U$ for $0<\lambda<1$ and $D_{n,m}^{\lambda}U$ for $0<\lambda<2$ exist and satisfy problem (13). By theorem2 and substituting U_1,U_2 and their Caputo fractional derivatives in problem(13), we get the (n,m)-system corresponding to (n,m)-solution.

Theorem4: Let $n,m\in\{1,2\}$, $\lambda\in\{\gamma,\beta,\delta\}$ and $U_1(x,t,\alpha)$ and $U_2(x,t,\alpha)$ solve the (n,m)-system on I×J, for every $\alpha\in[0,1]$. Let $[U(x,t)]^{\alpha}=[U_1(x,t,\alpha),U_2(x,t,\alpha)]$ If U has valid level sets on I×J and $D_{n,m}^{\lambda}U$ for $0<\lambda<1$ and $D_{n,m}^{\lambda}U$ for $1<\lambda<2$ exists, then U is an (n,m)-solution for the fuzzy problem(13).
Proof.
Since $[U(x,t)]^{\alpha}=[U_1(x,t,\alpha),U_2(x,t,\alpha)]$ is (n,m)-Caputo fractional differentiable, by theorem2 we can compute D_n^{λ} for $0<\lambda<1$ and $D_{n,m}^{\lambda}U$ for $0<\lambda<2$ according to $D_n^{\lambda}U_1(x,t,\alpha)$,$D_n^{\lambda}U_2(x,t,\alpha)$, for $0<\lambda\leq 1$ and $D_{n,m}^{\lambda}U_1(x,t,\alpha)$, $D_{n,m}^{\lambda}U_2(x,t,\alpha)$, for $0<\lambda<2$. Due to the fact that $U_1(x,t,\alpha)$, $U_2(x,t,\alpha)$ solve (n,m)-system, from definition9, it comes that U is an (n,m)-solution for (13).

Remark1: Let $\lambda=\{\delta,\beta,\delta\}$
If $U(x,t)$ is (n,m)-Caputo fractional differentiable for $0<\lambda<2$, $\lambda\neq 1$ then $U(x,t)$ is (n,m)-Caputo fractional differentiable with respect to x,t.

If $U(x,t)$ is (n,m)-solution on $I\times J$ for n, m ∈ {1,2}, then $U(x,t)$ is (n,m)-Caputo fractional differentiable on $I\times J$.

Remark2: Let $\lambda=\{1,2\}$

If $U(x,t)$is (n,m)-solution on $I\times J$ for $n,m\in\{1,2\}$, then $U(x,t)$ is (n,m)-differentiable on $I\times J$ see [12,5].

Remark3: We see that the solution of the fuzzy space-time fractional telegraph equations (13) depends upon the selection of derivatives.

It is clear that in this new procedure, the uniqueness of the solution is lost, an expected situation in the fuzzy context. Nonetheless, we can contemplate the existence of at most four solutions as shown in the examples of the next section.

V. EXAMPLE

Consider the following fuzzy space-time fractional nonhomogeneous telegraph equation [1]:

$$\begin{cases} D_x^{\beta}\mathrm{U(x,t)}+\mathrm{f(x,t,K)}=\mathrm{D_t^2\,U(x,t)}+\mathrm{D_t^{\gamma}\,U(x,t)}+\mathrm{U(x,t)} \quad \mathrm{t}\geq 0, \qquad \mathrm{x}\in[0,1] \\ \mathrm{U(0,t)=Ct,} \quad \mathrm{D_x U(0,t)=0} \end{cases} \tag{18}$$

Where $0<\beta\leq 2, 0<\gamma\leq 1, [\bar{K}]^{\alpha}=[\bar{C}]^{\alpha}=[\alpha,2-\alpha]$

$$[\bar{f}(x,t,\bar{K})]^{\alpha}=[f_1(x,t,\alpha),f_2(x,t,\alpha)]=[x^2+\alpha t-1,x^2+(2-\alpha)t-1]$$

And $[\bar{U}(0,t)]^{\alpha}=[U_1(0,t,\alpha),U_2(0,t,\alpha)]=[\alpha t,(2-\alpha)t]$

