Properties on Some Types of Generalized Inverses of Matrices-A Review

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Abstract

This paper aim to study some types of generalized inverse for every finite matrix A (square or rectangular) of real or complex elements with the property that the general solution of the equation $A^n x = 0$ *for all positive integer n .also study Existence and Construction of Generalized , Inverses like pseudo inverse , Moore–Penrose inverse, and Properties of {1}-Inverses Drazin inverse , another types of inverses of matrices and gives some important proofs of theories .with some examples about this subject .*

Keywords - *existence and construction ; Drazin inverse; generalized inverse ; matrices ; properties*

I. INTRODUCTION

An idea of a popularize or Generalized inverse or" reverse" appears for the first time and printed in (1903) by Fredholm that a special generalized inverse was named "pseudo inverse" of an integration processor was given. Popularize inverses of d differentiation processor, formerly implied in Hilbert's argument in (1904) of generalized Green a functions [1]. Popularize inverse of differentiation and integration processors so antedated the general inverses of array (matrix) , which subsistence was first observe by Moore, who cleared a unique reverse and 'named as "general reciprocal" for each fixed array (square or rectangular). though his first review of a topic . The group of all pseudo inverse was inspected in (1912) by Hurwitz who used the finite distances of the void area of the Fredholm processor to give a easy algebra structure.

Although his first publication in the subject [2], Moore (1920) formulated the general reverse of array in an algebraic setting but his first publication on the subject papered in (1920) Generalized inverses for operators were given by Murray & Von Neumann (1936) and another. Big expansion of benefit in the reign came in the 1950s by the study of the least squares properties. Penrose (1955) show that the Moore's reverse would satisfy the four equations and is unique inverse is yet called the Moore-Penrose inverse. after then many papers on this topic have appeared [3] , [4]. Bjerhammar (1958) recognize these properties and he rediscovered Moore's inverse additionally renowned the relation of general reverse to solutions of linear systems [5].

The notion of general inverse has a possibility for its wide application, especially general inverse of array on top finite domain. This topic is studies by many author some of them are Pearl (1968) Fulton (1978) and Wu & Dawson (1998a) [9]. Finite fields play fundamental role in applications including error correcting codes and Cryptography. For instance, Wu & Dawson (1998b) used generalized inverses in public key cryptosystem design. of generalized inverses in error correction [11] , [14].

Jovan D.keckid (1989) pointed on some types of General inverse of matrix, and some linear array equations , with the property that the public solution of the neutralization $A^n x = 0$ it can expressed by means of A_G for all positive integer n . also satisfied by the strong spectral inverse of Greville [12] . Campbell (1977) stated that the major outcome : if matrix come "close " to satisfactory the introduction of the Drazn reverse A^D of A, then $\left\| X - A^D \right\|$ is also small norm estimates are given which rank precise what is meant by close [10] .

Wel Yimin &wang Guorong (1997) submitted by Rebert E. Mutwig title is " The perturbation :Theory for the drazin inverse and its applications" studying two matrices A ,and E be (nxn) , B =A+E denoted the Drazin inverse of A by A^D, an error bound . we grant an top limited for proportional mistake $\left\| B^D - A^D \right\| / \left\| A^D \right\|$ under certain circumstances . An error bound for solution for singular equations A.x = b , [$b \in R(A)$] is also considered . [13]. The theory of generalized inverses has its genetic roots essentially in the context of so called " ill-posed " linear issues. It is fully renowned that if A is a non-single (square) matrix , then there occurs a singular array B, whom is invited the inverse of A. If A is a unique or a rectangular (however not square) array, no such as matrix B occur. Now if A^{-1} occurs, then the system of linear neutralization has the singular resolution $X = A^{-1.} B$. Henryk Hudzikand & Yuwen Wang (2011) [17] .

