Dhage Iteration Method For Nonlinear First Order Measure Integro-Differential Equations With Linear Perturbation

[1]D. M. Suryawanshi , [2]S.S. Bellale

[1]*Research Scholor*

[1,2]*Dayanand Science College, Latur (MS)*

Abstract : *In this paper, the results of author proves the existence and uniqueness of solutions for the approximation of solutions to a nonlinear first order integro-differential equations using abstract measure theory. The approximation of the solutions are obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and abstract results a real so solved by some numerical examples.*

Keywords and Phrases: *Abstract measure differential equations, Dhage iteration methods, existence theorem, extremal solutions, approximation of solution, Abstract measure integro differential equation. Hybrid fixed theorem.*

2010 Mathematics Subject Classification: 34A12, 34A45, 34G99

1. INTRODUCTION

The study of abstract measure differential equations is initiated by Sharma [6] and subsequently developed by Joshi [19] Shendge and Joshi [14], Dhage [1–2], Dhage et al. [10] and Dhage and Bellale [3], Bellale [6] for different aspects of the solutions, Similarly, the study of abstract measure integro differential equations is studied by Dhage [12], Dhage and Bellale [4-5], Bellale and Dapke [7] , Bellale and Birajdar [8-9] varius aspects of solutions in such model of differential equations involve the derivative of the unknown set-function with respect to the σ-finite complete measure.

The existence and uniqueness of solutions of the nonlinear abstract measure differential equation under usual compactness and Lipschitz type conditions have been discussed at length in the literature. These conditions are considered to be very strong assumptions in the study of nonlinear differential and integral equations. Similarly, upper and lower solution method and monotone iterative technique also require the assumption that both the lower as well as upper solution exist and preserve the order relation. However, a recent trend for the existence of solution for such nonlinear problem is to assume only one of lower and upper solutions. In the present paper, we prove existence and uniqueness of solutions of the abstract measure integro differential equations under the weaker partial compactness and partial Lipschitz type conditions via the Dhage iteration method by assuming one of lower and upper solutions to exist.

II. PRELIMINARIES

A mapping $T: X \to X$ is called D-Lipschitz if there exists a continuous and non-decreasing function $\phi: R^+ \to R^+$ such that

$$\| Tx - Ty \| \le \phi(\| x - y \|)$$

for all $x, y \in X$, where $\phi(0) = 0$. In particular if $\phi(r) = \alpha r, \alpha > 0, T$ is called a Lipschitz function with a Lipschitz constant α. Further if $\alpha < 1$, then T is called a contraction on X with the contraction constant α.

Let X be a Banach space and let $T: X \to X$ T is called compact if $\overline{T(X)}$ is a compact subset of X. T is called totally bounded if for any bounded subsets S of X, $T(S)$ is abounded subset of X. T is called completely continuous if T is continuous and bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of X. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [17].

III. STATEMENT OF THE PROBLEM

Let X be a real Banach algebra with a convenient norm $\|.\|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \le r \le 1\} \tag{3.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{rx \mid -\infty < r < 1\}, \tag{3.2}$$

and
$$\overline{S}_x = \{rx \mid -\infty < r \le 1\} \tag{3.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let ca(X, M) be the space of all vector measures (real signed measures) and define a norm $\| \cdot \|$ on ca(X, M) by

$$\|p\| = |p|(X), \tag{3.4}$$

where $|p|$ is a total variation measure of p and is given by

$$| p |(X) = \sup \sum_{i=1}^{\infty} | p(E_i) |, \quad E_i \subset X, \tag{3.5}$$

where the supremum is taken over all possible partitions $\{E_i : i \in N\}$ of X. It is known that $ca(X,M)$ is a Banach space with respect to the norm $\| \cdot \|$ given by (3.4).

Let μ be a σ-finite positive measure on X, and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ-algebra on S_{x_0} . Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $S_x, x \in \overline{x_0 z}$ .Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \| \cdot \|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a non decreasing (resp. non increasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$)for all $n \in$ N. The conditions guaranteeing the regularity of E may be found in Heikkiländ and Lakshmikantham [18] and the references therein. We need the following definitions (see Dhage [12] and the references therein) in what follows.

Definition 3.1. A mapping $T : E \to E$ is called **isotone or non-decreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in E$. Similarly, T is called **non-increasing** if $x \preceq y$ implies $Tx \succeq Ty$ for all $x, y \in E$. Finally, T is called **monotonic** or simply **monotone** if it is either non decreasing or non increasing on E.

Definition 3.2. A mapping $T : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\| Tx - Ta \| < \varepsilon$ whenever x is comparable to a and $\| x - a \| < \delta$. T called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 3.3. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator T on a partially normed linear space E into itself is called **partially bounded** if $T(E)$ is a partially bounded subset of E. T is called **uniformly partially bounded** if all chains C in $T(E)$ are bounded by a unique constant.

Definition 3.4. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $T: E \to E$ is called **partially compact** if $T(E)$ is a partially relatively compact subset of E. T is called **uniformly partially compact** if T is a uniformly partially bounded and partially compact operator on E. T is called **partially totally bounded** if for any bounded subset S of E, $T(S)$ is a partially relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Definition3.5. An upper semi-continuous and monotone non decreasing function $\psi: R_+ \to R_+$ is called a

D-function provided $\psi(0)=0$. An operator $T: E \to E$ is called partially nonlinear D-contraction if there exists a D-function ψ such that

$$\| Tx - Ty \| \le \psi(\| x - y \|) \tag{3.6}$$

For all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = kr$, $k > 0$, T is called a partial Lipschitz operator with a Lipschitz constant k and more over,if $0 < k < 1$, T is called a partial linear contraction on E with a contraction constant k.

