On The Normal Response And Buckling Load Of A Toroidal Shell Segment Pressurized By A Static Compressive Load

[1]A. M. Ette, [1*]J. U. Chukwuchekwa, [2]I. U. Udo-Akpan, [3]G. E. Ozoigbo

[1.] *Department of mathematics, federal university of technology, owerri. Imo state, nigeria.*
[2.] *Department of mathematics and statistics, university of port harcourt, port harcourt, rivers state, nigeria*
[3.] *Department of mathematics /computer science / statistics & informatics, federal university ndufu - alike, ikwo. Ebonyi state, nigeria.*

Abstract

This investigation is concerned with analytical determination of the out-of-plane normal response, the associated Airy stress function and the static buckling load of an imperfect, finite but simply-supported toroidal shell segment that is statically pressurized. Regular perturbation procedures and asymptotic expansions are freely utilized. In the final analysis, a simple implicit equation for obtaining the static buckling load is obtained and the result is asymptotic in nature.

Keywords: *Toroidal and Cylindrical shells, Static buckling, Airy Stress Function, Asymptotic and perturbation techniques.*

I. INTRODUCTION

The toroidal shell segment is an imperfection-sensitive elastic structure that has been studied for sometime now. Though its structural configuration resembles that of a cylindrical shell, it however differs from cylindrical shell segment in the possession of an inner radius namely a, and an outer radius b. Earlier studies on the structure were done by Stein and McElman [1] who investigated the buckling of segments of toroidal shell while Hutchinson [2] similarly investigated the initial post buckling of toroidalshell segments. Relatively recent but insightful investigations on the subject matter were done by Oyesanya [3, 4], who used asymptotics and perturbation techniques to analytically study the various restrictions of the structure. Related studies, though not strictly on toroidal shell segments, were initiated by Kriegesman et al. [5], while Hu and Burgueño [6] investigated elastic post buckling response of axially loaded cylindrical design. In the same token, Kubiak [7, 8] made substantial contributions to the subject matter through his investigations on thin-walled structures, while Kolakowski [9, 10] treated similar subjects in his investigations on thin-walledcomposite structures.

Though the formulation here is situated purely on static settings, the technique adopted in this work is similar to an earlier study by Lockhart and Amazigo [11], who investigated the dynamic buckling of externally pressurized imperfect cylindrical shells. As in Hilburger and Starnes [12], our investigation intends to determine the out-of-plane normal displacement of the finite but imperfect toroidal shell segment. We shall, in the static setting, determine the static buckling load of the structure, assuming that the structure is pressurized by a compressive static load that is either axially or hydrostatically applied. As in [11, 13], our attention shall be focused on a simply-supported toroidal shell segment where we shall employ expansions in double Fourier series.

II. FORMULATION OF THE PROBLEM

From [3, 4], the normal out-of-plane displacement W(X, Y) and Airy stress function F(X, Y) of a finite imperfect toroidal shell segment of length L, satisfies the following equilibrium equation and compatibility equation

$$D\nabla^4 W + \frac{1}{a}F_{,XX} + \frac{1}{b}F_{,YY} + p\left[\frac{1}{2}(W+\bar{W})_{,XX} + \left(1-\frac{1}{2}\frac{a}{b}\right)(W+\bar{W})_{,YY}\right] = S(W+\bar{W},F) \quad (1)$$

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{a}W_{,XX} - \frac{1}{b}F_{,YY} = -\frac{1}{2}S(W+\bar{W},W) \quad (2)$$

$$0 < X < L, \quad 0 < Y < 2\pi \quad (3)$$

$$W = W_{,XX} = F = F_{,XX} = 0 \; at \; X = 0,\pi \quad (4)$$

Here, X and Y are the axial and circumferential coordinates respectively, E and h are the Young's modulus and thickness respectively, while p is the area density and the bending stiffness D is given by $D = \frac{Eh^3}{12(1-\vartheta^2)}$, where ϑ is the Poisson's ratio. $\bar{W}$ is the stress-free time-independent continuously differentiable function of X and Y while the symmetric bilinear functional S is such that

$$S(P,Q) = P_{,XX}Q_{,YY} + P_{,YY}Q_{,YY} - 2P_{,XY}Q_{,XY} \quad (5)$$

Similarly, the symbol ∇^4 is the two-dimensional biharmonic operator defined by

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2 \quad (6)$$

III. NONDIMENSIONALIZATION OF THE GOVERNING EQUATIONS

We now introduce the following quantities

$$x = \frac{\pi X}{L}, \quad y = \frac{Y}{a}, \quad \epsilon \bar{w} = \frac{\bar{W}}{h}, \quad w = \frac{W}{h} \quad (7)$$

$$\lambda = \frac{L^2 ap}{\pi^2 D}, \quad A = \frac{L^2\sqrt{12(1-\nu^2)}}{\pi^2 ah}, \quad H = \frac{h}{a} \quad (8)$$

