A Note on Isomorphy and Unitary Isomorphy of Hilbert Space Frames

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Abstract:
In this paper we explore isomorphy and unitary isomorphy of frames for Hilbert spaces with a view to investigating properties of some frames left invariant under these equivalence relations. We will also define new equivalences of Hilbert space frames.

Keyword: *analysis operator, dual frame, frame, frame operator, isomorphy, unitary isomorphy, tight frame*

I. INTRODUCTION

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [10] who used frames as a tool in the study of non-harmonic Fourier series. In 1986 Daubechies et al [9] reintroduced the notion of frames and observed that frames can be used to find series expansions of functions in the Hilbert space $L^2(\mathbb{R})$. Frames are generalizations of orthonormal bases for Hilbert spaces. The main property of frames which makes them so useful is their redundancy. Representation of signals using frames is advantageous over basis expansions in a variety of practical applications. Recently, many generalizations of frames have been introduced and studied. Casazza and Kutyniok [7] introduced the notion of frames for subspaces or fusion frames to have many more applications of frames in sensor networks and packet encoding.

Frames are usually preferred because of their redundancy, yet providing stable decompositions, resilience or robustness to additive noise and erasure (see [7], [8]), resilience to quantization (see [11]), their numerical stability of reconstruction (see [8]), and greater freedom to capture signal characteristics (see [3], [4], [7]). A special type of frames, the equal-norm Parseval tight frame has found applications in the design of multiple-antenna codes (see [12]).

II. PRELIMINARIES

Let H be a Hilbert space and $B(H)$ denote the Banach algebra of bounded linear operators. If $T \in B(H)$, then T^* denotes the adjointof T , while $Ker(T), Ran(T), \bar{M}$ and $M^{\perp}$ stands for the kernel of T , range of T , closure of M and orthogonal complement of a closed subspace M of H, respectively. Two operators $A \in B(H)$ and $B \in B(K)$ are said to be similar if there exists an invertible operator $N \in B(H,K)$ such that $NA = BN$ or equivalently, $A = N^{-1}BN$, and are unitarily equivalent if there exists a unitary operator $U \in B(H,K)$ such that $A = U^*BU$. Two operators $A \in B(H)$ and $B \in B(H)$ are said to be metrically equivalent if $\|Ax\| = \|Bx\|$, (equivalently, if $A^*A = B^*B$,), for all $x \in H$ (see [13]).

An operator $T \in B(H)$ is said to be positive if it is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

III. HILBERT SPACE FRMAES AND THEIR ASSOCIATED OPERATORS

Theorem 3.1(Parseval Identity) Let $\{f_k\}_{k=1}^n$ be an orthonormal basis for an n-dimensional Hilbert space H. Then for any $f \in H$,

$$\sum_{k=1}^{n} \left|\langle f, f_k\rangle\right|^2 = \|f\|^2 .$$

The Parseval Identity also holds in infinite dimensional Hilbert spaces.

A subset $\{f_k\}_{k\in J}$ (where J is an indexing set) of a Hilbert space H is said to be complete if every element $f \in H$ can be represented arbitrarily well in norm by linear combinations of the elements in $\{f_k\}_{k\in J}$. A complete set $\{f_k\}_{k\in J}$ is said to be over-complete or redundant if removal of an element f_j from the set results in a complete set or system. That is, if $\{f_k\}_{k\in J\setminus\{j\}}$ is still complete.

Definition 3.2 A sequence of vectors $\{f_k\}_{k\in J}$ in a Hilbert space H is a frame for H if there exists real numbers $0<\alpha\le\beta<\infty$ called frame bounds such that for all $f\in H$,

$$\alpha\|f\|^2 \le \sum_{k\in J}\left|\left\langle f, f_k\right\rangle\right|^2 \le \beta\|f\|^2 .$$

The numbers α and β are called the lower bound and upper bound of the frame, respectively. They are, respectively, the smallest and largets eigenvalues of the frame operator. The numbers $\left\langle f, f_k\right\rangle$ are called the frame coefficients. A frame is a redundant or over-complete (that is, not linearly independnet) coordinate system for a vector space that satisfies a Parseval-type norm inequality. Clearly, as set of vectors in a finite dimensional Hilbert space is a frame if and only if it is (just) a spanning set.

If $\alpha=\beta$, then the frame $\{f_k\}_{k\in J}$ is called tight and if $\alpha=\beta=1$, the frame is called a normalized tight frame or Parseval. If $\|f_i\|=\|f_j\|$, for all $i, j\in J$, then the frame $\{f_k\}_{k\in J}$ is called an equal-norm or uniform-norm frame, and if in addition $\alpha=\beta=1$, it is called a uniform normalized tight frame. If a frame is equal-norm and if there exists a $c\ge 0$ such that $\left|\left\langle f_j, f_k\right\rangle\right|=c$, for all $j,k\in J$, with $j\ne k$, then the frame is said to be equiangular.

Clearly, a sequence $\{f_k\}_{k\in J}$ in a Hilbert space H is a tight frame if there exists a number $\alpha>0$ such that

$$\sum_{k\in J}\left|\left\langle f, f_k\right\rangle\right|^2=\alpha\|f\|^2, \quad \text{for all } f\in H.$$

Definition 3.3. Given a frame $\{f_k\}_{k\in J}$ for a Hilbert space H, another frame $\{g_k\}_{k\in J}$ is said to be a dual frame of

$\{f_k\}_{k\in J}$ if the following reproducing or reconstruction formula holds

$$f=\sum_{k\in J}\left\langle f, f_k\right\rangle g_k=\sum_{k\in J}\left\langle f, g_k\right\rangle f_k, \text{ for all } f\in H.$$

We call $\{f_k\}_{k\in J}$ and $\{g_k\}_{k\in J}$ a pair of dual frames or a dual frame pair. Dual frames are not unique. However, if the frame is exact (number of frame vectors is equal to the dimension of the space), then the dual is unique.

Definition 3.4 Let $\{f_k\}_{k\in J}$ be a frame for a Hilbert space H. The operator $A: H\to l^2(\square)$ defined by $Af=\{\left\langle f, f_k\right\rangle\}$, for all $f\in H$ is called the analysis operator of the frame $\{f_k\}_{k\in J}$.