The **Dhage iteration method** or Dhage iteration principle embodied in the following applicable hybrid fixed point theorem of Dhage [10] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of the Dhage iteration method or principle is given in, Dhage *et al.*[1,16] and the references therein.

Theorem 3.1. *Let* $(E, \preceq, \|\cdot\|)$ *be a regular partially ordered complete normed linear space such that every compact chain C of E. Let* $A, B: E \to E$ *be two non decreasing operators such that*

(a) *A is partially bounded and partially nonlinear D-contraction,*

(b) *B is partially continuous and partially compact, and*

(c) *there exists an element* $x_0 \in E$ *such that* $x_0 \preceq Ax_0 + Bx_0$ *or* $x_0 \succeq Ax_0 + Bx_0$.

Then the operator equation Ax+ Bx = x has a solution x in E and the sequence* $\{x_n\}$ of *successive iterations defined by* $x_{n+1} = Ax_n + Bx_n$, $n = 0, 1, \ldots$, *converges monotonically to x*.*

Theorem 3.2. Let $(E, \preceq, \|.\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $T: E \to E$ be a partially continuous, non decreasing, and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq Tx_0$ or $Tx_0 \preceq x_0$, then the operator equation.

$Tx = x$ has a solution x^* in E, and the sequence $\{T^n x_0\}$ of successive iterations converges monotonically to x^*.

Theorem 3.3. Let $(E, \preceq, \|.\|)$ be a partially ordered Banach space and let $T: E \to E$ be a non decreasing and partially nonlinear D-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$. If T is continuous or E is regular, then T has a fixed point x^*, and the sequence $\{T^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.

The following useful lemma is obvious and may be found in Dhage [1].

Lemma 3.1. For any function $\sigma \in L^1(J, R)$, x is a solution to the differential equation

$$\left.\begin{array}{r} x'(t)+\lambda x(t)=\sigma(t),\ t\in J, \\ x(0)=x(T), \end{array}\right\} \tag{3.7}$$

if and only if it is a solution of the integral equation

$$x(t)=\int_0^T G\lambda(t,s)\sigma(s)ds \tag{3.8}$$

where,
$$G_\lambda(t,s)=\begin{cases} \dfrac{e^{\lambda s-\lambda t+\lambda T}}{e^{\lambda T}-1}, & if\ 0\le s\le t\le T, \\ \dfrac{e^{\lambda s-\lambda t}}{e^{\lambda T}-1}, & if\ 0\le t< s\le T. \end{cases} \tag{3.9}$$

Notice that the Green's function G_λ is continuous and non negative on $J\times J$ and therefore, the number

$$K_\lambda := \max\{|G_\lambda(t,s)| : t,s\in[0,T]\}$$

exists for all $\lambda\in R^+$. For the sake of convenience, we write $G_\lambda(t,s)=G(t,s)$ and $K_\lambda=K$.

Lemma 3.2. If there exists a function $u\in C(J,R)$ such that

$$\left.\begin{array}{l} u'(t)+\lambda u(t)\le\sigma(t),\quad t\in J, \\ u(0)\le u(t). \end{array}\right\} \tag{3.10}$$

then
$$u(t)\le\int_0^T G(t,s)\sigma(s)ds \tag{3.11}$$

for all $t\in J$, where $G(t,s)$ is a Green's function given by (3.9).

IV. MAIN RESULT

In this section, we prove an existence and approximation result for the AMDE (4.3) on a closed and bounded interval $J=[a,b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the AMDE (4.3) in the function space $C(J,R)$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J,R)$ by

$$\|x\|=\sup_{t\in J}|x(t)| \tag{4.1}$$

and
$$x\le y \Leftrightarrow x(t)\le y(t) \text{ for all } t\in J \tag{4.2}$$

Clearly, $C(J,R)$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,R)$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it.

Given a $p\in ca(S_z,M_z)$ with p is absolutely continuous with respect to the measure μ if $\mu(E)=0\Rightarrow p(E)=0, E\in M$ consider the periodic boundary value problem (PBVP) for the first order ordinary nonlinear abstract measure integro- differential equation (AMIDE),

$$\left.\begin{aligned} \frac{dp}{d\mu}+\lambda p(S_x) &= f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right), x\in\overline{x_0 z},\\ \textit{And} \qquad p(E) &= q(E), E\in M_0 \end{aligned}\right\} \tag{4.3}$$

Where $\frac{dp}{d\mu}$ is Radon–Nikodym derivative of p with respect to μ for some $\lambda\in R, \lambda>0$, where $g:S_z\times R\to R$ and $f:S_z\times R\times R\to R$ are continuous functions. q is given known vector measure. A solution of the AMIDE (4.3), we mean a differentiable function $u\in ca(S_z, M_z)$ that satisfies problem (4.3), where $ca(S_z, M_z)$ is the space of all vector measures .