Meltem (2016) studied General inverse of matrix and enforcement to coding theory This thesis deals with the application of general reverses of matrices over finite fields and the method of least squares in linear codes. It is proven that if the Moore-Penrose inverse of a generator array of a linear code exists, a unique word approaching to a received word near the code words of the code can be found [18] . Also Wel Yimin & Wang Guorong & Qiao SanZhehg (2018) studied the Generalized Inverses appear different uses and fields such as in science , statistics and engineering , such as least squares parataxis, unique differential and vary neutralization , Markov chains, its impact problems [19] . Also they discussed the Moore-Penrose reverse and the { *i, j, k*} inverse that possession some "inverse-like" property. The {i , j, k} inverse supply some kinds of resolution, or the least-square resolution, for linear system of neutralization solely as the uniform inverse supply a singular resolution for a non-singular of linear system neutralization . Now the { i , j, k} inverse are invited neutralization resolving inverse.[19].

Michael Friendly (,2018) studied Generalized inverse ,In array algebra, the reverse of array is defined but for *square* array, and if array is *unique*, it does not have an reverse. The generalized inverse (or *pseudo inverse*) is an extension of the idea of a matrix inverse, which has some but not all the properties of an ordinary inverse. A combined utilize of the pseudo inverse is to compute a " better fits" (least squares) solution to a system of linear neutralization that reduction a singular solution .A generalized inverse does exist for any matrix, but unlike the regular inverse, the generalized inverse is not unique, in the feeling that there are various ways of defining a generalized inverse with various inverse-like properties [20] . Jeffrey Uhlmann (2018) derived a new generalized matrix inverse which is consistent with respect to arbitrary non-singular diagonal transformations .The new inverse complements the Drazin inverse (which is consistent with respect to similarity transformations) and the Moore-Penrose inverse (which is consistent with respect to unitary/orthonormal transformations) to complete a trilogy of generalized matrix inverses that exhausts the standard family of analytically-important linear system transformations . Results are generalized to obtain unit-consistent and unit-invariant matrix decompositions [21] .

II.INVERSE OF A NON- SINGULAR MATRIX

It can be fully renowned that any non-singular matrix A has a singular inverse, indicate by A^{-1}, like that " $A.A^{-1} = A^{-1}.A = I$ ".

When I is the identities matrix of that various property of the inverse matrix, we remind a little.

Thus $(A^{-1})^{-1} = A$, $(A^T)^{-1} = (A^{-1})^T$, $(A^*)^{-1} = (A^{-1})^*$,
$(A.B)^{-1} = B^{-1}.A^{-1}$, where A^T and A^* , respectively, denote the transpose and conjugate transpose of matrix *A*.

III. GENERALIZED INVERSES OF MATRICES

A matrix has an inverse only if it is square, and even then only if it is non singular or, in other words, if its columns (or rows) are linearly independent.. Recently different applications in various regions of apply mathematics have been used for different types of part converse of a matrix that is single or even of a matrix that is singular or even rectangular. By a general inverse of a given matrix *A* (matrix *X)* related in several behaviour together *A* as follows :

- It presents for a group of array big than the group of non singular matrix.
- It have several of the property of the typical inverse.
- It reduce to the typical reverse when *A* is non-unique.

Several writer have use the tenure " pseudo inverse " ' instead generalized inverse [10].

IV. EXISTENCE AND CONSTENCE OF GENERALIZED INVERSES (" THE PENROSE EQUATIONS ")

Penrose showed in (1955) that, for each finite array A (square or rectangular) of actual or compound items, there is a singular array X satisfactory the 4 neutralization (so as to called the Penrose neutralization).

$AXA = A$ (1) , $XAX = X$ (2) , $(AX)^* = AX$ (3) , $(XA)^* = XA$ (4)

Where A^T and A^* respectively, denote the transpose and conjugate transpose of matrix *A*. It will be recalled . Cause this singular generalized inverse have previous been study (during defined in a vary via) by E.H. Moore [3] . it is generally known such as the Moore–Penrose inverse, and is oftentimes indicate via A . If A is non-unique, then $X = A^{-1}$ immaterially satisfy the four neutralization. It follow up that the Moore–Penrose inverse of a non-unique array is the selfsame such as the average inverse . This paper will be many interested with general inverses that satisfying some, only not all, of the four Penrose neutralization. As we will wish to transact together a number of various subsets of the collection of four neutralization, we need a suitable marking for a general inverse satisfying sure specify neutralization [5], [15] .