Definition 3.5 Let $\{f_k\}_{k\in J}$ be a frame for a Hilbert space H with analysis operator A. The operator

$A^*: l^2(\mathbb{N}) \to H$ defined by $A^*(\{\langle f, f_k \rangle\}) = \sum_{k\in J} \langle f, f_k \rangle$ is called the synthesis operator of the frame $\{f_k\}_{k\in J}$.

The analysis and frame operators play a central role in the analysis, reconstruction and recovery of any function or signal $f \in H$. The analysis operator analysis a signal in terms of the frame by computing its frame coefficients.

In an n-dimensional Hilbet space, the synthesis operator of a finite frame $\{f_k\}_{k=1}^N = \{f_1, f_2, ..., f_N\}$ can be represented as an $n \times N$ matrix $\left[f_1 \quad f_2 \quad \ldots \quad f_N \right]$. That is, A^* has columns the frame operators f_k. This operator is usually identified with the frame itself. We call the quotient or ratio $\rho = \frac{N}{n}$ the redundancy of the frame and is a traditional measure of over-completeness of the frame.

We note that a sequence $\{f_k\}_{k\in J}$ is a frame for a Hilbert space H if and only if the analysis operator $A: H \to l^2(\mathbb{N})$ is well-defined and is a topological isomorphism onto a closed subspace of $l^2(\mathbb{N})$. The following result is a consequence of this fact.

Theorem 3.6 If $\{f_k\}_{k\in J}$ is a frame for a Hilbert space H with analysis operator $A: H \to l^2(\mathbb{N})$, then the following conditions hold.

(a). A is injective.

(b). $Ran(A)$ is closed.

(c). A^* is surjective.

Definition 3.7 Let $\{f_k\}_{k\in J}$ be a frame for a Hilbert space H with analysis operator A. The operators $S = A^*A$ and $G = AA^*$ are called the frame operator and Grammian operator, respectively.

By Theorem 3.6, the frame operator $S: H \to H$ is positive and invertible while the Grammian $G: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is positive but need not be invertible, since its range need not be all of $l^2(\mathbb{N})$. The Grammian operator nd its pseudo-inverse play a crucial role in the process of recovery of a signal $f \in H$ from frame representation.

Proposition 3.8[Frame Reconstruction Formula] Let $\{f_k\}_{k\in J}$ be a frame for a Hilbert space H with analysis operator A and frame operator S. Then for all $f \in H$

$$f = \sum_{k\in J} \langle S^{-1} f, f_k \rangle f_k = \sum_{k\in J} \langle f, S^{-1} f_k \rangle = \sum_{k\in J} \langle f, f_k \rangle S^{-1} f_k = \sum_{k\in J} \langle f, S^{-1/2} f_k \rangle S^{-1/2} f_k .$$

The reconstruction formula shows that all information about a given vector or signal $f \in H$ is contained in the sequence $\{\langle f, S^{-1} f_k \rangle\}$. We note that the choice of coefficients in Proposition 3.8 is not unique, in general. If the frame is redundant or over-complete, a typical phenomenon in applications, then there exists infinitely many choices of coefficients $c_k = \langle f, S^{-1} f_k \rangle$ in the expansion of $f = \sum_k c_k f_k$. The possibility ensures resilience to erasures or noise in a signal $f \in H$. A new approach (see [6]) has emerged recently, and has received increasing attention, namely choose the coefficient sequence to be sparse in the sense of having only few non-zero entries, thereby allowing data compression while preserving perfect reconstruction or recoverability. The frame $\{\tilde{f}_k\} = \{S^{-1} f_k\}_{k\in J}$ is called the canonical dual of $\{f_k\}_{k\in J}$ and its frame operator is denoted by S^{can}.

Proposition 3.9 Let $\{f_k\}_{k\in J}$ be a frame for a Hilbert space H and suppose that $\{g_k\}_{k\in J}$ is its dual frame. Then

$$f = \sum_{k\in J}\langle f, f_k\rangle g_k = \sum_{k\in J}\langle f, g_k\rangle f_k, \text{ for all } f \in H.$$

Lemma 3.10 Let $\{f_k\}_{k=1}^{n}$ be a frame for a finite dimensional Hilbert space H with analysis operator A and frame operator S. If $T \in B(H)$, then the analysis operator for the sequence $\{Tf_k\}_{k=1}^{n}$ equals AT^*.

Proof. Let B be the analysis operator for the sequence $\{Tf_k\}_{k=1}^{n}$. Then

$$Bf = \sum_{k=1}^{n}\langle f, Tf_k\rangle f_k = \sum_{k=1}^{n}\langle T^* f, f_k\rangle f_k = AT^* f, \ \forall f \in H.$$

That is, $B = AT^*$.

Theorem 3.11([2], Theorem 2.2) If $\{f_k\}_{k=1}^{n}$ is a frame for a N- dimensional Hilbert space H with frame operator S and

$T \in B(H)$, then the frame operator for the sequence $\{Tf_k\}_{k=1}^{n}$ equals TST^*.

Proof. The proof follows from the fact that the frame operator for $\{Tf_k\}_{k=1}^{n}$ is given by

$$\sum_{k=1}^{n}\langle f, Tf_k\rangle Tf_k = T\left(\sum_{k=1}^{n}\langle T^* f, f_k\rangle f_k\right) = TST^*.$$

Alternatively, from Lemma 3.10, the frame operator of $\{Tf_k\}_{k=1}^{n}$ is given by

$$B^*B = (AT^*)^*(AT^*) = TA^*AT^* = T(A^*A)T^* = TST^*.$$

Clearly,

$$TST^* f = T\left(\sum_{k=1}^{n}\langle T^* f, f_k\rangle f_k\right) = \sum_{k=1}^{n}\langle f, Tf_k\rangle Tf_k.$$

Theorem 3.11 leads to the following consequences.

Corollary 3.12 If $\{f_k\}_{k=1}^{n}$ is a tight frame for a N- dimensional Hilbert space H with frame operator S and

$T \in B(H)$, then the frame operator for the sequence $\{Tf_k\}_{k=1}^{n}$ is a scalar multiple of TT^*. Moreover, if $\{f_k\}_{k=1}^{n}$ is Parseval, then the frame operator for $\{Tf_k\}_{k=1}^{n}$ is TT^*.

