Zero free region & location of zero's of a polynomial with restricted coefficients

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Abstract: *Concerning the distribution of a polynomial a famous result is due to Enestrom – Kakeya which states that if* $P(z) = \sum_{i=0}^{n} a_i z^i$ *is an n^th^ degree polynomial, such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0s$$

Then all zero's of $P(z)$ *lie in* $|z| \leq 1$.

In this paper we relax the hypothesis in several ways and obtain a generalization of above result, we also present a result on the zero free region of certain polynomials.

Keywords: *zero's of polynomial, coefficients, bounds, zero free region.*

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I. Introduction

Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then according to a well known result of Enestrom – Kakeya [3] , deals with the location of zero's which states as follows:

Theorem A: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then all zero's of $P(z)$ lie in $|z| \leq 1$.

By applying the above result to the polynomial $z^n p\left(\frac{1}{z}\right)$, we get the following result.

Theorem B Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial such that

$$0 < a_n \leq a_{n-1} \leq \cdots \leq a_0$$

Then $P(z)$ does not vanish in $|z| < 1$.

In the literature many attempts have been made by various researchers to extend and generalize the Enestrom – Kakeya theorem, Joyal, Labelle and Rahman [2], dropped the non – negative condition of coefficients by proving the following:

Theorem C: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$$

Then $P(z)$ has all its zero's in $|z| < \frac{a_n - a_0 + |a_0|}{|a_n|}$

Applying the above result to the polynomial $z^n P\left(\frac{1}{z}\right)$, we get the following result.

TheoremD: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial such that

$$a_n \leq a_{n-1} \leq \cdots \leq a_0$$

Then $P(z)$ does not vanish in $|z| < \frac{|a_0|}{a_0 - a_n + |a_n|}$

On the other hand Govil and Rahman [1], extended the above results to the class of polynomials with complex coefficients by proved following result.

Theorem E: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial with complex coefficients such that for some real β

$$|arg a_i - \beta| \leq \alpha \leq \frac{\pi}{2} \quad , \quad 0 \leq i \leq n$$

And $\quad |a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|$, then $P(z)$ has all its zero's in

$$|z| \leq (sin\alpha + cos\alpha) + \frac{2sin\alpha}{|a_n|} \sum_{i=0}^{n-1} |a_i|$$

II. Main results

The aim of this paper is to obtain zero free region and generalization of theorem D, E and hence of theorem A. In fact we prove the following results.

Theorem 1: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial, if for some real numbers

$t \geq 1,\ 0 < S \leq 1$, such that

$$ta_n \geq a_{n-1} \geq \cdots \geq a_{p+1} \geq a_p \leq \cdots \leq a_{q+1} \leq a_q \geq a_{q-1} \geq \cdots \geq a_1 \geq Sa_o$$

Then $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{|a_0| - S(|a_0| + a_0) + 2(a_q - a_p) + t(a_n + |a_n|)}$$

Forp$= q = n$, $t = S = 1$, then theorem 1 reduces to the following result.

Corollary 1: If $P(z) = \sum_{i=0}^{n} a_i z^i$ is an n[th] degree polynomial, such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$$

Then $P(z)$ does not vanish in $|z| < \frac{|a_0|}{a_n + |a_n| - a_0}$

The bound is attained by the polynomial

$$P(z) = z^n + z^{n-1} + \cdots + z + 1$$

Example: Consider the polynomial

$$P(z) = 2z^{10} + 4z^9 + 3z^8 + 3z^7 + 2z^6 + 3z^5 + 3z^4 + 2z^3 + 2z^2 + 6$$

Here the coefficients satisfy the following

$$2(2) \geq 4 > 3 \geq 3 > 2 < 3 \leq 3 > 2 \geq 2 \geq \frac{1}{3}(6)$$

$$t = 2 \ , \ \alpha_p = 2 \ , \ \alpha_q = 3 \quad and s = \frac{1}{3}$$

For this polynomial $P(z)$, does not vanish in $|z| < \frac{1}{2}$

Theorem 2: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an n[th] degree polynomial, if for some real numbers

$t, S \geq 0$, such that

$$a_n - t \leq a_{n-1} \leq \cdots \leq a_{p+1} \leq a_p \geq a_{p-1} \geq \cdots \geq a_{q+1} \geq a_q \leq a_{q-1} \leq \cdots \leq a_1 \leq a_o + S$$

Then $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t}$$

Remarks: for $p = q = n$, $s = t = 0$,then theorem 2 reduces to theorem D.

Theorem 3: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ is an n[th] degree polynomial with complex coefficients such that

$$|arg a_i - \beta| \leq \alpha \leq \frac{\pi}{2} \quad , \quad i = 0, 1, 2 \ldots n$$

For some real β , $k \geq 1$, $0 < \rho \leq 1$ and

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_{p+1}| \geq |a_p| \leq |a_{p-1}| \leq \cdots \leq |a_{q+1}| \leq |a_q| \geq |a_{q-1}| \geq \cdots \geq \rho|a_0|$$

Then $P(z)$ has all its zeros in

$$|z+k-1| \le \frac{k|a_n|(cos\alpha+sin\alpha)+2cos\alpha\left(|a_q|-|a_p|\right)+2|a_0|-\rho|a_0|(cos\alpha-sin\alpha+1)+S}{|a_n|}$$

Where $S = 2sin\alpha \sum_{i=0}^{n-1} |a_i|$

Remark: For $k=\rho=1$, $p=q=n$, then theorem 3 reduces to theorem E.