Here, the symbol λ is the nondimensional load amplitude and

$$\xi = \frac{L^2}{(\pi a)^2}, \quad K(\xi) = -\left(\frac{A}{1+\xi}\right)^2, \quad 0 < \epsilon \ll 1 \quad (9)$$

Here, ϵ is a small parameter representing the amplitude of the imperfection and we shall neglect the boundary layer effect by assuming that the pre-buckling deflection is constant. In this way, we assume

$$F = -pa\left(X^2 + \frac{1}{2}\alpha Y^2\right) + \frac{Eh^2L^2}{\pi^2 a(1+\xi)^2}f \quad (10)$$

$$W = \frac{pa^2(1-\alpha\nu)}{Eh} + hw \quad (11)$$

The parameter α takes the value $\alpha = 1$, if the pressure contributes to axial stress through end plates, but similarly takes the value $\alpha = 0$, if pressure acts laterally.

On substituting (10) and (11) into (1) – (4), using (7) – (9), the following equations are easily derived

$$\overline{\nabla}^4 w - K(\xi)\left(f_{,xx} + \xi r f_{,yy}\right) + \lambda\left[\frac{\alpha}{2}(w+\epsilon\overline{w})_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)(w+\epsilon\overline{w})_{,yy}\right] = -K(\xi) H s(f, w+\epsilon\overline{w}) \quad (12)$$

$$\overline{\nabla}^4 f - (1+\xi)^2\left(w_{,xx} + \xi r w_{,yy}\right) = -\frac{1}{2} H(1+\xi) s(w+\epsilon\overline{w}, w) \quad (13)$$

$$0 < x < \pi, \quad 0 < y < 2\pi \quad (14)$$

$$w = w_{,xx} = f = f_{,xx} = 0 \; at \; x = 0, \pi. \quad (15)$$

$$where, r = \frac{a}{b}.$$

A subscript following a comma indicates partial differentiation, while

$$\overline{\nabla}^4 = \left(\frac{\partial^2}{\partial x^2} + \xi \frac{\partial^2}{\partial y^2}\right)^2, \quad s(p,q) = p_{,xx} q_{,yy} + p_{,yy} q_{,xx} - 2 p_{,xy} q_{,xy} \quad (16)$$

IV. CLASSICAL BUCKLING LOAD, λ_c

This is the load that is required to buckle the perfect linear structure and is obtained by neglecting all nonlinearities and imperfections in the governing equations. The relevant equations at this stage are

$$\overline{\nabla}^4 w - K(\xi)\left(f_{,xx} + \xi r f_{,yy}\right) + \lambda\left[\frac{\alpha}{2} w_{,xx} + \xi\left(1-\frac{\alpha}{2}\right) w_{,yy}\right] = 0 \quad (17)$$

$$\overline{\nabla}^4 f - (1+\xi)^2\left(w_{,xx} + \xi r w_{,yy}\right) = 0 \quad (18)$$

$$w = w_{,xx} = f = f_{,xx} = 0 \; at \; x = 0, \pi$$

For the solution of (17) and (18), we have

$$(w, f) = (a_{mk}, b_{mk}) sinmxsin(ky + \emptyset_{mk}) \quad (19)$$

where, $\emptyset_{mk}$ is an inconsequential phase and $(a_{mk}, b_{mk}) \neq (0,0)$. Substituting (19) in (18) yields

$$b_{mk} = \frac{-(1+\xi)^2 m^2 a_{mk}}{(m^2+\xi k^2)^2 + (1+\xi)^2 \xi r^2 k^2} \quad (20)$$

Now substituting (20) into (17) and simplifying, yields

$$(m^2+\xi k^2)^2 - \lambda\left\{\frac{\alpha m^2}{2} + \xi k^2\left(1-\frac{\alpha}{2}\right)\right\} - \frac{K(\xi)(m^2+\xi k^2 r)(1+\xi)^2}{(m^2+\xi k^2)^2 + (1+\xi)^2 \xi r k^2} = 0 \quad (21)$$

Batdorf, as cited in [11], had assumed that k varies continuously and so assumed the condition for classical buckling load as

$$\frac{d\lambda}{dk} = 0 \quad (22)$$

Thus, If $k = n$ is the value of k at the maximization (22), the classical buckling load λ_c is

$$\lambda_c = \frac{(m^2+\xi n^2)^2 - \frac{K(\xi)(m^2+\xi n^2 r)(1+\xi)^2}{(m^2+\xi n^2)^2+(1+\xi)^2\xi r n^2}}{\frac{\alpha m^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi r n^2} \quad (23)$$

On substituting for $K(\xi)$ from (9) and letting $\zeta = \xi n^2$, we get for $m = 1$,

$$\lambda_c = (1+\zeta)^2 + \frac{\frac{A^2(1+\zeta r)}{(1+\zeta)^2+(1+\xi)^2\zeta r}}{\frac{\alpha}{2} + \left(1-\frac{\alpha}{2}\right)\zeta \mathrm{r}} \quad (24)$$

