

Stability and Mean Consistent of Fourier Solutions of Nonlinear Stochastic Heat Equations in 1D

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Abstract

The main focus of this article is studying the stability of solutions of nonlinear stochastic heat equation and give conclusions in two cases: stability in probability and almost sure exponential stability. Also, We prove that the Fourier coefficient solution is mean consistent. The main tool is the study of related Lyapunov-type functionals. The analysis is carried out by a natural N -dimensional truncation in isometric Hilbert spaces and uniform estimation of moments with respect to N .

Keywords:

Nonlinear stochastic heat equation, Fourier solution, Stability, Mean Consistent.

1. Introduction

In this article we study the stability of solutions of semi-linear stochastic heat equations

$$u_t = \sigma^2 \Delta u + A(u) + B(u) \frac{dW}{dt}$$

with cubic nonlinearities $A(u)$ in one dimensions in terms of all systems parameters, i.e., with non-global Lipschitz continuous nonlinearities. Our study focusses on stability of analytic solution $u = u(x, t)$ under the geometric condition

$$\sigma^2 \frac{\pi^2}{l^2} - a_1 = \frac{\sigma^2 \pi^2 l^2 - a_1 l^2}{l^2} := \gamma > 0,$$

where $0 \leq x \leq l$ such that $\mathbb{D} = [0, l]$.

Many authors have treated stochastic heat equations (e.g. Chow [2] and DaPrato and Zabczyk [4]), semi-linear stochastic heat equations (e.g. Chow [2], DaPrato and Zabczyk [4], and Schurz [19]) or nonlinear stochastic evolution equations (e.g. Grecksch and Tudor [8] and Schurz [18]). Also, some authors study the stability of stochastic heat equations like Fournier and Printems [5] study the stability of the mild solution. Walsh [22] treats the stochastic heat equations in one dimension. Chow [2] studies that the null solution of the stochastic heat equation is stable in probability by using the definition. Recall that

$$\mathbb{L}^2(\mathbb{D}) = \{f : \mathbb{D} \rightarrow \mathbb{R} \mid \int_{\mathbb{D}} |f(x)|^2 d\mu(x) < \infty\},$$

where μ is the Lebesgue measure in one dimensions.

The paper is organized as follows. Section 2 states that the strong Fourier solution of equation (18) is proved. We write the solution using the finite-dimensional truncated system verifies properties of finite-dimensional Lyapunov functional. Section 3 discusses the stability of the strong solution of equation (18) is stable in probability and almost sure exponential stability. Eventually, Section 4 summarizes the most important conclusions on the well-posedness and behaviour of the original infinite-dimensional system (18).

2. Truncated Fourier Series Solution and Finite-dimensional Lyapunov Functional

Consider the stochastic nonlinear heat equation with additive noise

$$u_t = \sigma^2 u_{xx} + (a_1 - a_2 \|u\|_{\mathbb{L}^2(\mathbb{D})}^2)u + b \frac{dW}{dt} \quad (1)$$

with the initial condition $u(x, 0) = f(x)$ with $f \in \mathbb{L}^2(\mathbb{D})$ (initial position) and

$$W(x, t) = \sum_{n=1}^{\infty} \alpha_n W_n(t) e_n(x)$$

and $e_n = \sqrt{\frac{2}{l}} \sin(\frac{n\pi x}{l})$ driven by *i.i.d.* standard Wiener processes W_n with $\mathbb{E}[W_n(t)] = 0$, $\mathbb{E}[W_n(t)]^2 = t$. The solution of equation (18) in terms of Fourier series is proved by Schurz [19] and given by

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) e_n(x). \quad (2)$$

Theorem 2.1. Assume that $\sum_{n,m=1}^{\infty} \alpha_n^2 < \infty$, $\forall u \in \mathbb{L}^2(\mathbb{D}) \cap C^1(\mathbb{D} \times \mathbb{R}_+^1)$ with $u_x \in \mathbb{L}^2(\mathbb{D})$ and $W(x, t) = \sum_{n,m=1}^{\infty} \alpha_n W_n(t) e_n(x)$, then for all $t \geq 0$, $x \in \mathbb{D} = (0, l_x)$, the Fourier-series solutions (2) have Fourier coefficients c_n satisfying (a.s.)