The AMIDE (4.3) is well-known and includes

$$\left.\begin{aligned} \frac{dp}{d\mu}+\lambda p(S_x) &= f(x, p(S_x)), x\in\overline{x_0 z},\\ p(E) &= q(E), E\in M_0 \end{aligned}\right\} \tag{4.4}$$

$$\left.\frac{dp}{d\mu}+\lambda p(S_x) = \int_{S_x} g(x, p(S_x))ds, x\in\overline{x_0 z},\right\} \tag{4.5}$$

and $$\left.\frac{dp}{d\mu}+\lambda p(S_x) = f\left(x, \int_{S_x} p(S_x)dx\right), x\in\overline{x_0 z},\right\} \tag{4.6}$$

as special cases. That is, the results in this paper include results for the differential equations (4.4),(4.5), and (4.6) on $\overline{x_0 z}$ .The existence and uniqueness of solutions of the nonlinear AMIDE (4.3) under usual compactness and Lipschitz type conditions have been discussed in this paper.

Definition 4.1. An operator T from a normed linear space E into itself is compact if *T* (*E*) is a relatively compact subset of *E*. We say that *T* is totally bounded if for any bounded subset *S* of *E*, *T* (*S*) is a relatively compact subset of *E*. If *T* is continuous and totally bounded, then it is called completely continuous on *E*.

Remark 4.1. Suppose that *T* is a non decreasing operator on *E* into itself. Then *T* is a partially bounded or partially compact if *T* (*C*) is a bounded or relatively compact subset of *E* for each chain *C* in *E*.

Remark 4.2. Note that every compact mapping on a partially normed linear space is partially compact, and every partially compact mapping is partially totally bounded. However, the reverse implications do not hold. Every completely continuous mapping is partially completely continuous. Every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Remark 4.3. The regularity of *E* in Theorem 3.2 above may be replaced with a stronger continuity condition of the operator *T* on *E*.

The following hybrid fixed point theorems will be used to prove some of our existence and uniqueness results for the solutions of the AMIDE (4.3). We need the following notion of a D-function in these theorems.

Definition 4.2. An upper semi-continuous and non decreasing function $\psi:R_+\to R_+$ is called a D-function provided $\psi(0)=0$.

Remark 4.4. We remark that hypothesis (a) of Theorem 3.2 implies that operator *A* is partially continuous on *E*. The regularity of *E* in above Theorem3.2 may be replaced with a stronger continuity condition of the operators

A and B on E which is a result proved in Dhage [4]. Again, the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$.

4.1 . Existence Theorems and Uniqueness theorem

The equivalent integral form of the HIDE (4.3) is considered in the function space $C(J, R)$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,R)$by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and $\quad x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in J$.

Clearly, $C(J, R)$ is a Banach space with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation$\leq$. It is known that the partially ordered Banach space $C(J, R)$ is regular and is a lattice, so every pair of elements in the space has an upper and a lower bound in the space. The next lemma concerning the compatibility of sets in $C(J,R)$ follows by an application of the Arzella-Ascoli theorem.

We need the following definition.

Definition 4.3. A differentiable function $u \in ca(S_z, M_z)$ is a lower solution of the AMIDE (4.3) if it satisfies

$$\left.\frac{du}{d\mu} + \lambda u(x) \leq f\left(x, u(x), \int_{S_x} g(x, u(x))dx\right),\right\}$$

for all $x \in \overline{x_0 z}$. Similarly, $v \in ca(S_z, M_z)$ an upper solution to the AMIDE (3.1) is defined on $\overline{x_0 z}$ by reversing the above inequalities.

We consider the following set of hypothesis.

(H_1) There exists a constant $M_f > 0$ such that $|f(x, p(S_x), p(S_y)| \leq M_f$ for all $x \in \overline{x_0 z}$ and $x \in R$.

(H_2) The function $f(x, p(S_x), p(S_y)$ is a monotone non decreasing in x and y for each $x \in \overline{x_0 z}$.

(H_3) The function $g(x,p(S_x))$ is monotone non decreasing in x for each $x \in \overline{x_0 z}$.

(H_4) The AMIDE (4.1) has a lower solution $u \in ca(S_z, M_z)$.

(H_5) There exists a constant $L > 0$ such that $0 \leq g(t,x) - g(t,y) \leq L(x-y)$

for all $x \in \overline{x_0 z}$ and $x, y \in R$ with $x \geq y$.

(H_6) There exists D-functions ψ_1 and ψ_2 such that

$$0 \leq f(x, p(S_{x_1}), p(S_{x_2}) - f(x, p(S_{y_1}), p(S_{y_2})) \leq \psi_1(x_1 - y_1) + \psi_2(x_2 - y_2)$$

for all $x_1, x_2, y_1, y_2 \in R$ with $x_1 \geq y_1$ and $x_2 \geq y_2$. Moreover, $\psi(r) = KT[\psi_1(r) + \psi_2(LTr)] < r$ for each $r > 0$.

Our main existence result in this section is contained in the following theorem.

Theorem 4.1. Assume that conditions $(H_1)-(H_4)$ hold. Then the AMIDE (4.3) has a solution x^* defined on $\overline{x_0 z}$ and the sequence $\{p_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$x_0 = u,$$

$$p_{n+1}(x) = \int_0^T G(x, p(S_x)) f\left(x, p_n(S_x), \int_{S_x} g(x, p_n(x))dx \right) dx, \tag{4.7}$$

for all $x \in \overline{x_0 z}$, converges monotonically to x^*.