Thus, the displacement and corresponding Airy stress function are

$$(w,f) = \left(1, \quad \frac{-(1+\xi)^2}{(1+\zeta)^2+(1+\xi)^2\zeta r}\right) a_{1n} sinxsin(ny+\emptyset_{1n}) \quad (25)$$

V. STATIC DEFORMATION

We now let

$$\binom{w}{f} = \sum_{i=1}^{\infty} \binom{w^{(i)}}{f^{(i)}} \epsilon^i \quad (26)$$

Substituting (26) into (12) and (13) and simplifying yields

$$O(\epsilon)\begin{cases} \overline{\nabla}^4 w^{(1)} - K(\xi)\left(f^{(1)}_{,xx} + \xi r f^{(1)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}\left(w^{(1)}+\epsilon\overline{w}\right)_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)\left(w^{(1)}+\epsilon\overline{w}\right)_{,yy}\right] = 0 & (27) \\ \overline{\nabla}^4 f^{(1)} - (1+\xi)^2\left(w^{(1)}_{,xx} + \xi r w^{(1)}_{,yy}\right) = 0 & (28) \end{cases}$$

$$O(\epsilon^2)\begin{cases} \overline{\nabla}^4 w^{(2)} - K(\xi)\left(f^{(2)}_{,xx} + \xi r f^{(2)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}\left(w^{(1)}+\overline{w}\right)_{,xx} + \xi\left(1-\frac{\alpha}{2}\right) w^{(1)}_{,yy}\right] \\ = -K(\xi)H\left[s\left(f^{(1)},w^{(1)}\right) + s\left(f^{(1)},\overline{w}\right)\right] & (29) \\ \overline{\nabla}^4 f^{(1)} - (1+\xi)^2\left(w^{(2)}_{,xx} + \xi r w^{(2)}_{,yy}\right) = -\frac{1}{2}H(1+\xi)\left[s\left(w^{(1)},w^{(1)}\right) + s\left(w^{(1)},\overline{w}\right)\right] & (30) \end{cases}$$

$$O(\epsilon^3)\begin{cases} \overline{\nabla}^4 w^{(3)} - K(\xi)\left(f^{(3)}_{,xx} + \xi r f^{(3)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2} w^{(3)}_{,xx} + \xi\left(1-\frac{\alpha}{2}\right) w^{(3)}_{,yy}\right] \\ = -K(\xi)H\left[s\left(f^{(1)},w^{(2)}\right) + s\left(f^{(2)},w^{(1)}\right) + s\left(f^{(2)},\overline{w}\right)\right] & (31) \\ \overline{\nabla}^4 f^{(3)} - (1+\xi)^2\left(w^{(3)}_{,xx} + \xi r w^{(3)}_{,yy}\right) = -\frac{1}{2}H(1+\xi)\left[s\left(w^{(1)},w^{(2)}\right) + s\left(w^{(2)},w^{(1)}\right) + s\left(w^{(2)},\overline{w}\right)\right] & (32) \end{cases}$$

$$w^{(i)} = w^{(i)}_{,xx} = f^{(i)} = f^{(i)}_{,xx} = 0, at\ x = 0, \pi \quad (33)$$

As noted by Lockhart and Amazigo [11], any time independent stress-free normal displacement $\overline{w}(x,y)$, satisfying reasonable smoothness conditions, can be expanded in a double Fourier series. Thus, if the edge effects are neglected and the origin of the circumferential coordinate is appropriately chosen, such a series takes the form

$$\bar{w}(x,y) = \bar{a} sinxsiny + \sum_{\substack{m=1,k=0 \\ (m,k)\neq(1,n)}}^{\infty} \left(\bar{a}_{mk}\, sinky + \bar{b}_{mk} cosky\right) sinmx \qquad (34a)$$

or

$$\bar{w}(x,y) = \sum_{m=1,k=0}^{\infty} \left(\bar{a}_{mk}\, sinky + \bar{b}_{mk}\, cosky\right) sinmx \qquad (34b)$$

$$with\ \bar{b}_{1n} = 0 \qquad (34c)$$

However, in this work, we shall take

$$\bar{w}(x,y) = \bar{a} sinmxsinny \qquad (35)$$

Throughout the analysis, we shall use the fact that if, for example,

$$\bar{w}(x,y) = \sum_{p=1,q=0}^{\infty} (l_1 cosqy + l_2 sinqy) sinpx \qquad (36a)$$

then

$$\bar{\nabla}^4 w = \sum_{p=1,q=0}^{\infty} (p^2 + \xi q^2)^2 (l_1 cosqy + l_2 sinqy) sinpx \qquad (36b)$$

Any integration with respect to x shall have 0 and π as the lower and upper limits respectively while integration with respect to y shall have 0 and 2π as the lower and upper limits respectively.