V. DEFINITIONS

A. ***Definition***: for each $A \in \mathbb{C}^{mxn}$, let $A\{ i, j, . . . , k\}$ indicate the set of matrices $X \in Cn\times m$ which satisfy equations (I), (j), . . . , (k) from among equations (1)–(4). A matrix X ∈ A{ I, j, . . . , k} is called an { I , j, . . . , k}- *inverse* of *A* and also indicate by $A^{(i, j,..., k)}$[15] .

B. Definitions: Range and Null Space

For any $A \in C^{m\times n}$, denote by

Rang of a Matrix A is *:* R(*A*) = { y∈ C^m : y = Ax for some x ∈ C^n}, the range of *A (Column space)* .
Null Space of a Matrix *A is : N*(A) = { x ∈ C^n : Ax = 0} , dim R(A)+ dim N(A) = n

in many applications a basis for R(A) is useful in many applications , such as, in the numerical calculation of the Moore-Penrose inverse. The need for a basis of *N*(*A*) is clarify by the general solution of the linear in homogeneous equation that AX = B is the sum of any particular solution x_0 ,& the general solution of the homogeneous *formal that AX* = 0 The latter general solution consists of all linear combinations of the elements of any basis for *N*(*A*).[16] , [18] .

C. *Definition* : Nilpotent *of a Matrix is :* if A is non singular , Ind (A) =0 , A is nilpotent of index v , $A^v=0$, $A^{v-1} \neq 0$ then Ind(A) = v , for some positive integer v. Example : The matrix A $= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent , ago A^2 =0 , any triangular matrix with(0s) along the main diagonal is nilpotent.

D. *Example:1*:. If the set of inverse of A is = A*{* 1 , 2 , 3 , 4*}* is non empty then it depend of a single element [5].

solution.: Let *X, Y ∈ A{*1, 2, 3, 4*}*.
Then $X = X(AX)^* = XX^*A^* = X(AX)^*(AY)^*$
$= XAY = (XA)^*(YA)^*Y = A^*Y^*Y = (YA)^*Y = Y$.

VI. EXISTENCE AND CONSTENCE OF {1}- INVERSES

Any matrix X satisfying the equality A.X.A = A is called a {1} .-inverse of A and is by indicate $A^{(1)}$. it easy to Construction of (1*)* – Inverse of matrix R $\in C^{m\times n}$, given

$$R = \begin{bmatrix} I_r & K \\ O & O \end{bmatrix} \quad \ldots \quad (1\text{-}1)$$

for each $L \in C^{(n-r)x(m-r)}$, the (nxm) matrix $S = \begin{bmatrix} I_r & O \\ O & L \end{bmatrix}$

is a *{*1*}*-inverse of (1-1). If *R* is of full column (Row) rank, the two lower (right-hand) sub matrices are interpreted as absent.
The construction of *{*1*}*-inverses for an arbitrary $A \in Cm\times n$ is simplified by transforming *A* into a Hermite normal form, where E is product of elementary matrices in (1-2) and P is a permutation matrix , as shown in the following theorem [10], [15].

Theorem(1) : Let $A \in C_r^{mxn}$ and let E ∈ C_m^{mxm} and P∈ C_n^{nxn} be such that

$$\text{EAP} = \begin{bmatrix} I_r & K \\ O & O \end{bmatrix} \quad \ldots. \quad (1\text{-}2)$$

Then for each $L \in C^{(n-r)x(m-r)}$, (n × m) matrix $X = P\begin{bmatrix} I_r & O \\ O & L \end{bmatrix}E$ … (1-3)

is a {1}-inverse of A. The separate matrices in (1-2) and (1-3) must be suitably explicate in any event (r = m or r = n).

proof : Rewrite $EAP = \begin{bmatrix} I_r & K \\ O & O \end{bmatrix}$, as $A = E^{-1}\begin{bmatrix} I_r & K \\ O & O \end{bmatrix}P^{-1}$ … (1-4)

it is easily verified that any *X* given by (1-3) satisfies *"A.X.A = A*" .
In the trivial case of *r* = 0, when *A* is therefore the (*m*× *n)* null matrix, *any n* × *m* matrix is a *{*1*}*-inverse.
We note that since *P* and *E* are both non singular, the rank of *X* as taken by (1-3) is the rank of the partitioned matrix in the right member

rank $X = r$ + rank L ... (5)

Since L is tyrannical, it follows that a $\{1\}$-inverse of A exists having any rank between r and $\min\{m, n\}$, inclusive . By Theorem 1 showed that every finite matrix with elements in the complex field has a $\{1\}$-inverse, and suggests how such an inverse can be constructed [8].