Proof . The AMIDE (4.3) is equivalent to the nonlinear integral equation

$$p(x) = \int_0^T G(x, p(S_x)) f\left(x, p_n(S_x), \int_{S_x} g(x, p_n(x))dx \right) dx, x \in \overline{x_0 z} \tag{4.8}$$

Set $E = ca\ (S_z, M_z)$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E. Define the operator T by

$$Tp(x) = \int_0^T G(x, p(S_x)) f\left(x, p_n(S_x), \int_{S_x} g(x, p_n(x))dx \right) dx, x \in \overline{x_0 z}\ . \tag{4.9}$$

From the continuity of the integral, it follows that T maps E into itself. The AMIDE (3.1) is then equivalent to the operator equation

$$Tp(S_x) = p(S_x), \quad x \in \overline{x_0 z}. \tag{4.10}$$

Through a series of steps, we shall show that the operator T satisfies all the conditions of Theorem 3.2 on E.

Step I: T is a non decreasing operator on E.

Let $x, y \in E$ with $x \leq y$. Then, from (H_2), we obtain

$$\begin{aligned} Tp(x) &= \int_0^T G(x, p(S_x)) f\left(x, p_n(S_x), \int_{S_x} g(x, p_n(x))dx \right) dx, x \in \overline{x_0 z} \\ &\leq \int_0^T G(x, p(S_x) f\left(x, p(S_y), \int_{S_x} g(x, p(S_y))dx \right) dx \\ &= Tp(y), \end{aligned}$$

for all $x \in \overline{x_0 z}$. This shows that T is a non decreasing operator on E.

Step II: T is partially continuous operator on E.

Let $\{p_n\}$ be a sequence of points of a chain C in E such that $p_n \to p$ for all $n \in \mathrm{N}$. Then, by the dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} Tp_n(x) &= \lim_{n\to\infty} \left[\int_0^T G(x, p(S_x)) f\left(x, p_n(x), \int_{S_x} g(x, p_n(x))dx \right) dx \right] \\ &= \int_0^T G(x, p(S_x)) \left[\lim_{n\to\infty} f\left(x, p_n(x), \int_{S_x} g(x, p_n(x))dx \right) \right] dx \\ &= \int_0^T G(x, p(S_x)) f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx \right) dx \\ &= \mathrm{T}p(\mathrm{x})\ , \end{aligned}$$

for all $x \in \overline{x_0 z}$. This shows that $\{Tp_n\}$ converges to Tp point wise on $\overline{x_0 z}$.

Next, we show that $\{Tp_n\}$ is an equi continuous sequence of functions in *E*. Let $x_1, x_2 \in \overline{x_0 z}$ with $x_1 < x_2$.

Then $$|Tp_n(x_2) - Tp_n(x_1)|$$

$$\leq \left| \int_0^T G(x_2, p(S_x)) f\left(s, x_n(s), \int_{S_x} g(x, p_n(x))dx\right) dx \right.$$

$$\left. - \int_0^T G(x_1, p(S_x)) f\left(s, x_n(s), \int_{S_x} g(x, p_n(x))dx\right) dx \right|$$

$$\leq \left| \int_0^T [G(x_2, p(S_x)) - G(x_1, p(S_x)] f\left(x, p_n(x), \int_{S_x} g(x, p_n(x))\, dx\right) dx \right|$$

$$\leq \left| \int_0^T | G(x_2, p(S_x)) - Gx_1, p(S_x) | \left| f\left(x, p_n(x), \int_{S_x} g(x, p_n(x))\, dx\right)\right| dx \right|$$

$$\leq M_f \int_0^T |G(x_2, p(S_x)) - G(x_1, p(S_x))|\, dx$$

$$\to 0 \text{ as } x_1 \to x_2,$$

uniformly for all $n \in N$. This shows that the convergence $Tp_n \to Tp$ uniformly and hence, *T* is a partially continuous operator on *E*.

Step III: *T* is partially compact operator on *E*.

Let *C* be an arbitrary chain in *E*. We will show that *T* (*C*) is a uniformly bounded and equicontinuous set in *E*. To show that *T* (*C*) is uniformly bounded, let $x \in C$. Then,

$$|Tp(S_x)| \leq \left| \int_0^T G(x, p(S_x))\, f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right) dx \right|$$

$$\leq \int_0^T G(x, p(S_x)) \left| f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right)\right| dx$$

$$\leq K M_f T$$

$$= r,$$

for all $x \in \overline{x_0 z}$. Taking the supremum over *x*, we obtain $\|Tx\| \leq r$ for all $x \in C$. Hence, *T* (*C*) is a uniformly bounded subset of *E*.

To show that *T* (*C*) is an equicontinuous set in *E*, let $x_1, x_2 \in \overline{x_0 z}$ with $x_1 < x_2$. Then

$$|Tp(x_2) - Tp(x_1)|$$

$$\left|\int_0^T G(x_2, p(S_x)) f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right) dx\right| - \left|\int_0^T G(x_1, p(S_x)) f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right) dx\right|$$

$$\leq \left|\int_0^T [G(x_2, p(S_x) - G(x_1, p(S_x)] f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right) dx\right|$$

$$\leq \left|\int_0^T | G(x_2, p(S_x) - G(x_1, p(S_x) | \left| f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right)\right| dx\right|$$

$$\leq M_f \int_0^T | G(x_2, p(S_x)) - G(x_1, p(S_x)) | dx$$

$$\to 0 \text{ as } x_1 \to x_2,$$

uniformly for all $x \in C$. Hence $T(C)$ is compact subset of E and consequently T is a partially compact operator on E into itself.