Solution of Equations of Order ϵ

Let

$$\begin{pmatrix} w^{(1)} \\ f^{(1)} \end{pmatrix} = \sum_{p=1,q=0}^{\infty} \left[\begin{pmatrix} w_1^{(1)} \\ f_1^{(1)} \end{pmatrix} sinpxcosqy + \begin{pmatrix} w_2^{(1)} \\ f_2^{(1)} \end{pmatrix} sinpxsinqy \right] \qquad (37)$$

We now substitute (37) into (28) and get, using (36b)

$$\sum_{p=1,q=0}^{\infty} \left[\left\{ (p^2 + \xi q^2)^2 f_1^{(1)} + (1+\xi)^2 (q^2 r\xi - p^2) w_1^{(1)} \right\} sinpxcosqy + \left\{ (p^2 + \xi q^2)^2 f_2^{(1)} + (1+\xi)^2 (q^2 r\xi - p^2) w_2^{(1)} \right\} sinpxsinqy \right] = 0 \qquad (38)$$

Multiplying (38) first by $sinmxcosny$ and next by $sinmxsinny$, and for $p = m, q = n$ in each case, we simplify to get

$$f_1^{(1)} = \frac{-(1+\xi)^2 (n^2 r\xi - m^2) w_1^{(1)}}{(m^2 + \xi n^2)^2}, \qquad f_2^{(1)} = \frac{-(1+\xi)^2 (n^2 r\xi - m^2) w_2^{(1)}}{(m^2 + \xi n^2)^2} \qquad (39)$$

Next, assuming (37) in (27) and multiplying through, first by $sinmxcosny$ and after by $sinmxsinny$, we get, using (39) and for $p = m, q = n$,

$$w_1^{(1)} = 0, \quad w_2^{(1)} = \frac{\lambda\bar{a}\left\{\frac{\alpha m^2}{2} + \xi n^2\left(1 - \frac{\alpha}{2}\right)\right\}}{\varphi_1} \tag{40}$$

where,

$$\varphi_1 = \left[(m^2 + \xi n^2)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2 r\xi\right\}(1+\xi)^2\left\{\frac{n^2 r\xi - m^2}{(m^2 + \xi n^2)^2}\right\} - \lambda\left\{\frac{\alpha m^2}{2} + \xi n^2\left(1 - \frac{\alpha}{2}\right)\right\}\right] \tag{41}$$

So far, it follows that

$$\begin{pmatrix} w^{(1)} \\ f^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -\varphi_0 \end{pmatrix} w_2^{(1)} sinmxsinny \tag{42a}$$

where,

$$\varphi_0 = (1+\xi)^2\left\{\frac{n^2 r\xi - m^2}{(m^2 + \xi n^2)^2}\right\} \text{(42b)}$$

Solution of Equations of Order ϵ^2

We shall now let

$$\begin{pmatrix} w^{(2)} \\ f^{(2)} \end{pmatrix} = \sum_{p=1,q=0}^{\infty}\left[\begin{pmatrix} w_1^{(2)} \\ f_1^{(2)} \end{pmatrix} sinpxcosqy + \begin{pmatrix} w_2^{(2)} \\ f_2^{(2)} \end{pmatrix} sinpxsinqy\right] \tag{43}$$

On substituting on the right hand sides of equations (29) and (30), we get

$$\bar{\nabla}^4 w^{(2)} - K(\xi)\left(f_{,xx}^{(2)} + \xi r f_{,yy}^{(2)}\right) + \lambda\left[\frac{\alpha}{2}w_{,xx}^{(2)} + \xi\left(1 - \frac{\alpha}{2}\right)w_{,yy}^{(2)}\right]$$
$$= -K(\xi)H\varphi_0 mn^2\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right)(cos2mx + cos2ny) \tag{44}$$

$$\bar{\nabla}^4 f^{(2)} - (1+\xi)^2\left(w_{,xx}^{(2)} + \xi r w_{,yy}^{(2)}\right) = \frac{1}{2}H(1+\xi)mn^2\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right)(cos2mx + cos2ny) \tag{45}$$

Now substituting (43) into (45), using (36b) and simplifying, we get

$$\sum_{p=1,q=0}^{\infty}\left[\left\{(p^2 + \xi q^2)^2 f_1^{(2)} + q^2 r\xi^2 - (1+\xi)^2 p^2 w_1^{(2)}\right\} sinpxcosqy\right.$$
$$\left. + \left\{(p^2 + \xi q^2)^2 f_2^{(2)} + q^2 r\xi^2(1+\xi)^2 p^2 w_2^{(2)}\right\} sinpxsinqy\right]$$
$$= \frac{1}{2}H(1+\xi)mn^2\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right)(cos2mx + cos2ny) \tag{46}$$

We multiply (46) by $sinmxcos2ny$ and for $p = m, q = 2n$, integrate to get (for m odd)

$$f_1^{(2)} = \frac{\frac{-2H(1+\xi)mn^2}{\pi}\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right) - (1+\xi)^2(4n^2 r\xi - m^2)w_1^{(2)}}{(m^2 + 4n^2\xi)^2} \tag{47a}$$