$$\begin{aligned} \frac{d}{dt} c_n(t) = & \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{\infty} (c_k)^2 \right] c_n \\ & + b \alpha_n \frac{dW_n}{dt}. \end{aligned} \quad (3)$$

Proof. See Schurz [19]. □

We need to truncate the infinite series (2) for practical computations. So, we have to consider finite-dimensional truncations of the form

$$u_N(x, t) = \sum_{n=1}^N c_n e_n \quad (4)$$

with Fourier coefficients c_n satisfying the naturally truncated system of stochastic differential equations (SDEs).

$$\begin{aligned} \frac{d}{dt} c_n(t) = & \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{\infty} (c_k)^2 \right] c_n \\ & + b \alpha_n \frac{dW_n}{dt}. \end{aligned} \quad (5)$$

where $\lambda_n = (\frac{n\pi}{l})^2$.

Assume that $\sigma^2 \frac{\pi^2}{l^2} > a_1$. Define the Lyapunov functional V_N as follows

$$\begin{aligned} V_N(c) = & V_N((c_n)_{n=1, \dots, N}) \\ := & \sum_{n=1}^N (\sigma^2 \lambda_n - a_1) (c_n)^2 + \frac{a_2}{2} \left(\sum_{n=1}^N c_n \right)^2 \end{aligned} \quad (6)$$

for $N \in \mathbb{N}$.

This functional is a modification of a functional appeared in Schurz [20]. It is clear that this function is of Lyapunov-type because it is nonnegative and smooth as long as $a_2 \geq 0$, radially unbounded if additionally $\sigma^2 \pi^2 > a_1 l^2$. Equipped with Euclidean norm

$$|c|_{l_N^2} = \sqrt{\sum_{n=1}^N c_n^2}$$

Lemma 2.2. Consider the Lyapunov functional defined in equation (6), and let

$$\sigma^2 \frac{\pi^2}{l^2} - a_1 = \frac{\sigma^2 \pi^2 - a_1 l^2}{l^2} =: \gamma > 0.$$

Then $\forall u \in \mathbb{L}^2(\mathbb{D})$:

$$V_N(u) \geq \gamma \|u\|_{\mathbb{L}^2(\mathbb{D})}^2 \quad (7)$$

Proof. See [9]. □

Lemma 2.3. Assume that $a_2 \geq 0$. Then, $\forall N \in \mathbb{N}$, the functional V_N is

(a) nonnegative and positive semi-definite if $\sigma^2 \pi^2 \geq a_1 l^2$ or $a_2 \geq 0$,

(b) positive-definite if $\sigma^2 \pi^2 > a_1 l^2$,

and

(c) satisfies the condition of radial unboundedness

$$\lim_{|c|_{l_N^2} \rightarrow +\infty} V_N(c) = +\infty, \text{ if } [\sigma^2 \pi^2 - a_1 l^2]_+ + a_2 > 0.$$

Proof. See [9]. □

3. Stability of Fourier Solutions

Recall equation (5) governed by

$$\frac{d}{dt} c_n(t) = \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{\infty} (c_k)^2 \right] c_n \quad (8)$$

$$\begin{aligned} &+ b\alpha_n \frac{dW_n}{dt} \\ &= [-\sigma^2 \lambda_n + a_1 - a_2 \|u_N\|^2] c_n + b\alpha_n \frac{dW_n}{dt}. \end{aligned} \quad (9)$$

To simplify, let

$$f(u_N) = -\sigma^2 \lambda_n + a_1 - a_2 \|u_N\|^2$$

and

$$g(u_N) = b\alpha_n$$

Definition. The trivial solution of system (8) (in terms of norm $\|u\|_{\mathbb{L}^2(\mathbb{D})}$) is said to be *stochastically stable* or *stable in probability*, if for $0 < \epsilon < 1$ and $r > 0$, \exists a $\delta = \delta(\epsilon, r)$ such that, $\forall t \geq \delta$, we have

$$\mathbb{P}\left\{\|u(t)\|_{\mathbb{L}^2(\mathbb{D})} < r\right\} \geq 1 - \epsilon. \quad (10)$$

whenever $\delta > 0$.