Corollary 3.12 **Let** $\{f_k\}_{k=1}^{n}$ be a frame for a Hilbert space H with frameoperator S . The canonical dual frame operator satisfies $S^{can} = S^{-1}$.

Proof. For every $f \in H$, we have

$$S^{can} f = \sum_{k=1}^{n} \langle f, \tilde{f}_k \rangle \tilde{f}_k = \sum_{k=1}^{n} \langle f, S^{-1} f_k \rangle S^{-1} f_k = S^{-1} \sum_{k=1}^{n} \langle S^{-1} f, f_k \rangle f_k = S^{-1} S S^{-1} f = S^{-1} f.$$

Therefore $S^{can} = S^{-1}$.

IV. ISOMORPHY AND UNITARY ISOMORPHY OF FRAMES

There are several commonly used notions of equivalence among frames. There are frames, although they are technically different, are considered to be the "same" in some sense. First we explore the more general notions of isomorphy and unitary isomorphy of frames, associated operators and their properties.

Definition 4.1 Two frames $\Phi = \{f_k\}_{k=1}^{n}$ and $\Psi = \{f_k\}_{k=1}^{n}$ for an N-dimensional Hilbert space H are said to be isomorphic if there is an invertible operator $T : H \to H$ such that for all $g_k \in \Psi,$ we have $g_k = Tf_k$, for all $k = 1, 2, ..., n$.

Definition 4.2 Two frames $\Phi = \{f_k\}_{k=1}^{n}$ and $\Psi = \{f_k\}_{k=1}^{n}$ for an N-dimensional Hilbert space H are said to be unitarily isomorphic if there is a unitary operator $U : H \to H$ such that for all $g_k \in \Psi,$ we have $g_k = Uf_k$, for all $k = 1, 2, ..., n$.

Remark. Balan in [1] has used the term F-equivalent to mean isomorphic. We note also that in the literature, thevterms similarity and unitary equivalence have been used in place of isomorphy and unitary isomorphy, respectively. In this paper, we adopt the latter and reserve the terms similar and unitary equivalence to bounded linear operators. We also note that isomorphy and unitary isomorphy are equivalence relations in the set of frames F. We also note that isomorphy of frames is order-dependent, is the sense that the order in which the frames vectors are arranged matters.

Example 4.3 If $\{e_n\}$ is an orthonormal basis for an N-dimensional Hilbert space H, then the sets $\{0, e_1, e_2, ..., e_n\}$ and $\{e_1, 0, e_2, ..., e_n\}$ are two non-similar frames for H, although they are the same set.

Definition 4.4 Let $\Phi = \{f_k\}_{k=1}^{n}$ be a frame for a Hilbert space H with frame operator S. The sequence $\Phi^{can} = \{S^{-1/2} f_k\}_{k=1}^{n}$ is also a frame, called the canonical tight frame.

Remark. We note that the canonical tight frame $\Phi^{can} = \{S^{-1/2} f_k\}_{k=1}^{n}$ is a Parseval frame that inherits properties of the original frame $\Phi = \{f_k\}_{k=1}^{n}$.An interesting result in the context of frame isomorphy is that any Parseval frame derived from a frame is isomorphic to it.

Theorem 4.5 Let $\Phi = \{f_k\}_{k=1}^{n}$ be a frame for a Hilbert space H with frame operator S. Then the Parseval frame $\Phi^{can} = \{S^{-1/2} f_k\}_{k=1}^{n}$ is isomorphic to $\Phi = \{f_k\}_{k=1}^{n}$.

Proof. The proof follows from Definition 4.1 by letting $T = S^{-1/2}$.

Lemma 4.6 Let $\Phi = \{f_k\}_{k=1}^{n}$ and $\Psi = \{g_k\}_{k=1}^{n}$ be isomorphic Parseval frames for a Hilbert space H. Then they are unitarily isomorphic.

Lemma 4.6 says that the notions of isomrphy and unitary isomorphy coincide for Parseval frames.

Theorem 4.7 Every tight frame $\Phi = \{f_k\}_{k=1}^{n}$ for a Hilbert space H with frame bound $\alpha \neq 1$ can be rescaled to a Parseval frame.

Proof. Suppose that $\Phi = \{f_k\}_{k=1}^{n}$ is a tight frame with frame bound $\alpha \neq 1$. Then

$$\sum_{k=1}^{n} \left|\langle f, f_k \rangle\right|^2 = \alpha \|f\|^2, \text{ for all } f \in H.$$

Thus,

$$\frac{1}{\alpha} \sum_{k=1}^{n} \left|\langle f, f_k \rangle\right|^2 = \|f\|^2, \text{ for all } f \in H.$$

Pulling the factor $\frac{1}{\alpha}$ into the sum, we have

$$\sum_{k=1}^{n} \left|\left\langle \frac{1}{\sqrt{\alpha}} f, f_k \right\rangle\right|^2 = \sum_{k=1}^{n} \left|\left\langle f, \frac{1}{\sqrt{\alpha}} f_k \right\rangle\right|^2 = \|f\|^2 \text{ for all } f \in H.$$

Remark. Theorem 4.7 says that given a frame, it is always possible to find a frame isomorphic to it.

Theorem 4.8 The Grammian matrix G for an frame $\Phi = \{f_k\}_{k=1}^{n}$ is given by

$$G = (\langle f_j, f_k \rangle)_{m.n\in} = \begin{bmatrix} \langle f_1, f_1 \rangle & \langle f_1, f_2 \rangle & \cdots & \langle f_1, f_n \rangle \\ \langle f_2, f_1 \rangle & \langle f_2, f_2 \rangle & \cdots & \langle f_2, f_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_n, f_1 \rangle & \langle f_n, f_2 \rangle & \cdots & \langle f_n, f_n \rangle \end{bmatrix}$$

Proof. The proof follows from the fact that $G = AA^*$, where A^* has columns the frame operators f_k.

Remark. Theorem 4.8 says that the entries of the Grammian matrix are the inner products between the frame elements.