Step IV: u satisfies the operator inequality $u \leq Tu$.

Since condition (H_4) holds, u is a lower solution of (3.1) defined on $\overline{x_0 z}$ so that

$$\left.\begin{aligned} \frac{du}{d\mu} + \lambda u(x) \leq f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right) \\ u(E) \leq q(E), E \leq M_0 \end{aligned}\right\} \tag{4.11}$$

for all $x \in \overline{x_0 z}$. Applying Lemma 3.2 to the inequality (4.11), we obtain

$$u(x) \int_{S_x} G(x, p(S_x)) f\left(x, p(S_x), \int_{S_x} g(x, p(S_x))dx\right) dx, \tag{4.12}$$

for all $x \in \overline{x_0 z}$. This shows that u is a lower solution of the operator equation $x = Tx$.

Thus, T satisfies all the conditions of Theorem 3.2, and in view of Remark 2.11, we can conclude that the operator equation $Tx = x$ has a solution. Thus, the integral equation and the AMIDE (4.3) has a solution x^* defined on $\overline{x_0 z}$. Furthermore, the sequence $\{p_n\}$ of successive approximations defined by (4.7) converges monotonically to x^*. This completes the proof of the theorem.

We illustrate our result with the following example.

Example 3.8. Consider the following AMIDE

$$\left.\begin{aligned} \frac{dp}{d\mu} + p(S_x) = \tanh xp(S_x) + \tanh\left(\int_{S_x} g(x, p(S_x))dx\right), \; x \in \overline{x_0 z}, \\ p(E) = q(E), E \in M_0 \end{aligned}\right\} \tag{4.13}$$

where $g: S_z \times R \to R$ is the function defined by

$$g(x, p(S_x)) = \begin{cases} x+1, & if \quad x \leq 0, \\ 1+\log(x+1), & if \quad x > 0 \end{cases}$$

Here, $\lambda = 1$, $c = 1$, and $f(x, p(S_x), p(S_y)) = \tanh p(S_x) + \tanh p(S_y)$. Clearly, the functions f and g are continuous on $S_z \times R$, and f satisfies (H_1) with $M_f = 2$. Moreover, $g(x, p(S_x))$ a is non decreasing in x for each $x \in \overline{x_0 z}$,and $f(x, p(S_y))$ is non decreasing in x and y for each $x \in \overline{x_0 z}$, so conditions (H_2) and (H_3) are satisfied.

Finally, the AMIDE (4.13) has a lower solution u defined by $u(x) = -2e^x$ on. Thus, all the hypotheses of Theorem 4.1 are satisfied, and so (4.13) has a solution x^* defined on $\overline{x_0 z}$, and the sequence $\{p_n\}$ defined by

$$x_0 = u,$$

$$p_{n+1}(x) = \int_0^1 G(x, p(S_x)) \tanh p_n(S_x)\, dx + \int_0^1 G(x, p(S_x)) \tanh\left(\int_{S_x} g(x, (p_n(S_x)) dx \right) dx$$

for all $x \in \overline{x_0 z}$, converges monotonically to p^*, where $G(x, p(S_x))$ is a Green's function associated with the homogeneous PBVP

$$\left.\begin{array}{l} \dfrac{dp}{d\mu} + p(S_x) = 0, \quad x \in \overline{x_0 z}, \\ \quad p(E) = 0, \end{array}\right\} \tag{4.14}$$

given by

$$G(x, p(S_x)) = \begin{cases} \dfrac{e^{S_x - x + 1}}{e - 1}, & if \;\; 0 \le S_x \le x \le 1 \\ \dfrac{e^{S_x - x}}{e - 1}, & if \;\; 0 \le x < S_x \le 1. \end{cases} \tag{4.15}$$

Next, we prove a uniqueness theorem for the AMIDE (4.3) under the weaker partially Lipschitz condition. We will need the following conditions.

Theorem 4.2. Assume that conditions (H_4) **-** (H_6) hold. Then the AMIDE (4.3) has a unique solution x^* defined on $\overline{x_0 z}$, and the sequence $\{p_n\}$ of successive approximations defined by (4.7) converges monotonically to x^*.

Proof. Set $E = ca(S_z, M_z)$. Clearly, E is a lattice w.r.to. the order relation $\le$ and so lower and upper bounds exist for every pair of elements in E. Define the operator T by (4.9). Then, the AMIDE (4.3) is equivalent to the operator equation (4.10). We shall show that T satisfies all the conditions of Theorem3.3.