Lemma 3.1. If \exists a positive-definite function $V \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [0, \infty), \mathbb{R}_+)$ such that $\mathcal{L}V(x, t) \leq 0$ and $\forall (x, t) \in \mathbb{R}^d \times [0, \infty)$, then the trivial solution of the equation

$$dX(t) = f(x(t), t) dt + g(x(t), t) dW(t) \quad (11)$$

is stochastically stable.

Proof. See Arnold [1]. □

Theorem 3.2. *Let*

$$V(u(t)) = \sigma^2 \|\nabla u\|_{L^2(\mathbb{D})}^2 - a_1 \|u\|_{L^2(\mathbb{D})}^2 + \frac{a_2}{2} \|u\|_{L^2(\mathbb{D})}^4.$$

If $(1 - a_2 \sum_{n=1}^N c_n^2) \sum_{n=1}^N c_n > 0$, then the trivial solution of equation (8) is stochastically stable i.e., stable in probability.

Proof. From Lemma 2.3, we know that $V_N(u(t))$ is positive-definite if $\forall n \in \mathbb{N}, \sigma^2 \lambda_n - a_1 > 0$. Define the linear operator \mathcal{L} as in Schurz [19]

$$\mathcal{L} = \sum_{n=1}^N \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^N (c_k)^2 \right] c_n \frac{\partial}{\partial c_n} + \frac{b^2}{2} \sum_{n=1}^N \alpha_n^2 \frac{\partial^2}{\partial c_n^2}.$$

The first and second partial derivative of $V_N(t)$ with respect to c_n are

$$\frac{\partial V_N}{\partial c_n} = 2 \sum_{n=1}^N \left[(\sigma^2 \lambda_n - a_1) + a_2 \left(\sum_{n=1}^N c_n^2 \right) \right] c_n$$

and

$$\frac{\partial^2 V_N}{\partial c_n^2} = 2 \sum_{n=1}^N (\sigma^2 \lambda_n - a_1) c_n + 4a_2 \left(\sum_{n=1}^N c_n \right)^2 + 2a_2 \sum_{n=1}^N c_n^2.$$

Then

$$\begin{aligned} \mathcal{L}V_N(c_n(t)) &= -2 \left(\sum_{n=1}^N (\sigma^2 \lambda_n - a_1) c_n \right)^2 \\ &\quad - 2a_2 \sum_{n=1}^N (\sigma^2 \lambda_n - a_1) \left(\sum_{n=1}^N c_n^2 \right)^2 \\ &\quad - 2a_2 \sum_{n=1}^N c_n^2 \left(1 - a_2 \sum_{n=1}^N c_n^2 \right) \sum_{n=1}^N [(\sigma^2 \lambda_n - a_1)] c_n. \end{aligned}$$

But by our assumption that

$$(1 - a_2 \sum_{n=1}^N c_n^2) \sum_{n=1}^N c_n > 0,$$

then Thus

$$\mathcal{L}V_N(c_n(t)) \leq 0.$$

So by Lemma 3.1, applied to truncation of (8), the trivial solution of system (8) is *stochastically stable*. □

Corollary 3.3. *Let $p \geq 2$ and let V be as above. Imposing the same assumptions as in Theorem 3.2 with $N \rightarrow +\infty$, then we have $\forall 0 \leq t \leq T$,*

$$\mathbb{E} \|u(t)\|_{L^2(\mathbb{D})}^p \leq \frac{1}{\min(1, \gamma)} \mathbb{E} V^{\frac{p}{2}}(u(0)).$$