V. MAIN RESULTS

Theorem 5.1 Two frames for a Hilbert space H are unitarily isomorphic if and only if their Grammians are equal.

Proof. Suppose that $\Phi = \{f_k\}_{k=1}^{n}$ and $\Psi = \{g_k\}_{k=1}^{n}$ are unitarily isomorphic. Then $g_k = Uf_k$, for some unitary operator $U \in B(H)$. By Theorem 4.8, the Grammian

$$G_\Psi = \langle g_j, g_k \rangle = \langle Uf_j, Uf_k \rangle = \langle f_j, f_k \rangle = G_\Phi.$$

Conversely, suppose $G_\Psi = G_\Phi$. Then $\langle g_j, g_k \rangle = \langle f_j, f_k \rangle = \langle Uf_j, U_K \rangle$, for some unitary operator $U \in B(H)$. Therefore $g_k = Uf_k$. This proves proves that the frames are unitarily isomorphic.

Remark. Theorem 5.1 shows that unitary isomorphy preserves the Grammian of a frames. In fact, the Grammian characterizes the equivalence class of a frame.

Corollary 5.2 Let $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ be frames for an N-dimensional Hilbert space H with analysis operators A and B, respectively. Then the following conditions are equivalent.

(a). Φ and Ψ are unitarily isomorphic.

(b). $Ran(A) = Ran(B)$.

(c). $Ker(A^*) = Ker(B^*)$.

Proof. (a)$\Rightarrow$(b): Suppose that Φ and Ψ are unitarily isomorphic. By Theorem 5.1, $AA^* = BB^*$. Therefore,

$$Ran(A) = Ran(AA^*) = Ran(BB^*) = Ran(B).$$

(b)$\Rightarrow$(c): We use the fact that $Ker(T^*) = Ran(T)^\perp$ for any $T \in B(H)$. So if $Ran(A) = Ran(B)$, then $Ker(A^*)^\perp = Ker(B^*)^\perp$, which implies that $Ker(A^*) = Ker(B^*)$.

(c)$\Rightarrow$(a): We prove by contradiction. Suppose that $Ker(A^*) = Ker(B^*)$ but Φ and Ψ are not unitarily isomorphic. Then $G_\Psi \neq G_\Phi$. This implies that $Ker(BB^*) \neq Ker(AA^*)$. This implies that $Ker(B^*) \neq Ker(A^*)$, which is a contradiction to the assumption that $Ker(A^*) = Ker(B^*)$. This proves the claim.

Remark. Unlike the Grammian operator, we note that unitary isomorphy need not preserve the frame operator of a frame. The next result characterizes the unitary isomorphy of two frames in terms of their frame operators.

Theorem 5.3 Unitarily isomorphic frames have unitarily equivalent frame operators.

Proof. Suppose $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ are unitarily isomorphic frames and suppose that $\Phi = \{f_k\}_{k=1}^n$ has frame operator S. Then we have $g_k = Uf_k$, for some unitary operator $U \in B(H)$. By Theorem 3.11, the frame operator of Ψ is USU^*, which is unitarily equivalent to S.

The following result gives a condition when unitarily isomorphic frames have the same frame operator.

Theorem 5.4 Unitarily isomorphic tight frames for a Hilbert space H the same frame operator.

Proof. Suppose $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ are unitarily isomorphic tight frames with frame operators S_Φ

and S_Ψ, respectively. Tightness of the frames implies that $S_\Phi = \alpha_1 I$ and $S_\Psi = \alpha_2 I$ for some $0 < \alpha_1, \alpha_2 < \infty$. Using Theorem 5.3, unitary isomorphy of the frames implies unitary equivalence of the frames operators. This means that $S_\Phi = \alpha_1 I = U(\alpha_2 I)U^* = \alpha_2 I = S_\psi$. This proves the claim.

Remark. We define a new relation, called duality of finite frames. Recall that $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ are a dual pair if and only if $f = \sum_{k\in J} \langle f, f_k \rangle g_k = \sum_{k\in J} \langle f, g_k \rangle f_k$, for all $f \in H$. We denote this relation by $\Phi \overset{dual}{\square} \Psi$.

Theorem 5.5 Duality of frames $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ for a Hilbert space H is an equivalence relation.

Proof. Clearly $\Phi \overset{dual}{\square} \Phi$, since $f = \sum_{k\in J} \langle f, f_k \rangle f_k$. This shows that $\overset{dual}{\square}$ is reflexive. Now suppose $\Phi \overset{dual}{\square} \Psi$. Then $f = \sum_{k\in J} \langle f, f_k \rangle g_k = \sum_{k\in J} \langle f, g_k \rangle f_k$, for all $f \in H$. This shows that $\Phi \overset{dual}{\square} \Psi$ implies that $\Psi \overset{dual}{\square} \Phi$. This shows that $\overset{dual}{\square}$ is symmetric. Finally, suppose $\Omega = \{h_k\}_{k=1}^n$ is another frame for H. Suppose that $\Phi \overset{dual}{\square} \Psi$ and $\Psi \overset{dual}{\square} \Omega$. Then $f = \sum_{k\in J} \langle f, f_k \rangle g_k = \sum_{k\in J} \langle f, g_k \rangle f_k$, and $f = \sum_{k\in J} \langle f, g_k \rangle h_k = \sum_{k\in J} \langle f, h_k \rangle g_k$. This implies that $f = \sum_{k\in J} \langle f, f_k \rangle g_k = \sum_{k\in J} \langle f, h_k \rangle g_k$. Equating the coefficients, we have that $\langle f, f_k \rangle = \langle f, h_k \rangle$ for all $k \in J$. Therefore $f = \sum_{k\in J} \langle f, h_k \rangle f_k$, which proves that $\Phi \overset{dual}{\square} \Omega$. Thus $\overset{dual}{\square}$ is transitive. This proves that $\overset{dual}{\square}$ is an equivalence relation.□

Remark. The Grammian of a tight frame $\Phi = \{f_k\}_{k=1}^n$ is an orthogonal projection $G = AA^* = P =: P_\Phi$. The columns of P_Φ give a canonical copy of Φ and so the kernel of P_Φ is the space of linear dependence between vectors in Φ. This leads to the following result.