According to the LVIM, a correct functional for (18) in form (14) from (7,9,11,12) can be constructed solution as follows

$$U_1(x,t,\alpha)=\alpha t+x^2\left(1-2E_{\beta,3}(x^{\beta})\right)+\left[1+\alpha\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\left(E_{\beta,1}(x^{\beta})-1\right)$$

$$U_2(x,t,\alpha)=(2-\alpha)t+x^2\left(1-2E_{\beta,3}(x^{\beta})\right)+\left[1+(2-\alpha)\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\left(E_{\beta,1}(x^{\beta})-1\right)$$

If U is a (1,1)-solution for the problem, then

$$[D_xU_1(x,t,\alpha),\ D_xU_2(x,t,\alpha)]\,,[D_x^2U_1(x,t,\alpha),D_x^2U_2(x,t,\alpha)]$$

And $[D_tU_1(x,t,\alpha),\ D_tU_2(x,t,\alpha)]\,,[D_t^2U_1(x,t,\alpha),D_t^2U_2(x,t,\alpha)]$

and they satisfy the (14) associated with (13). On the other hand, by direct calculation, the corresponding solution of the (14) has necessarily the following expression of standard telegraph equation i.e for $\beta=2$ and $\gamma=1$:

$$U_1(x,t,\alpha)=\alpha t+x^2+(\alpha-1)[\cosh(x)-1]$$

$$U_2(x,t,\alpha)=(2-\alpha)t+x^2+(1-\alpha)[\cosh(x)-1] \tag{19}$$

We see $U_1(x,t,\alpha)$ and $U_2(x,t,\alpha)$ represent a valid fuzzy number for $t\geq 0, x\geq 0$. The (1)-derivative of (19) in that case is given by:

$$D_xU_1(x,t,\alpha)=2x+(\alpha-1)\sinh(x)\,,D_xU_2(x,t,\alpha)=2x+(1-\alpha)\sinh(x)$$

And

$$D_tU_1(x,t,\alpha)=\alpha\,,D_tU_2(x,t,\alpha)=2-\alpha$$

Then it is again (1)-differentiable and

$$D_x^2U_1(x,t,\alpha)=2+(\alpha-1)\cosh(x)\,,D_x^2U_2(x,t,\alpha)=2+(1-\alpha)\cosh(x)$$

We thus see that $U(x,t)$ defined by (19) is (1,1)-differentiable.

Hence $U(x,t), D_xU(x,t), D_tU(x,t)$ and $D_x^2U(x,t)$ have valid level sets for $t\geq 0, x\geq 0$, then $U(x,t)$ is (1,1)-differentiable and defines a (1,1)-solution of the fuzzy space-time fractional telegraph equation for $t\geq 0, x\geq 0$.

For (1,2)-solution, we get the solution for (15) of (18) using the LVIM i.e by(7,9,11,12):

$$U_1(t,x,\alpha)=\alpha(t-1)+x^2[1-2E_{\beta,3}(x^{\beta})]+[E_{\beta,1}(x^{\beta})-1]+\left[\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}(x^{\beta})\right]$$
$$+(\alpha-1)[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}]E_{\beta,1}(-x^{\beta})$$

$$U_2(t,x,\alpha)=(2-\alpha)(t-1)+x^2[1-2E_{\beta,3}(x^{\beta})]+[E_{\beta,1}(x^{\beta})-1]+\left[\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}(x^{\beta})\right]$$
$$+(1-\alpha)\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}(-x^{\beta}) \tag{20}$$

for $\beta = 2$ and $\gamma = 1$

$$U_1(t,x,\alpha) = \alpha(t-1) + x^2 + 1 + (\alpha-1)\cos(x)$$
$$U_2(t,x,\alpha) = (2-\alpha)(t-1) + x^2 + 1 + (1-\alpha)\cos(x)$$

Where $U(x,t)$ has valid level sets for $t \geq 1, x \in [0,1]$ and we have

$$D_x U_1(x,t,\alpha) = 2x + (\alpha-1)\sin(x)\,, D_x U_2(x,t,\alpha) = 2x + (1-\alpha)\sin(x)$$

and

$$D_x^2 U_1(x,t,\alpha) = 2 + (\alpha-1)\cos(x)\,, D_x^2 U_2(x,t,\alpha) = 2 + (1-\alpha)\cos(x)$$

Since $[D_x U_2(x,t,\alpha), D_x U_1(x,t,\alpha)], [D_x^2 U_2(x,t,\alpha), D_x^2 U_1(x,t,\alpha)]$ and $[D_t U_1(x,t,\alpha), D_t U_2(x,t,\alpha)]$ then $U(x,t)$ is (1,2)-differentiable on $x \in [0,1], t \geq 1$, hence, no (1,2)-solution exists for $x \in [0,1], t \geq 1$.