Proof. We know, from the definition of $V(u)$, and Lemma 2.2 that $\|u\|_{L^2(\mathbb{D})}^2 \leq \frac{V(u)}{\gamma}$. it is easy to show that

$$\mathbb{E} \|u(t)\|_{L^2(\mathbb{D})}^p \leq \frac{1}{\min(1, \gamma)} \mathbb{E} V^{\frac{p}{2}}(u(0)).$$

□

Corollary 3.4. $\forall p \geq 2$ and $\forall 0 \leq t \leq T$, with $\sigma^2 \lambda_1 - a_1 > 0$, we have $\forall 0 \leq t \leq T$.
1) If $a_2 \geq 0$, then

$$\mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^p \leq \frac{\mathbb{E} V^{\frac{p}{2}}(u(0))}{\left[\sigma^2 \lambda_1 - a_1\right]^{\frac{p}{2}}}.$$

2) If $a_2 > 0$, then

$$\mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^p \leq \left(\frac{2}{a_2}\right)^{\frac{p}{4}} \mathbb{E} V^{\frac{p}{4}}(u(0)).$$

Proof. 1) Note that we have $(\sigma^2 \lambda_1 - a_1) \|u(., t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq V_N(u(t))$. Since λ_n is increasing in n ,

$$[\sigma^2 \lambda_1 - a_1] \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq V_N(u(t)).$$

So,

$$\|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \frac{V_N(u(t))}{\sigma^2 \lambda_1 - a_1}$$

Pull over expectation, then

$$\mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \frac{\mathbb{E} V(u(t))}{\sigma^2 \lambda_1 - a_1}.$$

By using Corollary 3.3, we have

$$\mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \frac{\mathbb{E} V(u(0))}{\sigma^2 \lambda_1 - a_1}.$$

2) From the definition of $V(u(t))$, it is clear that $\frac{a_2}{2} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^4 \leq V(u(t))$, so

$$\left(\frac{a_2}{2}\right) \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^4 \leq V_N(u(0)).$$

Now, take the expectation to both sides, and we get

$$\left(\frac{a_2}{2}\right) \mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^4 \leq \mathbb{E} V_N(u(0)),$$

i.e., $\forall 0 \leq t \leq T$,

$$\mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \left(\frac{2}{a_2}\right)^{\frac{1}{2}} \mathbb{E} V^{\frac{1}{2}}(u(0)).$$

□

Remark: The corollary 3.4 means that $\forall t \geq 0$:

$$\mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \min \left\{ \frac{\mathbb{E} V_N(u(0))}{\left[\sigma^2 \lambda_1 - a_1\right]}, \left(\frac{2}{a_2}\right)^{\frac{1}{2}} \mathbb{E} V^{\frac{1}{2}}(u(0)) \right\}.$$

Definition. The trivial solution of system (8) is said to be *a.s. exponentially stable* if

$$\theta(u_N) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|u(t)\|_{\mathbb{L}^2(\mathbb{D})} < 0 \quad (a.s.) \quad (12)$$

$\forall u(0) \in \mathbb{D}$. The quantity of the left hand side of (12) is called *the sample top Lyapunov exponent of u*.

Lemma 3.5. [Øksendal] [16]. Let $v(t)$ be a nonnegative integrable function such that

$$v(t) \leq C + A \int_0^t v(s) ds, \quad 0 \leq t \leq T \quad (13)$$

for some constants C, A . Then $C \geq 0$ and

$$v(t) \leq C \exp(At), \quad 0 \leq t \leq T. \quad (14)$$

Theorem 3.6. Let $V(u(t))$ as in Theorem 3.2. If $(1 - a_2 \sum_{n=1}^N c_n^2) \sum_{n=1}^N c_n > 0$, then the norm of the trivial solution of N -dimensional system (8) is a.s. exponentially stable with sample top Lyapunov exponent