Proposition 5.6 Let $\Phi = \{f_k\}_{k=1}^n$ be a tight frame for an N-dimensional Hilbert space H with Grammian $G = P_\Phi$. Then $dep(\Phi) = Ker(P_\Phi)$.

Proof. Since the Grammian of a tight frame $\Phi = \{f_k\}_{k=1}^n$ is an orthogonal projection, we have

$$Ker(P_\Phi) = \{a = (a_1, a_2, \cdots, a_n) \in F^n : Pa = \sum_{k=1}^n a_k P e_k = 0\} = \{a \in F^n : Pa = \sum_{k=1}^n a_k f_k = 0\} =: dep(\Phi).$$

Remark. Proposition 5.6 says that for a tight frame the Grammian is determined by its kernel. It also says that P is the orthogonal projection onto $dep(\Phi)^\perp$. We also note that from the definition that if Φ is a basis for H, then $dep(\Phi) = \{0\}$.

Theorem 5.7 Let $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ be unitarily isomorphic frames for a Hilbert space H. Then $dep(\Phi) = dep(\Psi)$.

Proof. The proof follows easily from Theorem 5.1 and the definition of the notion of linear dependence.

Remark. Theorem 5.7 can be relaxed as follows.

Theorem 5.8 Let $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ be finite frames for a Hilbert space H with analysis operator A_Φ and A_Ψ, respectively. Then the following conditions are equivalent.

(a). Φ and Ψ are isomorphic.

(b). $dep(\Phi) = dep(\Psi)$.

Proof. (a)$\Rightarrow$(b): Suppose that Φ and Ψ are isomorphic. Then there exists an invertible operator $Q \in B(H)$ such that $g_k = Qf_k$ for all $k = 1, 2, \cdots, n.$ Thus the synthesis operator for Ψ is $A_\Psi^* = [g_k] = [Qf_k] = Q[f_k]$. Using the fact that the Grammian of Ψ is a projection $P = P_\Psi$ and the fact that $Ker(AA^*) = Ker(A^*)$ for any bounded linear operator A and the definition, we have

$$dep(\Psi) = Ker(A_\Psi^*) = Ker(Q[f_k] = Ker([f_k]) = Ker(A_\Phi^*) = Ker(P_\Phi) = dep(\Phi).$$

(b)$\Rightarrow$(a): Suppose that $dep(\Phi) = dep(\Psi)$. Using the fact that $A_\Phi^* = A_\Phi^* P_\Phi$ and $Ran(P_\Phi) = Ker(A_\Phi^*)^\perp$, we have that $A_\Phi^* : Ran(P_\Phi) \to H$ is invertible. Similarly, $A_\Psi^* : Ran(P_\Psi) \to H$ is invertible. This means that $Q := (A_\Phi^* |_{Ran(P_\Phi)})(A_\Psi^* |_{Ran(P_\Psi)})^{-1} : H \to H$ is invertible. Using the fact that $f_k = A_\Phi^* e_k = A_\Phi^* P_\Phi e_k$, where $\{e_k\}$ is an orthonormal basis for $l^2(J)$, $J = \{1, 2, \cdots n\}$, we have

$$Qf_k = QA_\Phi^*(P_\Phi e_k) = A_\Psi^* P_\Psi e_k = A_\Psi^* e_k = g_k.$$

This shows that $g_k = Qf_k$. Therefore the frames are isomorphic.

Theorem 5.9 Unitarily isomorphic frames have the same frame bounds.

Proof. Suppose that $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ are unitarily isomorphic frames for a Hilbert space H. Then $g_k = Uf_k$ some unitary operator $U \in B(H)$. Suppose $\Phi = \{f_k\}_{k=1}^n$ has S as its frame operator. By Corollary 3.11, $\Psi = \{g_k\}_{k=1}^n$ has frame operator USU^*. Corollary 5.3 shows that the frame operators are unitarily equivalent and hence have the same spectrum. That is $\sigma(S) = \sigma(USU^*)$ and therefore the lower and upper frame bounds are the same.

Remark. We note that Theorem 5.9 need not be true if we replace unitary isomorphy with isomorphy. This is because isomorphy of frames need not imply similarity of their frame operators. The following example illustrates this fact.

Example 5.10 Consider the frame $\Phi = \{f_k\}_{k=1}^3 = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ for $H = \mathbb{R}^2$. Let $U = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ and $Q = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$. Clearly both U and Q are invertible and in addition, U is unitary. So the sequences

$\Psi = \{Uf_k\}_{k=1}^{n}$ and $\Omega = \{Qf_k\}_{k=1}^{n}$ are frames for H. The frames Φ and Ψ are unitarily isomorphic while Φ and Ω are isomorphic. A simple computation gives the corresponding frame operators as

$$S_\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \; S_\Psi = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } S_\Omega = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

By inspection, the frame bounds for Φ are $\alpha = 1, \beta = 2$ and are the same for Ψ. But the frame bound bound for Ω is $\alpha = \beta = 4$. This shows that unlike unitary isomorphy of frames, frame isomorphy need not preserve frame bounds. From Example 5.10, we also note that frame isomorphy need not preserve tightness. In Example 5.10 Ω is a tight frame, while Φ is not, although they are similar frames.

Remark. For frames which are not tight, isomorphy of frames is weaker than unitary isomorphy of frames. Thus

$$\text{Unitary Isomorphy} \Rightarrow \text{Isomorphy},$$

But the converse is not true, in general.

Theorem 5.11 A frame $\Phi = \{ f_k \}_{k=1}^{n}$ for a Hilbert space H with a frame operator S, its canonical dual $\tilde{\Phi} = \{S^{-1} f_k\}_{k=1}^{n}$ and its canonical tight frame $\Phi^{can} = \{S^{-1/2} f_k\}_{k=1}^{n}$ are isomorphic frames. Moreover, they are unitarily isomorphic if and only if $\Phi = \{ f_k \}_{k=1}^{n}$ is Parseval.

Proof. The proof of the first claim follows from Theorem 5.8 and Theorem 5.9. The proof of the second claim follows from Lemma 4.6.

Theorem 5.12 Two Parseval frames $\Phi = \{ f_k \}_{k=1}^{n}$ and $\Psi = \{g_k\}_{k=1}^{n}$ are unitarily isomorphic and only if they are isomorphic.