Let

$$\varphi_2 = \frac{2H(1+\xi)mn^2}{\pi(m^2+4n^2\xi)^2}, \quad \varphi_7 = \frac{(1+\xi)^2(4n^2r\xi - m^2)}{\pi(m^2+4n^2\xi)^2} \qquad (47b)$$

$$\therefore \quad f_1^{(2)} = -\left[\varphi_2\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right) + \varphi_7 w_1^{(2)}\right] \qquad (47c)$$

Next, we multiply (46) by $sinmxsin2ny$ and simplify to get

$$f_2^{(2)} = -\varphi_7 w_2^{(2)} \qquad (48)$$

Using (36b), we next multiply (44) by $sinmxcos2ny$ and integrate and for $p = m, q = 2n$ (m odd) we get, after simplification

$$w_1^{(2)} = \frac{-\left[\left(\frac{2A}{(1+\xi)}\right)^2\frac{H\varphi_0 mn^2}{\pi} + \varphi_2\left\{\left(\frac{Am}{(1+\xi)}\right)^2 + 4n^2\xi\right\}\right]\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right)}{(m^2+4n^2\xi)^2 + \varphi_7\left\{\left(\frac{Am}{(1+\xi)}\right)^2 + 4n^2\xi\right\} - \lambda\left\{\frac{\alpha m^2}{2} + 4\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}} \qquad (49a)$$

We now assume the following:

$$\varphi_9 = \left[\left(\frac{2A}{(1+\xi)}\right)^2\frac{H\varphi_0 mn^2}{\pi} - \varphi_2\left\{\left(\frac{Am}{(1+\xi)}\right)^2 + 4n^2\xi\right\}\right] \qquad (49b)$$

$$\varphi_{10} = \left[(m^2+4n^2\xi)^2 + \varphi_7\left\{\left(\frac{Am}{(1+\xi)}\right)^2 + 4n^2\xi\right\} - \lambda\left\{\frac{\alpha m^2}{2} + 4\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\right] \qquad (49c)$$

$$\therefore \quad w_1^{(2)} = \frac{\varphi_9}{\varphi_{10}}\left({w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right) \qquad (49d)$$

Similarly, multiplying (44) by $sinmxsin2ny$and integrating as usual, yields

$$w_2^{(2)} = 0 \qquad (50)$$

It follows at this stage that

$$\begin{pmatrix} w^{(2)} \\ f^{(2)} \end{pmatrix} = \begin{pmatrix} w_1^{(2)} \\ f_1^{(2)} \end{pmatrix} sinmxcos2ny \qquad (51)$$

Solution of Equations of Order ϵ^3

We next substitute on the right hand sides of (31) and (32) and simplify to get

$$\overline{\nabla}^4 w^{(3)} - K(\xi)\left(f_{,xx}^{(3)} + \xi r f_{,yy}^{(3)}\right) + \lambda\left[\frac{\alpha}{2}\left(w^{(3)} + \bar{w}\right)_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)w_{,yy}^{(3)}\right]$$
$$= -m^2n^2K(\xi)H[9sin3ny - sinny - cos2mxsin3ny + 9cos2mxsinny]\left(2w_2^{(1)}w_1^{(2)} + \bar{a}w_2^{(1)}\right) \qquad (52)$$

$$\overline{\nabla}^4 f^{(3)} - (1+\xi)^2\left(w_{,xx}^{(3)} + \xi r w_{,yy}^{(3)}\right)$$
$$= -\frac{1}{8}m^2n^2H(1+\xi)[9sin3ny - sinny - cos2mxsin3ny + 9cos2mxsinny]\left(2w_2^{(1)}w_1^{(2)} + \bar{a}w_2^{(1)}\right) \qquad (53)$$

Let

$$\begin{pmatrix} w^{(3)} \\ f^{(3)} \end{pmatrix} = \sum_{p=1,q=0}^{\infty} \left[\begin{pmatrix} w_1^{(3)} \\ f_1^{(3)} \end{pmatrix} sinpxcosqy + \begin{pmatrix} w_2^{(3)} \\ f_2^{(3)} \end{pmatrix} sinpxsinqy \right] \quad (54)$$

On substituting (54) into (53), multiplying through by $sinmxsinny$, we get, for $p = m, q = n$ (and for m odd)

$$f_{2(n)}^{(3)} = \frac{H(1+\xi)\frac{mn^2}{2\pi}\left(2w_2^{(1)}w_1^{(2)} + \bar{a}w_1^{(2)}\right) - (1+\xi)^2(n^2r\xi - m^2)w_{2(n)}^{(3)}}{(m^2+n^2\xi)^2} \quad (56a)$$

Let

$$\varphi_3 = \frac{H(1+\xi)mn^2\varphi_9}{2\pi(m^2+n^2\xi)^2\varphi_{10}} \quad (56b)$$