Clearly, T is a non decreasing operator from E into itself. We wish to show that the operator T is a partially nonlinear D-contraction on E, so let $x, y \in E$ with $x \ge y$. Then, by (H_5) and (H_6),

$$|Tp(x) - Tp(y)|$$

$$\le \left| \int_0^T G(x, p(S_x)) f\left(x, p(S_x), \int_{S_x} g(x, p(S_x)) dx \right) dx - \int_0^T G(x, p(S_x)) f\left(y, p(S_y), \int_{S_x} g(y, p(S_y)) dy \right) dy \right|$$

$$\le \int_0^T G(x, p(S_x)) \left| f\left(x, p(S_x), \int_{S_x} g(x, p(S_x)) dx \right) - f\left(x, p(S_y), \int_{S_x} g(x, p(S_y)) dx \right) \right| dx$$

$$\le \int_0^T G(x, p(S_x)) \left[\psi_1(p(S_x) - p(S_y)) + \psi_2\left(\int_{S_x} [g(x, p(S_x)) - g(x, p(S_y))] x \right) \right] ds$$

$$\leq \int_0^T G(x, p(S_x)\left[\psi_1(p(S_x)-p(S_y))+\psi_2\left(\int_{S_x}[L(p(S_x)-p(S_y))dx]\right)\right]dx$$

$$\leq \int_0^T G(x, p(S_x))\left[\psi_1(|p(S_x)-p(S_y)|)+\psi_2\left(\int_{S_x} L\,|p(S_x)-p(S_y)|\,dx\right)\right]dx$$

$$\leq \int_0^T K\left[\psi_1(\|x-y\|)+\psi_2\left(\int_{S_x} L\,\|x-y\|\,dx\right)\right]dx$$

$$\leq \int_0^T K[\psi_1(\|x-y\|)+\psi_2(LT\,\|x-y\|)]dx$$

$$\leq \psi(\|x-y\|)$$

for all $x \in \overline{x_0 z}$, where $\psi(r) = KT[\psi_1(r)+\psi_2(LTr)] < r, r > 0$.

Taking the supremum over *x*, we obtain

$$\|Tx - Ty\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$ with $x \geq y$. As a result, *T* is a partially nonlinear D-contraction on *E*. Furthermore, as in the proof of Theorem 4.1, it can be shown that the function *u* given in condition (H_4) satisfies the operator inequality $u \leq Tu$ on $\overline{x_0 z}$. Now a direct application of Theorem 3.3 yields that the AMIDE (4.3) has a unique solution *x**, and the sequence $\{p_n\}$ of successive approximations defined by (4.7) converges monotonically to *p**.

To illustrate this theorem, we present the following example.

Example4.1.We consider the following AMIDE

$$\left.\begin{array}{c} \dfrac{dp}{d\mu} + p(S_x) = \dfrac{1}{2}\left[\tan^{-1} p(S_x) + \tan^{-1}\left(\displaystyle\int_{S_x} g(x, p(S_x))dx\right)\right], x \in \overline{x_0 z}, \\ p(E) = q(E), E \in M_0 \end{array}\right\} \tag{4.16}$$

where $g : S_z \times R \to R$ is the function defined by

$$g(x, p(S_x)) = \begin{cases} 1 & if \quad x \leq 0, \\ 1 + \dfrac{x}{1+x}, & if \quad x > 0. \end{cases}$$

Here, $\lambda = 1, c = 1, f(x, p(S_x), p(S_y)) = \frac{1}{2}[\tan^{-1} p(S_x) + \tan^{-1} p(S_y)]$. Clearly, the functions *f* and *g* are continuous on $S_z \times R \times R$ and $S_z \times R$, respectively. The function *f* satisfies (H_1) with $M_f = \frac{\pi}{2}$ and it is easy to show that *g* satisfies (H_5) with $L = 1$. Moreover, $f(x, p(S_x), p(S_y))$ is non decreasing in *x* and *y* for each $x \in \overline{x_0 z}$. To show that *f* satisfies (H_6) on $S_z \times R \times R$, let $x_1, x_2, y_1, y_2 \in R$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then,

$$0 \leq f(x, p(S_{x_1}), p(S_{x_2})) - f(x, p(S_{y_1}), p(S_{y_2}))$$

$$\leq \frac{1}{2}\left[\tan^{-1} p(S_{x_1}) - \tan^{-1} p(S_{y_1}) + \tan^{-1} p(S_{x_2}) - \tan^{-1} p(S_{y_2})\right]$$

$$\leq \frac{1}{2}.\frac{x_1 - y_1}{1+\xi_1^2} + \frac{1}{2}.\frac{x_2 - y_2}{1+\xi_2^2}$$

$$\leq \psi_1(x_1 - \psi_1) + \psi_2(x_2 - y_2)$$

for all $x \in \overline{x_0 z}$ and for some $x_1 > \xi_1 > y_1$ and $x_2 > \xi_2 > y_2$, where ψ_1 and ψ_2 are D-functions defined by $\psi_1(r) = \frac{1}{2}\frac{r}{1+\xi_1^2}$ and $\psi_2(r) = \frac{1}{2}\frac{r}{1+\xi_2^2}$ for $0 < \xi_1, \xi_2 < r$ Furthermore,

$$KT[\psi_1(r) + \psi_2(LTr)] \leq \frac{1}{2}.[\psi_1(r) + \psi_2(r)] = \frac{r}{1+\xi^2} < r,$$

where $\xi = \min\{\xi_1, \xi_2\}$.Finally, the AMIDE (4.17) has a lower solution $u(x) = -4e^x$ defined on $\overline{x_0 z}$. Thus, all the hypotheses of Theorem 4.2 are satisfied and so we conclude that the AMIDE (4.16) has a unique solution x^* defined on $\overline{x_0 z}$.. In addition, the sequence $\{p_n\}$ defined by

$$x_0 = u,$$

$$p_{n+1}(x) = \frac{1}{2}\int_0^1 G(x, p(S_x)) \tan^{-1} p_n(S_x)dx + \frac{1}{2}\int_0^1 G(x, p(S_x)) \tan^{-1}\left(\int_0^s g(x, p_n(S_x))dx\right)dx$$

for all $x \in \overline{x_0 z}$, converges monotonically to x^*, where $G(x, p(Sx))$ is a Green's function associated with the homogeneous PBVP (4.14) given by (4.15).