Proof. We prove the converse. The other direction is clear for all frames. Isomorphy of $\Phi = \{ f_k \}_{k=1}^{n}$ and $\Psi = \{g_k\}_{k=1}^{n}$ implies existence of an invertible operator $T \in B(H)$ such that $g_k = Tf_k$, for all $k = 1, 2, ..., n$. By ([2], Theorem 2.2) and Corollary 3.11 and the fact that the frames are Parseval implies that the frame operator of $\Psi = \{g_k\}_{k=1}^{n}$ is $T(I)T^* = TT^* = I$. This implies that T is an isometry. Invertibility of T then implies that T where U is a unitary operator. Therefore $g_k = Uf_k$. This proves that the frames are unitarily isomorphic.

Remark. The Proof in Theorem 5.12 is equivalent to the following direct one: For all $f \in H$, we have

$$\left\|T^* f\right\|^2 = \sum_{k=1}^{n} \left|\left\langle T^* f, f_k \right\rangle\right|^2 = \sum_{k=1}^{n} \left|\left\langle f, Tf_k \right\rangle\right|^2 \sum_{k=1}^{n} \left|\left\langle f, g_k \right\rangle\right|^2 = \| f \|^2 .$$

This proves that T^* is an isometry. That is, $TT^* = I$. Invertibility then implies that T is unitary. The result now follows by letting $T = U$, for some unitary $U \in B(H)$.

Remark. We note that if a frame is unitarily isomorphic to a Parseval frame, then it is also a Parseval frame. We also note that every Riesz basis is isomorphic to an orthonormal basis for a Hilbert space.

The following result characterizes finiteisomorphic frames in terms of the Grammians of their canonical tight frames.

Proposition 5.13 Let $\Phi = \{f_k\}_{k=1}^n$ and $\Psi = \{g_k\}_{k=1}^n$ be finite frames for a Hilbert space H with analysis operators A and B, respectively. Then the following are equivalent.

(a). Φ and Ψ are isomorphic.

(b). $Gram(\Phi^{can}) = Gram(\Psi^{can})$.

(c). $Ran(A) = Ran(B)$. Equivalently, $Ker(A^*) = Ker(B^*)$.

Proof. (a)$\Rightarrow$ (b): Suppose that Φ and Ψ are isomorphic and that the frame operator of $\Psi = \{g_k\}_{k=1}^n$ is S_Ψ. Then there exists an invertible operator Q such that $g_k = Qf_k$ for all k. So

$$g_k = Qf_k = QS^{1/2}S^{-1/2} f_k = (QS^{1/2})S^{-1/2} f_k = Tf_k^{\ can} = T\Phi^{can}, \text{ for all } k,$$

Where $T = QS^{1/2}$ is invertible. Hence $\Psi = \{Tf_k^{\ can}\}$. For convenience, we denote the synthesis operator by C^{can}. Then

$$S_\Psi = T(C^{can})^*(C^{can})T^* = TT^* = I.$$

Using the fact that $\Psi = \{Tf_k^{\ can}\}$ is a tight frame if and only if $S_\Psi = c^2 I$ for some $c > 0$, we conclude that T is a unitary operator. Thus Ψ Is unitarily isomorphic to Φ^{can}. Therefore $Gram(\Phi^{can}) = Gram(\Psi^{can})$.
(b)$\Rightarrow$(c): Since $P_\Phi^{\ can} := Gram(\Phi^{can})$ and $P_\Psi^{\ can} := Gram(\Psi^{can})$ are orthogonal projections, they are determined by the ranges, then $P_\Phi^{\ can} = P_\Psi^{\ can}$ if and only if $Ran(A) = Ran(B)$, or equivalently, $Ker(A^*) = Ker(B^*)$.

(c)$\Rightarrow$ (a): This follows immediately from Corollary 5.2, since unitary isomorphy implies isomorphy. We give a rigorous proof. First note that

$$Bf = \sum_{k=1}^n \langle f, g_k \rangle f_k = \sum_{k=1}^n \langle f, Tf_k \rangle f_k = \sum_{k=1}^n \langle T^* f, f_k \rangle f_k = AT^* f.$$

This shows that $Ran(B) = Ran(AT^*) = Ran(A)$. For convenience, we let

$$Ran(A) = Ran(B) = M,$$

where M is a closed subspace of $l^2(\mathbb{N})$. Since A^* and B^* are invertible when restricted to M, the operator $T = B^*(AA^*|_M)^{-1}A : H \to M$ is onto and hence invertible on M. Therefore $A^*(M^\perp) = B^*(M^\perp) = \{0\}$. Thus $f_k = A^* e_k = A^* P e_k$ and $g_k = B^* e_k = B^* P e_k$, where $\{e_k\}$ is the standard basis for $l^2(\mathbb{N})$. Therefore

$$Tf_k = TA^* e_k = TA^* P e_k = B^*(AA^*|_M)^{-1}(AA^*|_M)Pe_k = B^* P e_k = g_k.$$

That is, $g_k = Tf_k$, which implies that the frames are isomorphic.

Remark. Proposition 5.13 shows that finite frames are isomorphic if and only if their canonical Grammians are equal. It also says that finite frames are isomorphic if and only if their analysis operators have the same range.

We define the traditional notion of redundancy $\operatorname{Re}d(\Phi)$ of a frame $\Phi = \{f_k\}_{k=1}^{N}$ for an n-dimensional Hilbert space H as the quotient $\rho = \frac{N}{n}$. This however, is a customary and crude quantitative notion of redundancy. For literature on other quantitative notions of redundancy (see [6], [7]). In any case, it is known that the redundancy of an over-complete frame is greater than 1.

Remark. We note that unitary isomorphy of frames preserves redundancy of frames. However, equality of redundancy does not, in general, translate to unitary isomorphy.