Then, we have

$$f_{2(n)}^{(3)} = \varphi_3\left(2{w_2^{(1)}}^3 + 3\bar{a}{w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right) - \varphi_0 w_{2(n)}^{(3)} \quad (56c)$$

Similarly, by multiplying (53) by $sinmxsin3ny$ and for $p = m, q = 3n$, we integrate to get

$$f_{2(3n)}^{(3)} = -\left[\frac{\frac{9H(1+\xi)mn^2}{2\pi}\left(2w_2^{(1)}w_1^{(2)} + \bar{a}w_1^{(2)}\right) + (1+\xi)^2(9n^2r\xi - m^2)w_{2(3n)}^{(3)}}{(m^2+9n^2\xi)^2}\right] \quad (57a)$$

Let

$$\varphi_{11} = \frac{9H(1+\xi)mn^2\frac{\varphi_9}{\varphi_{10}}}{2\pi(m^2+9n^2\xi)^2}, \quad \varphi_{12} = \frac{(1+\xi)^2(9n^2r\xi - m^2)}{(m^2+9n^2\xi)^2} \quad (57b)$$

$$\therefore \quad f_{2(3n)}^{(3)} = \left[-\varphi_{11}\left(2{w_2^{(1)}}^3 + 3\bar{a}{w_2^{(1)}}^2 + \bar{a}w_2^{(1)}\right) + \varphi_{12}w_{2(3n)}^{(3)}\right] \quad (57c)$$

To determine $w^{(3)}$, we now substitute into (52), multiply through by $sinmxsinny$ and for $p = m, q = n$ (m odd), we simplify to get

$$w_{2(n)}^{(3)} = \frac{-HK(\xi)mn^2}{\pi\varphi_1}\left(f_2^{(1)}w_1^{(2)} + f_1^{(2)}w_2^{(1)} + f_1^{(2)}\bar{a}\right) + \frac{H(1+\xi)mn^2}{2\pi(m^2+n^2\xi)^2\varphi_1}\left\{\left(\left(\frac{Am}{(1+\xi)}\right)^2 + n^2r\xi\right)\left(2w_2^{(1)}w_1^{(2)} + \bar{a}w_1^{(2)}\right)\right\} \quad (58a)$$

After carefully simplifying (58a), we get

$$w_{2(n)}^{(3)} = \frac{1}{\varphi_1}\left[\left(\varphi_{14} + \frac{2\varphi_{13}\varphi_9}{\varphi_{10}}\right){w_2^{(1)}}^3 + \bar{a}\left(2\varphi_{14} + \frac{3\varphi_{13}\varphi_9}{\varphi_{10}}\right){w_2^{(1)}}^2 + \bar{a}^2\left(\varphi_{14} + \frac{\varphi_{13}\varphi_9}{\varphi_{10}}\right)w_2^{(1)}\right] \quad (58b)$$

where,

$$\varphi_{13} = \frac{H(1+\xi)mn^2}{2\pi(m^2+n^2\xi)^2}\left(\left(\frac{Am}{(1+\xi)}\right)^2 + n^2r\xi\right) \quad (58c)$$

$$\varphi_{14} = \frac{H\left(\frac{A}{(1+\xi)}\right)^2 mn^2\left(\varphi_2 + \frac{\varphi_7\varphi_9}{\varphi_{10}}\right)}{\pi(m^2+n^2\xi)^2} \qquad (58d)$$

Following (58b – d), we further let

$$Q_1 = \left(\varphi_{14} + \frac{2\varphi_{13}\varphi_9}{\varphi_{10}}\right), \quad Q_2 = \left(2\varphi_{14} + \frac{3\varphi_{13}\varphi_9}{\varphi_{10}}\right), \quad Q_3 = \left(\varphi_{14} + \frac{\varphi_{13}\varphi_9}{\varphi_{10}}\right) \qquad (58e)$$

In this case, we get

$$w_{2(n)}^{(3)} = \frac{1}{\varphi_1}\left(Q_1 {w_2^{(1)}}^3 + Q_2\bar{a}{w_2^{(1)}}^2 + Q_3\bar{a}^2 w_2^{(1)}\right) \qquad (58g)$$

Next, we substitute into (52), multiply by $sinmxsin3ny$ and for $p = m, q = 3n$ (m odd), simplify to get

$$w_{2(3n)}^{(3)} = \left[\frac{\frac{-9HK(\xi)mn^2}{\pi}\left(f_2^{(1)}w_1^{(2)} + f_1^{(2)}w_2^{(1)} + \bar{a}f_1^{(2)}\right) + 9Hmn^2(1+\xi)^2\left\{\left(\left(\frac{Am}{(1+\xi)}\right)^2 + 9n^2r\xi\right)\left(2w_2^{(1)}w_1^{(2)} + \bar{a}w_1^{(2)}\right)\right\}}{\varphi_{15}}\right] \qquad (59a)$$

where,

$$\varphi_{15} = (m^2+9n^2\xi)^2 + \left(\left(\frac{Am}{(1+\xi)}\right)^2 + 9n^2r\xi\right)(1+\xi)^2(9n^2\xi - m^2) - \lambda\left\{\frac{\alpha m^2}{2} + 9\xi n^2\left(1-\frac{\alpha}{2}\right)\right\} \qquad (59b)$$