4.2. Linear Perturbations of the First Type

we consider the nonlinear AMIDE

$$\left.\begin{aligned} \frac{dp}{d\mu} + \lambda p(S_x) &= f_1\left(x, p(S_x), \int_0^t g(x, p(S_x))dx\right) \\ &+ f_2\left(x, p(S_x), \int_0^t g(x, p(S_x))dx\right), \\ p(E) &= q(E), \end{aligned}\right\} \tag{4.17}$$

for all $x \in \overline{x_0 z}$, , where $f_1, f_2 : S_z \times R \times R \to R$ and $g : S_z \times R \to R$ are continuous functions.

By a solution of the AMIDE (4.1) we mean a function $P \in ca(S_z, M_z)$ that satisfies equation (4.1), where $ca(S_z, M_z)$ is the usual Banach space of continuously differentiable real-valued functions defined on $\overline{x_0 z}$.

We will need the following definition.

Definition 4.4. A differentiable function $u \in ca(S_z, M_z)$ is said to be a lower solution of the AMIDE (4.1) if it satisfies

$$\left.\begin{aligned} \frac{du}{d\mu}+\lambda u(x) &\le f_1\left(x,u(x),\int_0^t g(x,u(x))dx\right) \\ &\quad + f_2\left(x,u(x),\int_0^t g(x,u(x))dx\right), \\ u(0) &\le u(T) \end{aligned}\right\}$$

for all $x\in\overline{x_0 z}$. Similarly, an upper solution $v\in ca(S_z,M_z)$ to the AMIDE (4.1) is defined on $\overline{x_0 z}$ by reversing the above inequalities.

Theorem 4.3. Assume that (H_1)-(H_3) hold with f replaced by f_2, and let (H_1) and (H_5) - (H_6) hold with f replaced by f_1. If (H_7) holds, then the AMIDE (4.1) has a solution x^* defined on E and the sequence $\{p_n\}$ of successive approximations defined by

$$x_0=u,$$

$$p_{n+1}(x)=\int_0^T G(x,p(S_x))\,f_1\left((x,p_n(x),\int_{S_x} g(x,p_n(x))dx\right)dx$$

$$+\int_0^T G(x,p(S_x))\,f_2\left((x,p_n(S_x),\int_{S_x} g(x,p_n(x))dx\right)dx, \tag{4.18}$$

for $x\in\overline{x_0 z}$, converges monotonically to x^*, where $G(x,p(S_x))$ is a Green's function defined by (3.5) on E.

Proof. Set $E = ca(S_z, M_z)$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E. By Lemma 3.1, the AMIDE (4.1) is equivalent to the nonlinear integral equation

$$p(x)=\int_0^T G(x,p(S_x))\,f_1\left(s,x(s),\int_0^s g(x,p(S_x))dx\right)dx$$

$$+\int_0^T G(x,p(S_x)\,f_2\left(x,p(S_x),\int_0^s g(x,p(S_x))dx\right)dx,\ x\in\overline{x_0 z}, \tag{4.19}$$

where $G(x, p(S_x))$ is a Green's function defined by (3.5) on E. Define the operators A and B on E by

$$Ap(x)=\int_0^T G(x,p(S_x))\,f_1\left(x,p(S_x),\int_0^s g(x,p(S_x))dx\right)dx,\ x\in\overline{x_0 z}, \tag{4.20}$$

and

$$Bp(x)=\int_0^T G(x,p(S_x))\,f_2\left(x,p(S_x),\int_0^s g(x,p(S_x))dx\right)dx,\ x\in\overline{x_0 z}, \tag{4.21}$$

Clearly, $A,B:E\to E$. Also, the AMIDE (4.1) is equivalent to the operator equation

$$Ap(x)+Bp(x)=p(x),\quad x\in\overline{x_0 z}. \tag{4.22}$$

it can be shown that the operator A is a partially bounded and nonlinear D-contraction and B is a partially continuous and partially compact operator on E. Furthermore, as in the proof of Theorem4.1, it can be shown that the function u given in condition (H_4) satisfies the operator inequality $u\le Au+Bu$ on E. A direct application of Theorem 3.1 yields that the operator equation $Ax+Bx=x$ has a solution x. Consequently, the

AMIDE (4.1) has a solution x^*, and the sequence $\{p_n\}_{n=1}^{\infty}$ defined by (4.2) converges monotonically to p. Hence the result.