Example 5.14 Consider the frames $\Phi = \{e_1, e_1, e_2\}$ and $\Psi = \{e_1, e_2, e_2\}$, where $\{e_k\}$ denotes the orthonormal basis for $H = \square^2$. Then $\operatorname{Re}d(\Phi) = \operatorname{Re}d(\Psi) = \frac{3}{2}$. However, the frames are not unitarily isomorphic, since

$$Gram(\Phi) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = Gram(\Psi).$$

Note that although the Grammians are not equal, they are similar as operators, with the similarity being implemented by $N = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

Let $F = \{\Phi : \Phi \text{ is a frame}\}$ be the set of frames for a finite dimensional Hilbert space H.

Define a relation $\overset{red}{\square}$ on F by $\Phi \overset{red}{\square} \Psi$ if and only if $\operatorname{Re}d(\Phi) = \operatorname{Re}d(\Psi)$.

Theorem 5.15 $\overset{red}{\square}$ is an equivalence relation on F .

Proof. Trivial and hence omitted.

Theorem 5.15 says that searching for a frame which possesses a predetermined redundancy function is equivalent to searching for the equivalence class. Frames which have the same redundancy in a Hilbert space will be called $\overset{red}{\square}$ -equivalent.

Corollary 5.16 Let Φ and Ψ be frames for a finite dimensional Hilbert space H. Suppose Φ and Ψ are normalized with frame operators S_Φ and S_Ψ, respectively. Then the following conditions are equivalent.

(a). $\operatorname{Re}d(\Phi) = \operatorname{Re}d(\Psi)$.

(b). $S_\Phi = S_\Psi$.

Corollary 5.16 says that $\overset{red}{\square}$ -equivalent frames must have the same number of non-zero frame vectors.

Theorem 5.17 Unitarily isomorphic and isomorphic frames are $\overset{red}{\square}$ -equivalent.

In particular, orthonormal bases and Riesz bases are $\overset{red}{\square}$ -equivalent.

Definition 5.18 Let $\Phi = \{f_k\}_{k=1}^{N}$ be a collection of unit vectors for an n-dimensional Hilbert space H. The frame potential of Φ is the number

$$\widehat{P}_{\Phi} = \sum_{i=1}^{k}\sum_{j=1}^{k}\left|\langle f_{i,} f_j \rangle\right|^2 .$$

The frame potential gives an intuitive idea of the configurations of the vectors in a tight frame. This notion can be extended to any collection of vectors with varied norms. The inner product between vectors gives a quantity describing the orthogonality of the vectors. For more literature on frame potential (see [4]).

Theorem 5.19 All tight frames $\Phi = \{f_k\}_{k=1}^{N}$ in a finite dimensional Hilbert space has the same frame potential.

Lemma 5.20 If $\Phi = \{f_i\}_{i=1}^{k}$ is a tight frame of unit vectors in $\square^{n}$, then the frame potential of Φ is $\widehat{P}_{\Phi} = \frac{k^2}{n}$.

Proof. Since Φ is a tight frames of unit vectors, the frame bound is $\alpha = \frac{k}{n}$. By definition of a tight frame, we then have

$$\widehat{P}_{\Phi} = \sum_{i=1}^{k}\sum_{j=1}^{k}\left|\langle f_{i,} f_j \rangle\right|^2 = \sum_{i=1}^{k}(\sum_{j=1}^{k}\left|\langle f_{i,} f_j \rangle\right|^2) = \sum_{i=1}^{k}\alpha \| f_i \|^2 = \sum_{i=1}^{k}\alpha = \frac{k^2}{n}.$$

Remark. First, recall that $tr(S) = tr(G)$, where $S = A^*A$ and $G = AA^*$. The frame potential of a frame $\Phi = \{f_i\}_{i=1}^{k}$ can be described in terms of the trace and the Grammian matrix G as

$$\widehat{P}_{\Phi} = \sum_{i=1}^{k}\sum_{j=1}^{k}\left|\langle f_{i,} f_j \rangle\right|^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\left|G_{i,j}\right|^2 = tr(G^2) = \sum_{i=1}^{k}\lambda_i^2,$$

Where λ_i are the eigenvalues of G .

The next result shows that unitary isomorphy preserves frame potential.

Lemma 5.21 Unitarily isomorphic frames in a finite dimensional Hilbert space have equal frame potential.

Proof. Suppose that $\Phi = \{f_i\}_{i=1}^{k}$ is a frame for a Hilbert space H. Let $\Psi = \{g_i\}_{i=1}^{k}$, where $g_i = Uf_i$ for all $i = 1, 2, \cdots, n$ and some unitary operator $U \in B(H)$. Then

$$\widehat{P}_{\Psi} = \sum_{i=1}^{k}\sum_{j=1}^{k}\left|\langle Uf_{i,} Uf_j \rangle\right|^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\left|\langle f_{i,} f_j \rangle\right|^2 = \widehat{P}_{\Phi}.$$

Given two $M \times N$ matrices A and B, we define the Hilbert-Schmidt trace inner product as

$\langle A,B\rangle_{H.S}=tr(AB^*)$. This inner product induces the Hilbert-Schmidt norm $\|.\|_{H.S}$ or the Frobenius norm on the vector space of all $M\times N$ matrices. Using this distance notion we define a distance function on the space of frames $F=\{\Phi:\Phi \text{ is a frame}\}$.

Definition 5.22 Let $\Phi=\{f_i\}_{i=1}^k$ and $\Psi=\{g_i\}_{i=1}^k$ be finite frames for an n-dimensional Hilbert space H. The frame distance between them is

$$d_F(\Phi,\Psi)=\|\Phi-\Psi\|_{H.S}.$$

Clearly, the frame distance is a metric on the space of frames F.

Definition 5.23 Let $\Phi=\{f_i\}_{i=1}^k$ and $\Psi=\{g_i\}_{i=1}^k$ be finite frames for an n-dimensional Hilbert space H with analysis operators A and B, respectively. The Grammian distance between them is

$$d_G(\Phi,\Psi)=\|Gram(\Phi)-Gram(\Psi)\|_{H.S},$$

where $Gram(\Phi)=A^*A$ and $Gram(\Psi)=B^*B.$

Clearly, the Grammian distance is a pseudo-metric on the space of frames F because unitary isomorphy of Φ and Ψ implies $Gram(\Phi)=Gram(\Psi)$, which means that $d_G(\Phi,\Psi)=0.$

We also define a distance in terms of unitary isomorphy of frames.