After simplifying (59a), we get

$$w_{2(3n)}^{(3)} = \left\{2\varphi_{16} - \varphi_{17}\left(\varphi_2 + \frac{\varphi_9}{\varphi_{10}}(\varphi_0+\varphi_7)\right)\right\}{w_2^{(1)}}^3 + \left\{3\varphi_{16} - \varphi_{17}\left(\frac{\varphi_9\varphi_7}{\varphi_{10}} + \frac{\varphi_9}{\varphi_{10}}(\varphi_0+\varphi_7) + 2\varphi_2\right)\right\}\bar{a}{w_2^{(1)}}^2 + \left\{\varphi_{16} - \varphi_{17}\left(\frac{\varphi_9\varphi_7}{\varphi_{10}} + \varphi_2\right)\right\}\bar{a}^2 w_2^{(1)} \qquad (60a)$$

where,

$$\varphi_{16} = \frac{9Hmn^2(1+\xi)^2\left\{\left(\left(\frac{Am}{(1+\xi)}\right)^2 + 9n^2r\xi\right)\right\}\frac{\varphi_9}{\varphi_{10}}}{\pi\varphi_{15}} \qquad (60b)$$

$$\varphi_{17} = \frac{-9H\left(\frac{A}{1+\xi}\right)^2 mn^2}{\pi\varphi_{15}} \qquad (60c)$$

We now let

$$Q_4 = 2\varphi_{16} - \varphi_{17}\left(\varphi_2 + \frac{\varphi_9}{\varphi_{10}}(\varphi_0+\varphi_7)\right) \qquad (61a)$$

$$Q_5 = \left\{3\varphi_{16} - \varphi_{17}\left(\frac{\varphi_9\varphi_7}{\varphi_{10}} + \frac{\varphi_9}{\varphi_{10}}(\varphi_0 + \varphi_7) + 2\varphi_2\right)\right\} \qquad (61b)$$

$$Q_6 = \varphi_{16} - \varphi_{17}\left(\frac{\varphi_9\varphi_7}{\varphi_{10}} + \varphi_2\right) \qquad (61c)$$

It follows that

$$w_{2(3n)}^{(3)} = Q_4 {w_2^{(1)}}^3 + Q_5\bar{a}{w_2^{(1)}}^2 + Q_6\bar{a}^2 w_2^{(1)} \qquad (62)$$

So far, the normal displacement $w^{(3)}(x,y)$ and corresponding Airy stress function $f^{(3)}(x,y)$ at this order of perturbation are

$$\begin{pmatrix} w^{(3)} \\ f^{(3)} \end{pmatrix} = \begin{pmatrix} w_{2(n)}^{(3)} \\ f_{2(n)}^{(3)} \end{pmatrix} sinmxsinny + \begin{pmatrix} w_{2(3n)}^{(3)} \\ f_{2(3n)}^{(3)} \end{pmatrix} sinmxsin3ny \qquad (63)$$

Generally, it follows that

$$\begin{pmatrix} w(x,y) \\ f(x,y) \end{pmatrix} = \epsilon \begin{pmatrix} 1 \\ \varphi_0 \end{pmatrix} w_2^{(1)} sinmxsinny + \epsilon^2 \begin{pmatrix} w_1^{(2)} \\ f_1^{(2)} \end{pmatrix} sinmxcos2ny + \epsilon^3 \left[\begin{pmatrix} w_{2(n)}^{(3)} \\ f_{2(n)}^{(2)} \end{pmatrix} sinmxsinny + \begin{pmatrix} w_{2(3n)}^{(3)} \\ f_{2(3n)}^{(2)} \end{pmatrix} sinmxsin3ny\right] + \cdots \qquad (64)$$

VI. STATIC BUCKLING LOAD, λ_S

To determine the static buckling load λ_S, we shall use only the buckling modes that are strictly in the shape of imperfection as in (35). As in Lockhart and Amazigo [11], the condition for static buckling is

$$\frac{d\lambda}{dw} = 0 \qquad (65)$$

where, it is tacitly implied that the displacement $w(x,y)$ is an embodiment of the load parameter λ. Amazigo[12] and Amazigo and Ette [13] had earlier shown that the application of equation (65) should be preceded by a reversal of the series (64) which we now embark by first letting