Example 4.2. Consider the following AMIDE

$$\left.\begin{array}{c} \frac{dp}{d\mu} + p(S_x) = \tan^{-1} p(S_x) + \tanh\left(\int_{S_x} g(x, p(S_x))dx\right), \ x \in \overline{x_0 z}, \\ p(E) = q(E), E \in M_0 \end{array}\right\} \tag{4.23}$$

where $g : S_z \times R \to R$ is the function defined by

$$g(x, p(S_x)) = \begin{cases} x+1, & if \quad x \le 0 \\ x^2+1, & if \quad x > 0 \end{cases}$$

Here, $\lambda = 1, c = 1, f_1(x, p(S_x), p(S_y)) = \tan^{-1} x$ and $f_2(x, p(S_x), p(S_y)) = \tanh y$. Then the function f_1 satisfies (H_1) with $M_{f_1} = \frac{\pi}{2}$ and satisfies (H_6) with $\psi_1(r) = \frac{r}{1+\xi^2}, 0 < \xi < r$, and $\psi_2(r) = 0$. Now f_2 satisfies (H_1) with $M_{f_2} = 1$ and is non decreasing in y, so (H_2) holds. Similarly, g satisfies (H_3). Finally, $u(x) = -3e^x$ for all $x \in \overline{x_0 z}$ is a lower solution of the AMIDE (4.7) on E, and so (H_7) is satisfied. Therefore, by Theorem 4.3, the AMIDE (4.7) has a solution x^* on E, and the sequence $\{p_n\}_{n=1}^{\infty}$ defined by

$$p(x) = -3e^{-x},$$

$$p_{n+1}(x) = \int_0^1 G(x, p(S_x)) \tan^{-1} p_n(x)\,dx + \int_0^1 G(x, p(S_x)) \tanh\left(\int_0^s g(x, p_n(x))dx\right)dx$$

for each $x \in \overline{x_0 z}$, converges monotonically to x^*, where $G(x,p(S_x))$ is a Green's function associated with the homogeneous PBVP (4.14) given by (4.15).

Remark 4.5. We note that if the AMIDE (4.3) or (4.1) has a lower solution u as well as an upper.

Solution v such that $u \le v$, then the corresponding solutions x_* and x^* of the AMIDE (4.3) or (4.1) satisfy $x_* \le x^*$ and they are the minimal and maximal solutions in the vector segment $[u, v]$ of the Banach space

$E = ca\ (S_z, M_z)$. This is because the order relation $\le$ defined by (3.2) is equivalent to the order relation defined by the order cone $K = \{p \in ca(S_z, M_z) \mid p(E) \ge 0\ forall E \in M_z\}$ which is a closed set in $ca(S_z, M_z)$. Thus, Dhage iteration method is also useful for proving the maximal and minimal solutions in a vector segment of the partially ordered Banach space E.

References

[1] B.C. Dhage, *Dhage iteration method for PBVPs of nonlinear first order hybrid integro-differential equations,Int .J.Nonlinear anal.* Appli. 8 (2017) no.1, 95-112.

[2] B.C. Dhage, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, Differ. Equ.* Appl. 2 (2010), 465-486.

[3] B.C. Dhage, and S.S. Bellale, *Exitence theorem for perturbed Abstract measure differential equations*. Nonlinear Analysis 71 (2009) 319-328.

[4] B.C. Dhage, S.S. Bellale, *Abstract measure integro differential equations*. Global J. math Anal. 1(1-2) (2007) 91-108.

[5] B.C. Dhage, and S.S. Bellale, *Exitence theorem for perturbed Abstract measure integro differential equations.* Non Linear Theory; Methods and Applications (2008).11.057. The 15th International conference of International Academy of Physical Sciences Dec 9-13,2012, Pathumthani, Thailand.

[6] R. R. Sharma, *An Abstract Measure Differential Equations*, proc. Amer. Math. Soc.32(1972) 503-510.

[7] S.S. Bellale, G.B. Dapke, *Hybrid fixed point theorem for abstract measure Integro-differential equation*, Intern. J. Math. Stat., 3 (1) (2018), 101-106.

[8] S.S. Bellale, S.B. Birajdar, *On quadratic abstract measure integro differential equations*, J.Comput. Modeling, 5 (3) (2015), 99-122.

[9] S.S. Bellale, S.B. Birajdar, D.S. Palimkar, *Existence theorem for abstract measure delay integro differential equations*, Appl. Comput. Math., 4(4) (2015), 225-231.

[10] B. C. Dhage, *A new monotone iteration principle in the theory of nonlinear first order integro differential equations*, Nonlinear Studies, *22*(3)(2015), 397-417.

[11] B.C. Dhage, S.B. Dhage, J.R. Graef. *Dhage iteration method for initial value problems for nonlinear first order hybrid integro differential equations*, J. Fixed Point Theory Appl., 17(2016), 309-325.

[12] B.C.Dhage , *On abstract measure integro-differential equation* , J Maths. Phys .Sci. 20 (1986) 367-380.

[13] S. Leela, *stability of measure differential equation*. Pacific J. Math 52 (2) (1974) 489-498.

[14] G.R. Shendge, S.R. Joshi, *An abstract measure differential inequalities and applications*, Acta Math. Hun 41(1983) 53-54

[15] D. Otrocol, I.A. Rus, *Functional-differential equations with maxima of mixed type argument*, Fixed Point Theory, 9(2008), no. 1, 207-220.

[16] B. C. Dhage, *Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations*, Differ. Equ . Appl., 5(2013), 155-184.

[17] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer Verlag, 2003.

[18] S. Heikkila, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[19] S. R. Joshi, *A system of abstract measure delay differential equations* J. Math. Phys. Sci. 13 (1979) 497-506.