$$\theta(u_N) \leq 0.$$

Proof. Return to the analysis of finite N -dimensional equation (5). Recall that

$$\begin{aligned} V(u_N(t)) &= \sigma^2 \|\nabla u_N\|_{\mathbb{L}^2(\mathbb{D})}^2 - a_1 \|u_N\|_{\mathbb{L}^2(\mathbb{D})}^2 + \frac{a_2}{2} \|u_N\|_{\mathbb{L}^2(\mathbb{D})}^4 \\ &= \sum_{n=1}^N [\sigma^2 \lambda_n - a_1] (c_n(t))^2 + \frac{a_2}{2} \left(\sum_{n=1}^N (c_n(t))^2 \right)^2 \end{aligned}$$

where $V(u_N) = V_N(c)$ and from Theorem 3.2 we know that

$$\begin{aligned} \mathcal{L}V_N(c_n(t)) &= -2 \left(\sum_{n=1}^N (\sigma^2 \lambda_n - a_1) c_n \right)^2 \\ &\quad - 2a_2 \sum_{n=1}^N (\sigma^2 \lambda_n - a_1) \left(\sum_{n=1}^N c_n^2 \right)^2 \\ &\quad - 2a_2 \sum_{n=1}^N c_n^2 \left(1 - a_2 \sum_{n=1}^N c_n^2 \right) \sum_{n=1}^N [(\sigma^2 \lambda_n - a_1)] c_n. \end{aligned}$$

But by our assumption that

$$(1 - a_2 \sum_{n=1}^N c_n^2) \sum_{n=1}^N c_n > 0,$$

so

$$\mathcal{L}V_N(c_n(t)) \leq -k,$$

where $k \geq 0$.

Using Dynkin's formula, we find that

$$\begin{aligned} \mathbb{E}V_N(c(t)) &= \mathbb{E}V_N(c(0)) + \mathbb{E} \int_0^t \mathcal{L}V_N(c_n(s)) ds \\ &\leq \mathbb{E}V_N(c, v)(0) - k t \end{aligned}$$

so

$$\mathbb{E}\|u_N(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \mathbb{E}V_N(c(t)) \leq \mathbb{E}V_N(c(0)) - k t$$

using extended Gronwall lemma, Lemma 3.5, gives us

$$\mathbb{E}\|u_N(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \mathbb{E}V_N(c(0)) e^{-k t}$$

hence

$$\log \mathbb{E}\|u_N(t)\|_{\mathbb{L}^2(\mathbb{D})}^2 \leq \log \mathbb{E}V_N(c(0)) - k t$$

thus

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log \mathbb{E}\|u_N(t)\|_{\mathbb{L}^2(\mathbb{D})}^2}{t} &\leq \limsup_{t \rightarrow \infty} \frac{\log \mathbb{E}V_N(c(0))}{t} - k \\ &\leq b_2^2 \|\alpha\|_{l_{4 \times N}^2}^2 - 2\kappa \leq -k. \end{aligned} \quad (15)$$

If $(1 - a_2 \sum_{n=1}^N c_n^2) \sum_{n=1}^N c_n > 0$ then the left side of identity (12) is negative and the trivial solution of the velocity v of N -dimensional system (8) is a.s. exponential stable.

Finally, we observe that all the previous estimates are uniformly bounded as $N \rightarrow \infty$. Hence, we arrive at

$$\limsup_{t \rightarrow \infty} \frac{\log \mathbb{E} \|u(t)\|_{\mathbb{L}^2(\mathbb{D})}^2}{t} < 0. \quad (16)$$

□

Corollary 3.7. Let $V(u(t))$ as in Theorem 3.2. If $\left(1 - a_2 \sum_{n=1}^N c_n^2\right)$, then the norm of the v -component of the trivial solution of infinite-dimensional system (18) is a.s. exponentially stable with sample top Lyapunov exponent

$$\theta(v) \leq -k < 0.$$

Proof. Return to the proof of previous theorem, theorem 3.6 and take the limit N to $+\infty$ after the estimation process (16) in the sample Lyapunov exponent $\theta(v_N)$. □