Definition 5.24 Let $\Phi=\{f_i\}_{i=1}^k$ and $\Psi=\{g_i\}_{i=1}^k$ be finite frames for an n-dimensional Hilbert space H. Theisomorphy distance between them is

$$d_I(\Phi,\Psi)=\inf_{\substack{\Phi'\cong\Phi\\ \Psi'\cong\Psi}}\|\Phi'-\Psi'\|_{H.S},$$

where $\cong$ denotes unitary isomorphy. Clearly, the isomorphy distance is a pseudo-metric on the space of frames F since $d_I(\Phi,\Psi)=0$ whenever Φ and Ψ are unitarily isomorphic.

Definition 5.25 Two frames $\Phi=\{f_i\}_{i=1}^k$ and $\Psi=\{g_i\}_{i=1}^k$ for an n-dimensional Hilbert space H are said to be switching equivalent if there is a unitary operator $U\in B(H)$ and a permutation π of the set $J=\{1,2,\dots,k\}$ such

$f_j=Ug_{\pi(j)}$, for all $j\in J$.

Theorem 5.26 **Two**Parseval frames $\Phi=\{f_i\}_{i=1}^k$ and $\Psi=\{g_i\}_{i=1}^k$ for an n-dimensional Hilbert space H are switching equivalent if and only if there exists a permutation π of the index set $J=\{1,2,\dots,k\}$ such $Gram(\Phi)_{i,j}=Gram(\Psi)_{\pi(i),\pi(j)}$.

Proof. Define a matrix

$$P=P_{i,j}=\begin{cases}1, & if\ \pi(i)=j\\ 0, & otherwise\end{cases}$$

Let A an B be the analysis operators of the frames Φ and Ψ, respectively. Suppose also that Φ and Ψ are switching equivalent. Then there exists a unitary operator $U \in B(H)$ and a permutation π of the set $J = \{1, 2, ..., k\}$ such $f_j = Ug_{\pi(j)}$, for all $j \in J$. Thus $A^* = UB^*P^*$, where $U \in B(H)$ is unitary. This is equivalent to

$$Gram(\Phi) = AA^* = PBU^*UB^*P^* = PBB^*P = P(Gram(\Psi))P^*.$$

Thus the Grammians are identical up to conjugation by a permutation matrix. Therefore

$$Gram(\Phi)_{i,j} = Gram(\Psi)_{\pi(i),\pi(j)}.$$

Conversely, suppose that $Gram(\Phi)_{i,j} = Gram(\Psi)_{\pi(i),\pi(j)}$. Then

$$\langle f_i, f_j \rangle = \langle g_{\pi(i)}, g_{\pi(j)} \rangle = \langle Ug_{\pi(i)}, Ug_{\pi(j)} \rangle.$$

Therefore $f_j = Ug_{\pi(j)}$, for all $j \in J$. This proves that the frames are switching equivalent.

Remark. Note that if $P = I$ in the proof of Theorem 5.26, then the Grammians of the frames are equal, which by Theorem 5.1, means that the frames are unitarily isomorphic. This shows that switching equivalence is weaker than unitary isomorphy of frames.

The next result shows that unitary isomorphy preserves tightness of frames.

Theorem 5.27 **If t**wo tight frames $\Phi = \{f_i\}_{i=1}^k$ and $\Psi = \{g_i\}_{i=1}^k$ for an n-dimensional Hilbert space H are unitarily isomorphic then they have the same tightness.

Proof. Suppose $\Phi = \{f_i\}_{i=1}^k$ is $\alpha -$tight. Then $f = \frac{1}{\alpha}\sum_{i=1}^k \langle f, f_i \rangle f_i$, for all $f \in H$. Taking inner product with f we get

$$\| f \|^2 = \langle f, f \rangle = \frac{1}{\alpha}\sum_{i=1}^k \langle f, f_i \rangle \overline{\langle f, f_i \rangle} = \frac{1}{\alpha}\sum_{i=1}^k \left|\langle f, f_i \rangle\right|^2.$$

Now suppose $g_i = Uf_i$, for some unitary operator $U \in B(H)$. Then using the fact that $U^* = U^{-1}$, we get

$$f = \frac{1}{\alpha}\sum_{i=1}^k \langle f, f_i \rangle f_i = \frac{1}{\alpha}\sum_{i=1}^k \langle f, U^{-1}Uf_i \rangle f_i = \frac{1}{\alpha}\sum_{i=1}^k \langle (U^{-1})^* f, Uf_i \rangle f_i$$

$$= (U^{-1})^* \frac{1}{\alpha}\sum_{i=1}^k \langle f, Uf_i \rangle f_i$$

$$= \frac{1}{\alpha}\sum_{i=1}^k \langle f, Uf_i \rangle (U^{-1})^* f_i$$

$$=\frac{1}{\alpha}\sum_{i=1}^{k}\langle f, Uf_i\rangle Uf_i$$

$$=\frac{1}{\alpha}\sum_{i=1}^{k}\langle f, g_i\rangle g_i.$$

Taking inner product with f gives

$$\|f\|^2=\langle f, f\rangle=\frac{1}{\alpha}\sum_{i=1}^{k}\langle f, g_i\rangle\overline{\langle f, g_i\rangle}=\frac{1}{\alpha}\sum_{i=1}^{k}\left|\langle f, g_i\rangle\right|^2.$$

This shows that the frame $\Psi=\{g_i\}_{i=1}^{k}$ is also $\alpha-$tight.

Remark. Theorem 5.27 can easily be proved by invoking Corollary 3.1 and the fact that unitay equivalence of operators preserves norms:

$$\|S_\Psi f\|^2=\|USU^* f\|^2=\|Sf\|^2=\alpha\|f\|^2.$$

VI. CONCLUSION

Frame isomorphy and unitary isomorphy can be used to determine equivalence classes of some Hilbert space frames. There are at most finitely many frame equivalence classes, which means that the problem of determining, for instance, tight frames reduces to the problem of finding representatives for each equivalence class and determining which of these equivalence classes is optimal in application. Some classes of frames are enticing to frame theorists and experts because their properties make calculations easier. Knowledge about the frame operators and synthesis operators and their properties is crucial in classifying frames.

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