A. Properties of $\{1\}$-Inverses

Sure property of $\{1\}$-inverses are give in Lemma 1. For a give array A, we indicate any $\{1\}$-inverse by $A(1)$. Note that, uniquely defined matrix. For any scalar λ we define λ^+ by $(\lambda)^+ = (\lambda)^{-1}$, if $\lambda \neq 0$, 0, if $\lambda = 0$. … (6)

It can be called that a square array E is called idempotent if $E^2 = E$. Idempotent array are carefully interrelated to general inverses.

Lemma (1) : *given* $A \in Cm\times n$, $\lambda \in C$. *Then:*

(a) $(A^{(1)})^* \in A^*\{1\}$.

(b) If A is non-unique, $A(1) = A^{-1}$ singular.

(c) $(\lambda)^+ . A^{(1)} \in (\lambda .A).\{1\}$.

(d) Rank $A^{(1)} \geq$ Rank A.

(e) If G and T are non-unique, $T^{-1}.A(1).G^{-1} \in G.A^T\{1\}$.

(f) $A.A^{(1)}$ and $A^{(1)}A$ are idempotent and has the alike rank as A.

Proof. : These are important relative (a)- (d) are given direct conclusion and the latter piece of (f) relay on the actuality that the rank of a producer of matrix does not go beyond the rank of any issue. If an($m\times n$) matrix A is of full column rank, its $\{1\}$-inverse are its left reverse. If it is complete line rank, its $\{1\}$-inverses are its right inverse.

Lemma (2) : (Ben-Israel , 2003) , Let $A \in C_r^{mxn}$. Then ,

a) $A^{(1)} A = In$ iff $r = n$.

b) $A. A^{(1)} = Im$ iff $r = m$.

Proof. (a) If : Let $A \in C_r^{mxn}$. Then n × n matrix $A^{(1)} \cdot A$ is by Lemma 1(f), idempotent and nonsingular. Multiplying $(A^{(1)}A)^2 = A(1)\, A$ by $(A^{(1)}A)^{-1}$ gives $A^{(1)} \cdot A = In$.

Only if :: $A^{(1)} \cdot A = In \Rightarrow$ rank $A(1). A = n \Rightarrow$ rank $A = n$, by Lemma 1(f)

(b) Same prove**.**

Example 2 :. prove that B is non-unique iff it has a singular $\{1\}$-inverse, which then correspond with B^{-1}.

Proof.: For any $\mathbf{x1} \in N1(B)$ [$\mathbf{y1} \in N1(B^*)$] , adding $\mathbf{x1}$.[($\mathbf{y1}$)*] to any column (row) of an $X \in B(1)$ gives another $\{1\}$-inverse of B. The uniqueness of the $\{1\}$-inverse is therefore equivalent to $N1(B) = \{\mathbf{0}\}$, $N1(B^*) = \{\mathbf{0}\}$, i.e. to the non singularity of B .

VII. EXISTENCE AND CONSTENCE OF {1 , 2}- INVERSES

It can be first well-known by Bjerhamar [6] that the present of a $\{1\}$-inverse of a matrix A implies the reality of a $\{1, 2\}$-inverse. This simply verified surveillance is declared as a lemma for expediency of orientation.

Lemma (3) : (Ben-Israel, & Greville , 2003)

Let $X_2 , X_3 \in A\{1 \}$, *and let* $X_1 = X_2 AX_3$. *Then* $X_1 \in A\{1, 2\}$.

Since the matrix A and X_1 happen symmetrically in (1) and (2), $X_1 \in A\{1, 2\}$ and $A \in X_1\{1, 2\}$ are equal sentences, and in either case we can say that A and X_1 are $\{1, 2\}$-inverse *of every another*. , it follows at once that if A and X_1 are $\{1, 2\}$-inverses of every another, by (1) and (2) and the verity that the rank of a invention of matrix does not exceed the rank of any they have the similar rank. clear, first illustrious by Bjerhamar, that if X_1 is a $\{1\}$-inverse of A and of the similar rank as A, it is a $\{$'1, 2$\}$-inverse of A [6] , [15].