$$c_1 = w_2^{(1)} sinmxsinny, \quad c_3 = w_{2(n)}^{(3)} sinmxsinny$$

$$\therefore \quad w = \epsilon c_1 + \epsilon^3 c_3 + \cdots \qquad (66a)$$

To reverse the series, we write

$$\epsilon = wd_1 + w^3 d_3 + \cdots \qquad (66b)$$

By substituting for $w$from (66a) in (66b), and equating the coefficients of powers of ϵ, we get

$$d_1 = \frac{1}{c_1}, \qquad d_3 = -\frac{c_3}{c_1^4} \qquad (66c)$$

The maximization (65) is better accomplished using (66b) to get the value of the displacement at static buckling, namely w_S, as

$$w_S = \sqrt{\frac{c_1^3}{3c_3}} \qquad (67a)$$

The static buckling load λ_S is next determined by determining (66b) at static buckling to get

$$\epsilon = \frac{2}{3}\sqrt{\frac{c_1}{3c_3}} \qquad (67b)$$

On simplification, (67b) gives

$$\left[(m^2+n^2\xi)^2+\left(\left(\frac{Am}{(1+\xi)}\right)^2+n^2r\xi\right)(1+\xi)^2\left(\frac{n^2r\xi-m^2}{(m^2+n^2\xi)^2}\right)-\lambda_S\left\{\frac{\alpha m^2}{2}+\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\right]^{3/2} = \frac{3\sqrt{3}}{2}\lambda_S(\bar{a}\epsilon)\left\{\frac{\alpha m^2}{2}+\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\sqrt{Q_1Q_7} \qquad (68a)$$

where,

$$Q_7 = \left[1+\left(\frac{Q_2}{Q_1}\right)\left(\frac{\bar{a}}{w_2^{(1)}}\right)+\left(\frac{Q_3}{Q_1}\right)\left(\frac{\bar{a}}{w_2^{(1)}}\right)^2\right] \qquad (68b)$$

Equations (68a, b) give an implicit expression for determining λ_S. The result is asymptotic in nature, all depending on the smallness of the perturbation parameter ϵ in relation to unity. Similar results on cubic and quadratic – cubic elastic model structures were obtained by Budiansky and Hutchinson [14].

VII. CONCLUSION

We have employed regular perturbation procedures and asymptotic expansions to evaluate the out–of–plane normal deflection and corresponding Airy stress function of a statically loaded, finite, imperfect toroidal shell segment that has simply – supported boundary conditions. We have similarly determined the static buckling load λ_S of the structure and the result is asymptotic in nature. It is expected that similar procedures can be utilized to analyze cases of dynamic loading of the same and similar structures.

REFERENCES

[1] M. Steinand J. A. McElman, Buckling of segments of toroidal shells, AIAA Jnl.,1965, 3, 1704.

[2] J. W. Hutchison, Initial post buckling behavior of toroidal shell segment, Int. J. Solids Struct. 1967, 3, 97 -115.

[3] M. O. Oyesanya, Asymptotic analysis of imperfection sensitivity of toroidal shell segment modal imperfections, J. Nigerian Ass. Maths. Physics2002, 6, 197 – 206.

[4] M. O. Oyesanya, Influence of extra terms on asymptotic analysis of imperfection sensitivity of toroidal shell segment with random imperfection, Mechanics research Communications,2005, 32, 444 – 453.

[5] B. Kriegesman, M. Mohleand R. Rolfes, Sample size dependent probabilistic design of axially compressed cylindrical shells, Thin – Walled Structures2015, 96, 256 – 268.

[6] V. Hu andR. Burgueño,Elastic post buckling response of axially loaded cylindrical shells with seeded geometric imperfection design, Thin – Walled structures 2014, 74, 222 – 231.
[7] T. Kubiak, Criteria of dynamic buckling estimation of Thin – walled structures, Thin – walled Structures 2007, 45, 888 – 892.
[8] T. Kubiak, Static and dynamic buckling of Thin – walled plate structures, Springer – Verlag, London, UK, 2013.
[9] T. A. Kolakowski, Buckling of Thin – walled composite structures with intermediate stiffeners, Composite Structures 2005, 69, 421 – 429.
[10] T. A. Kolakowski, Coupled dynamic buckling of Thin – walled composite columns with open cross – section, Composite Structures 2013, 95, 28 – 34.
[11] D. Lockharts and J. C. Amazigo, Dynamic buckling of externally pressurized imperfect cylindrical shells, J. of App. Mech., 1975, 42 (2), 316 – 320.
[12] J. C. Amazigo, Asymptotic analysis of the buckling of externally pressurized cylinders with random imperfection, Quart. Appl. Math. 1974, 32, 429 – 442.
[13] J. C. Amazigo and A. M. Ette, (On a two – small parameter differential equation with application to dynamic buckling, J. Nig. Math. Soc. 1987, 6, 90 – 102.
[14] B. Budiansky and W. J. Hutchinson, Dynamic buckling of imperfection – sensitive structures, Proceedings of 11th Int. Congr.Of Appl. Mech., Munich (ed. H. Gortler, 1964), Springer – Verlag 636 – 651, 1966.