4. Explicit Representation

Define the linear-implicit Euler method as the following:

$$c_n(t_{k+1}) = c_n(t_k) + h_k f_n(c(t_k)) c_n(t_{k+1}) + b\alpha_n \Delta_k W_n \quad (17)$$

where $f_n(c(t_k)) = -\sigma^2 \frac{n^2 \pi^2}{l^2} + a_1 - a_2 \sum_{n=1}^N c_n^2$.

Theorem 4.1. Assume that $|c_{n,m}^{i,j}(t_k)| < \infty$, and

$$a_2 \geq 0, \text{ and } \forall n \in \mathbb{N} : \left[-\sigma^2 (\lambda_n + \beta_m) + a_1 \right] h_k^2 < 1.$$

Then the method (LIEM) governed by equation (17) has the non-exploding explicit representation

$$c_n(t_{k+1}) = \frac{c_n(t_k) + b\alpha_n \Delta_k W_n}{1 - h_k f_n(c(t_k))} \quad (18)$$

and where $f_n(c(t_k))$ as above, $\Delta_k W_{n,m}^{i,j} = W_n(t_{k+1}) - W_n(t_k)$ and $h_k = t_{k+1} - t_k$.

Proof. See Schurz [19]. □

5. Local Mean Consistency On Numerical Solutions

To do numerical solution of the Fourier coefficient of the heat equation, it is better to start with mean consistent. In [11], we proved that the solution of the nonlinear stochastic wave equation is mean consistent. Also, in [12], we proved that solution is mean square consistent.

Definition. A numerical approximation \hat{c}_n of the n -th Fourier coefficient c_n along the partition of $[0, T]$ is said be **mean consistent** with rate $r_0 > 0$ on \mathbb{D} if $\exists K_0^c =$ consistency constant $\forall 0 \leq t \leq t+h \leq T$, where h is sufficiently small, i.e., $0 < h \leq \delta \leq 1$, and \exists a positive function $V(c(t))$ such that

$$\|\mathbb{E}(c(t+h|t, \mathcal{F}_t) - \hat{c}(t+h|t, \mathcal{F}_t))\|_H \leq K_0^c (V(c(t))) h^{r_0}$$

provided $\hat{c} = (c_n)_{n=1, \dots, N} \forall u(t) \in H$, where $H := \{u \in \mathbb{L}^2(\mathbb{D})\}$, and $\|u\|_H = \|c\|_{\mathbb{L}^2(\mathbb{D})}^2 : (\mathcal{F}_t, \mathcal{B}(\mathbb{L}^2))$ -measurable, then

$$\|\mathbb{E}(c(t+h|t, \mathcal{F}_t) - \hat{c}(t+h|t, \mathcal{F}_t))\|_H \leq K_0^c (V(c(t))) h^{r_0}$$

where $\hat{c}(t+h|t, c(t)) = \hat{c}(t) + \int_t^{t+h} f(c(s)) ds + \int_t^{t+h} b\alpha_n dW(s) ds$.

Recall that, in the Fourier space $\|u\|_{\mathbb{L}^2(\mathbb{D})}^2 = \|c\|_{l^2}^2$. Now,

$$\begin{aligned} c_n(t+h) &= \frac{c_n(t_k) + b \alpha_n \Delta_k W_n}{1 - h_k f_n(c(t_k))} \\ &= \frac{c_n(t_k) - h_k f_n(c(t_k))c_n(t_k) + h_k f_n(c(t_k))c_n(t_k)}{1 - h_k f_n(c(t_k))} \\ &\quad + \frac{b \alpha_n \Delta_k W_n}{1 - h_k f_n(c(t_k))} \\ &= \frac{[1 - h_k f_n(c(t_k))] c_n(t_k)}{1 - h_k f_n(c(t_k))} + h_k \frac{f_n(c(t_k))}{1 - h_k f_n(c(t_k))} \\ &\quad + \frac{b \alpha_n}{1 - h_k f_n(c(t_k))} \Delta_k W_n \end{aligned}$$

write the above equation

$$\hat{c}_n(t_k + h_k) = c_n(t_k) + h_k \hat{f}_n(c(t_k))c_n(t_k) + \hat{g}_n(c(t_k))\Delta_k W_n.$$