For (2,1)-solution of (18) we use the LVIM in (16) we deduce

$$U_1(t,x,\alpha) = \alpha t + x^2\left[1 - 2E_{\beta,3}\left(x^\beta\right)\right] + \left[E_{\beta,1}\left(x^\beta\right) - 1\right] + \left[\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}\left(x^\beta\right)\right]$$
$$+(1-\alpha)\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}\left(-x^\beta\right) - (2-\alpha)$$
$$U_2(t,x,\alpha) = (2-\alpha)t + x^2\left[1 - 2E_{\beta,3}\left(x^\beta\right)\right] + \left[E_{\beta,1}\left(x^\beta\right) - 1\right] + \left[\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}\left(x^\beta\right)\right]$$
$$+(\alpha-1)\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]E_{\beta,1}\left(-x^\beta\right) - \alpha \qquad (21)$$

for $\beta = 2$ and $\gamma = 1$

$$U_1(t,x,\alpha) = \alpha t + x^2 + (1-\alpha)\,(\cos(x) - 1)$$
$$U_2(t,x,\alpha) = (2-\alpha)t + x^2 + (\alpha-1)\,(\cos(x) - 1)$$

We see that the fuzzy function $U(x,t)$ has valid level sets for $t \geq 1, x \in [0,1]$.
Since $[D_x U_1(x,t,\alpha), D_x U_2(x,t,\alpha)], [D_x^2 U_1(x,t,\alpha), D_x^2 U_2(x,t,\alpha)]$ and $[D_t U_1(x,t,\alpha), D_t U_2(x,t,\alpha)]$ then $U(x,t)$ is (1,1)-differentiable on $x \in [0,1], t \geq 1$, hence, no (2,1)-solution exists for $t \geq 1, x \in [0,1]$.
Finally to find (2,2)-solution we use the LVIM in (17) we get

$$U_1(t,x,\alpha) = \alpha t + x^2\left[1 - 2E_{\beta,3}\left(x^\beta\right)\right] + \left[1 + (2-\alpha)\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\right]\left[E_{\beta,1}\left(x^\beta\right) - 1\right]$$

$$U_2(t,x,\alpha) = (2-\alpha)t + x^2\left[1 - 2E_{\beta,3}\left(x^\beta\right)\right] + \left[1 + \alpha\left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)}\right]\right]\left[E_{\beta,1}\left(x^\beta\right) - 1\right]$$

for $\beta = 2$ and $\gamma = 1$

$$U_1(t,x,\alpha) = \alpha t + x^2 + (1-\alpha)\,(\cosh(x) - 1)$$
$$U_2(t,x,\alpha) = (2-\alpha)t + x^2 + (\alpha-1)(\cosh(x) - 1)$$

that $U(x,t)$ has valid level sets for $t \geq \frac{5}{9}, x \in [0,1]$ and we have
Since $[D_x U_2(x,t,\alpha), D_x U_1(x,t,\alpha)], [D_x^2 U_2(x,t,\alpha), D_x^2 U_1(x,t,\alpha)]$ and $[D_t U_1(x,t,\alpha), D_t U_2(x,t,\alpha)]$ then $U(x,t)$ is (2,1)-differentiable on $x \in [0,1]$, $t \geq \frac{5}{9}$, hence, no (2,2)-solution exists for $t \geq \frac{5}{9}, x \in [0,1]$.

VI. CONCLUSIONS

In the present work, the Caputo fuzzy fractional derivatives and the Laplace Variational Iteration Method (LVIM) applied to find the exact fuzzy solution of the fuzzy space-time fractional telegraph equations. The main purpose of the paper is to present a new concept of solutions. The efficiency of the proposed algorithm is illustrated by giving example.

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