Theorem(2) : (Bjerhammar) *Given* A and $X \in A\{1\}$, $X \in A\{1, 2\}$ iff rank X = rank A.

Proof.: If Clearly $R(XA) \subset R(X)$, But rank XA = rank A by Lemma 1(f) and so, if rank X = rank A , $R(XA) = R(X)$, Thus, $XAY = X$ for some Y . Premultiplication by A grants $AX = AXAY = AY$, and therefore $XAX = X$. Only if : from (1) and (2) An equipollent confirmation [6] .

Corollary (1) : Any two of the following three statements imply the third:

$X \in A\{1\}$, $X \in A\{2\}$,

Rank X = rank A

By Theorem 2, shows that the {1}-inverse obtained from the Hermite normal form is a {1, 2}-inverse if we take ' L = O '. In other words,

$$X = P\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}E \qquad \ldots \text{ (1-5)}$$

is a *{*1, 2*}*-inverse of *A* where *P* and *E* are nonsingular and satisfy (1-2).

Example 3: Let A = [aij] ∈ Cm×n be nonzero and upper triangular, i.e., $a_{ij} = 0$ if i > j. Find a {1, 2}-inverse of A.

Solution.: Let P,Q be permutation matrices such that

$$\text{QAP} = \begin{bmatrix} G & K \\ O & O \end{bmatrix} \qquad \ldots \text{ (1-6)}$$

where G is upper triangular and non singular (the K block, or the zero blocks, are absent if A is of full-rank.) Then $X = P\begin{bmatrix} G^{-1} & O \\ O & O \end{bmatrix}Q \qquad \ldots \text{ (1-7)}$

is a *{*'1, 2*}*-inverse of *A* (, some zero bulks are absent if *A* is of full-rank.)
make a note of that the inverse G^{-1} is obtained from *G* by back substitution [15] .

VIII. EXISTENCE AND CONSTENCE OF {1 , 2 , 3 }- , { 1. 2 , 4 }- , AND { 1 , 2 , 3 , 4 } -INVERSES

Bjerhammar display that the present of a {1}-inverse reveal the presented of a {1, 2}-inverse, Urquhart [9] has shown that the present of a {1}-inverse of each finite array with items in C reveal the presented of a {1, 2, 3}-inverse and a {1, 2, 4}-inverse of every each such as array. However, in order that shown the non emptiness of A{1, 2, 3} and A{1, 2, 4} for any give A, we will uses the {1}-inverse not of A itself but of a related array. It is a weak general reverse (Bjerhammar), or reflective genera reverse if X ∈ A{1, 2, 3} ,and if X ∈ A{1, 2, 3, 4} is called the general mutual (Moore - mutual inverse) [1], [5] .

Theorem (3) : ("Urquhart"): For each finite matrix A has of complex elements,

$A^{(1,4)}A\,.A^{(1,3)} = A$ ¨. (7)

Proof. : Let X_1 denote LHS(7). It follows at once from Lemma 3 that $X_1 \in A^{\{1,2\}}$. Moreover, (7) gives: $A.X_1 = AA^{(1,3)}$, $X_1.A = A^{(1,4)}A$.
But, both $A.\,A^{(1,3)}$) & $A^{(1,4)}.A$ are Hermitian, by the def. of *A*(1,3) and *A*(1,4). Thus $X_1 \in A\{1, 2, 3, 4\}$.
However, by (Ex. 1) , *A{*1, 2, 3, 4*}* contains at most a single element. Therefore, it contains exactly one element, namely A^+ and $X_1 = A^+$ [9] .

IX. THE DRAZIN INVERSE

Drazin Inverse has a weaker kind of " cancelation Law " for elementary and is sometime what harder to work with algebraic than the Moore – penrose inverse [13].
We have show that the class inverse does not be present for every square matrix, although barely for those of index 1. However, we shall show in this paper that every square matrix has a singular $\{1^k, 2, 5\}$-inverse, where k is index. This inverse, called the Drazin inverse of A and is indicate by A^D, was first study through Drazin [7] . There is a singular array satisfactory the equalities :

$AX = XA$ (8) , $A X^2 = X$ (9) , $A^{k+1} X = A^k$ (10) , (k = ind A)

The matrix *X* is thus a $\{1^k, 2, 5\}$-inverse of *A*, property that define it uniquely,. *X* is called the Drazin inverse of A, and is indicate by $X = A^D$. exacting: when Ind(A) =0 or 1 [12] ,[15].