Since

$$f - \hat{f} = -h \frac{f^2}{1 - hf}$$

and

$$g - \hat{g} = -h \frac{b\alpha_n f}{1 - hf}$$

So,

$$\begin{aligned} c(t_k + h_k) - \hat{c}_n(t_k + h_k) &= c_n(t_k) - \hat{c}_n(t_k) \\ &\quad + h_k [f_n(c(t_k)) - \hat{f}_n(c(t_k))] c_n(t_k) \\ &\quad + [g_n(c(t_k)) - \hat{g}_n(c(t_k))] \Delta_k W_n(t_k) \end{aligned}$$

but $c_n(t_k) = \hat{c}_n(t_k)$ and $\Delta_k W_n(t_k) = \int_{t_k}^{t_{k+1}} dW(s)$ and

$$\int_{t_k}^{t_{k+1}} [g_n(c(t_k)) - \hat{g}_n(c(t_k))] dW(s) = 0, \quad (\text{martingale}),$$

pulling the expectation, we get

$$\begin{aligned} &\|\mathbb{E}[c_n(t_k + h_k) - \hat{c}_n(t_k + h_k)]\|_N \\ &= h_k \|\mathbb{E}([f_n(c(t_k)) - \hat{f}_n(c(t_k))]c_n(t_k))\|_N \\ &= h_k^2 \left\| \mathbb{E} \left[\frac{f_n^2(c(t_k))}{1 - h_k f_n(c(t_k))} c_n(t_k) \right] \right\|_N \\ &= h_k^2 \left\| \mathbb{E} \left[\frac{(-\sigma^2 \lambda_n + a_1)^2}{1 - h_k f_n(c(t_k))} \right] \right\|_N \\ &\quad - 2 a_2 h_k^2 \left\| \mathbb{E} \left[\frac{\sum_{n=1}^N c_n^2(t_k)}{1 - h_k f_n(c(t_k))} \right] \right\|_N \\ &\quad + h_k^2 a_2^2 \left\| \mathbb{E} \left[\frac{\sum_{n=1}^N c_n^2(t_k)^2}{1 - h_k f_n(c(t_k))} \right] \right\|_N \\ &\leq K_0^c (V_N(c(t_k))) h_k^2. \end{aligned} \tag{19}$$

where

$$V_N = (-\sigma^2 \lambda_n + a_1)^2 + 2a_2(\sigma^2 \lambda_n - a_1)\|u\|_{\mathbb{L}^2(\mathbb{D})}^2 + a_2^2 \|u\|_{\mathbb{L}^2(\mathbb{D})}^4. \quad (20)$$

$\lambda_n = \frac{n^2 \phi^2}{l}$, and $\mathbb{D} = [0, l]$. Thus the approximation solution \hat{c} is mean consistent with rate $r_0 = 2$.

6. Conclusion

By analyzing appropriate N -dimensional truncations of the original semi-linear heat equations (18), we can verify the asymptotic stability of random Fourier series solutions with strongly unique, Markovian, continuous time Fourier coefficients under the presence of cubic nonlinearities. For this purpose, we introduced and studied an appropriate Lyapunov. The analysis is basically relying on the fact that all estimations of moments of Lyapunov functional are made independent of dimensions N of their finite-dimensional truncations. Thus, the techniques of our proof are finite-dimensional in character, however the conclusions can be drawn to the original infinite-dimensional semi-linear equation. Also, in this article, we showed that the approximation solution is mean consistent with rate $r_0 = 2$.

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