X. THE MATRIX INDEX

It is with pleasure show that the set of three neutralization (1^k), (2), and (5) is equal to the set : (8) , (9) and (10) .
It is evident also that if (10) holds for some positive integer *k*, then it holds for every integer *l*_ > *k*. It follows also from (10) that

rank A^k = rank A^{k+1} ... (11)

Therefore, a solution X for (10) (and, consequently, of the set (8), (9), (10) exists only if (11) holds. In this connection,.the def.(10-1) is useful .
.:

A. *Definition :* *Let A* $\in C^{nxn}$, The smallest nonnegative integer k ,

$R(A^k) = R(A^{k+1})$ is called guide (index) one of A , indicate by Ind(A) .

B. Some Properties of the Drazin inverse

(a) $(A^*)^D = (A^D)^*$.
(b) $(A^T)^D = (A^D)^T$.
(c) $(A^l\_)^D = (A^D)^l\_$ for $\_l = 1, 2, \ldots$.
(d) If A has index k , $A^l\_$ has index one & $(A^l\_)^\# = (A^D)^l\_$ for $\_l \geq k$.
(e) $(A^D)^D = A$ iff A has index 1.(*Drazin*).
,A^T: The statements about the transpose of A and A * conjugate transpose of A[7], [15] .

C. *Example 1:* let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **,** $E = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Then Ind(A) = Ind(A+E) =2

And $A^D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **,** $(A+E)^D = \begin{bmatrix} 1 & -\kappa & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D. *Example 2* : if AB = BA then (i) $A^DB = BA^D$, (ii) $AB^D = B^DA$, ,(iii) $A^DB^D = B^DA^D$

Proof : (i) $A^DB = A^DAA^D = A^DBA^D$
$= A^DABA^D = BA^DAA^D = BA^D$
$\therefore$ $A^DB = BA^D$

(ii) Similarly proved.

(iii) $A^DB^D = A^DAA^DB^D = A^DB^DAA^D = A^DB^D BB^D AA^D =$
$= B^DA^DBAA^DB^D = B^DA^DBB^D$
$= B^D BB^DA^D = B^DA^D$
$\therefore$ $A^DB^D = B^DA^D$

XI. REMARKS

- If $A = T\begin{bmatrix} C & O \\ O & N \end{bmatrix}T^{-1}$ … (1-8)

  C is sxs non singular matrix , N is txt nilpotent matrix, ind(A) = v , dim $R(A^v)$ = s , dim $N(A^v)$ = t , s+t = n

  The Drazin inverse of A is defined by $A^D = T\begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix}T^{-1}$ …(1-9)

- if $|A| \neq 0 \Rightarrow$ N block is missing in A $\Rightarrow A^D = A^{-1}$
- if A is nilpotent , the C block is missing in A $\Rightarrow A^D = 0$

- The Drazin inverse is unique , the following properties are useful if A and ind(A) = v then $AA^D = A^DA$, $A^DAA^D = A^D$, $A^{k+1}A^D = A^k$ for k> ind ≥ (A)
- The Drazin inverse, A^{-D}, satisfies $(XAX^{-1})^D = XA^DX^{-1}$ for any square matrix A and non singular matrix X [21] .
- If ind(A) = 0 or 1 then The Drazin inverse is called the group inverse and **indicate** by $A^{\#}$ this inverse has property $AA^{\#}A = A$.

XII. CONCLUSIONS

- In real and complex case the Moore-Penrose inverse exists and unique but in finite fields it requires additional conditions.
- It was found that the Moore–Penrose inverse of a non- unique array has similar behavior as the ordinary inverse which satisfies the four equations.
- An array has an reverse only if it is square, and even then only if it is non-unique or, if its columns (or rows) are linearly independent .
- Each finite array with items in the compound field has a *{*1*}*-inverse, and proposes that an reverse can be build.
- Bjerhammar shown that the subsistence of a *{*1*}*-inverse implies the' survival of a *{*1, 2*}*-inverse ' Urquhart shown that the survival of a *{*1*}*-inverse of each finite array with items in C implies the survival of a *{*1, 2, 3*}*-inverse and a *{*1, 2, 4*}*-inverse of each such as matrix.
- A ∈ X{1, 2} and X∈A{1, 2} are the same statements, and it could be said that A and X are {1, 2}-inverses of each other.
- if determinant of matrix A is $|A| \neq 0$, the N- block is missing in A $\Rightarrow A^D = A^{-1}$, and if A is nilpotent , the C- block is missing in A then the Drazin inverse of A is equal to zero .

REFERENCES

[1] I.Fredholm , Sur une classe d'équations fonctionnelles. Acta thematica, 27(1), 365–390 , 1903.
[2] W.A.,Hurwitz, On the pseudo-resolvent to the kernel of an integral equation. Transactions of the American Mathematical Society, 13(4) , 1912 .
[3] E.H Moore, On the reciprocal of the general algebraic matrix , Bulletin of the American Mathematical Society, 26,. 52, 1920 .
[4] F.J.,Murray & Von Neumann, J , On rings of operators Annals of Mathematics, Second Series, 37(1), 116–229 , 1936.
[5] R.,Penrose, A generalized inverse for matrices., Mathematical Proceedings of the Cambridge Philosophical Society, 51, 1955.
[6] Bjerhammar, A generalized matrix algebra , Trans. Roy. Inst. , no. 124, 32 pp. , 1958 .
[7] M.P. Drazin , Pseudo inverses in associative rings and semigroups, Amer. Math. Monthly 65 , 1958 .
[8] A.G.Fisher, On Construction &, properties of the Generalized inverses, SIAM J. , Application. Math. 15 , 1967 .
[9] N.Urquhart , Computation of generalized inverse matrices which satisfy specified conditions. SIAM Review, 10, 216–218 , 1968.
[10] S.L.Campbell , On continuity properties of the Moore – penrose and Drazin Generalized inverses , Ibid. 18, 53-57 , 1977 .
[11] J.D..,Fulton , Generalized inverses of matrices over a finite field., Discrete Mathematics, 21(1), 23–29 , 1978.
[12] J.D..keckic , on some generalized inverses of matrices and some linear matrix equations , Nouvelle serie tome 45(59) , 1989 .
[13] Y.Wel & G. wang , The parturbtion theory for the drazin inverse and its applications", shanghat Normal University,China , Elsevier , 1997.
[14] Wu, C. K., & Dawson, E. (1998b). Generalized inverses in public key crypto system key crypto system design , IEE-Proceedings-Computers and Digital Techniques, 145(1), 1998a .
[15] A.,Ben-Israel, .& . T. , Greville,. Generalized inverses: Theory and applications., 2nd edn. , Springer Verlag , New York , CMS Books in Mathematics , Rutgers University , 2003 .
[16] S.L.,Campbell, & Meyer, C. D. , Generalized , inverses of linear transformations. Classics in Applied Mathematics. Philadelphia: Society for Industrial and Applied Mathematics, 2009 .
[17] H.Hudzikand & Yu. Wang **,** Construction of some Generalized Inverses of Operators between Banach Spaces and their Selections, Perturbations and Applications , By Haifeng Ma Presented to the Faculty of Mathematics and Computer Science , Adam Mickiewicz University , 2011.
[18] GÜ.Meltem , (2016) **,** Generalized Inverses of Matrices and Applications to coding Theory, A Thesis Submitted to the Dokuz EYliil University , İZMİR. .
[19] G.R.Wang , Yimin Wei &, S.Z.Qiao, Generalized Inverses: Theory and Computations , Graduate Text book of Shanghae science press , Springer , 2018 . https://www.researchgate.net/publication/240015816_Generalized_Inverses_Theory_and_Computations
[20] Michael Friendly , (2018) , Generalized inverse , http://www.cran.proiect.org/_web/packages/maltb ., 2018
[21] J.Uhlmann , A Generalized Matrix Inverse that is Consistent with Respect to Diagonal Transformations , SIAM J. Matrix Analysis, Vol. 239, No. 2, IJMTT Journal , 2018.