III. Proof of theorems

For the proof of theorem 3, we need following lemma.

Lemma 1: If $|arga_i-\beta| \le \alpha \le \frac{\pi}{2}$, and for some $t>0$, then

$$|ta_i-a_{i-1}| \le [t|a_i|-|a_{i-1}|]cos\alpha+[t|a_i|+|a_{i-1}|]sin\alpha$$

Above Lemma is due toGovel and Rahman [1]

Proof of theorem 1: Consider the Polynomial

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_{p+1}z^{p+1}+a_pz^p+\cdots+a_{q+1}z^{q+1}+a_qz^q+\ \dots+a_1z+a_0$$

Let $Q(z)=z^np\left(\frac{1}{z}\right)$

$$=a_0z^n+a_1z^{n-1}+a_qz^{n-q}+a_{q+1}z^{n-q-1}+\cdots+a_p\ \ z^{n-p}+a_{p+1}z^{n-p-1}+\cdots+a_{n-1}z+a_n$$

And $R(z)=(1-z)Q(z)$

$$=-a_0z^{n+1}+(a_0-a_1)z^n+(a_1-a_2)z^{n-1}+\cdots+\left(a_{q-1}-a_q\right)z^{n-q+1}+ \left(a_q-a_{q+1}\right)z^{n-q}+\cdots+\left(a_{p-1}-a_p\right)z^{n-p+1}+\left(a_p-a_{p+1}\right)z^{n-p}+\cdots+(a_{n-1}-a_n)z+a_n$$

$$=-a_0z^{n+1}+(Sa_0-a_1)z^n+(1-S)a_0z^n+(a_1-a_0)z^{n+1}+\cdots+\left(a_{q-1}-a_q\right)z^{n-q+1} + \left(a_q-a_{q+1}\right)z^{n-q}+\cdots+\left(a_{p-1}-a_p\right)z^{n-p+1}+\left(a_p-a_{p+1}\right)z^{n-p}+\cdots +(a_{n-1}-ta_n)z+ (t-1)a_nz+a_n$$

$$|R(z)|=\left|-a_0z^{n+1}+(Sa_0-a_1)z^n+(1-S)a_0z^n+(a_1-a_0)z^{n+1}+\cdots+\left(a_{q-1}-a_q\right)z^{n-q+1} +\left(a_q-a_{q+1}\right)z^{n-q}+\cdots+\left(a_{p-1}-a_p\right)z^{n-p+1}+\left(a_p-a_{p+1}\right)z^{n-p}+\cdots+(a_{n-1}-ta_n)z +(t-1)a_nz+a_n\right|$$

$$|R(z)| \ge |a_0||z|^{n+1} -\left\{(1-S)|a_0||z|^n+|Sa_0-a_1||z|^n+\cdots+\left|a_{q-1}-a_q\right||z|^{n-q+1}+\left|a_q-a_{q+1}\right||z|^{n-q}+\cdots +\left|a_{p-1}-a_p\right||z|^{n-p+1}+\left|a_p-a_{p+1}\right||z|^{n-p}+\cdots+|a_{n-1}-ta_n||z|\ \ +(t-1)|a_n||z|+|a_n|\right\}$$

$$\ge |a_0||z|^n\left[|z|-\frac{1}{|a_0|}\left\{(1-S)|a_0|+|Sa_0-a_1|+|a_1-a_2|\frac{1}{|z|}+\cdots+\left|a_{q-1}-a_q\right|\frac{1}{|z|^{q-1}}+\left|a_q-a_{q+1}\right|\frac{1}{|z|^q}+\cdots +\left|a_{p-1}-a_p\right|\frac{1}{|z|^{p-1}}+\left|a_p-a_{p+1}\right|\frac{1}{|z|^p}+\cdots+|a_{n-1}-ta_n|\frac{1}{|z|^{n-1}}+(t-1)\frac{|a_n|}{|z|^{n-1}}+\frac{|a_n|}{|z|^n}\right\}\right]$$

For $|z| \ge 1$, so that $\frac{1}{|z|} \le 1$

$$|R(z)| \geq |a_0||z|^n \Big[|z| - \frac{1}{|a_0|}\{(1-S)|a_0| + |Sa_0 - a_1| + |a_1 - a_2| + \cdots + |a_{q-1} - a_q| + |a_q - a_{q+1}| + \cdots + |a_{p-1} - a_p| + |a_p - a_{p+1}| + \cdots + |a_{n-1} - ta_n| + (t-1)|a_n| + |a_n|\}\Big]$$

$$\geq |a_0||z|^n \Big[|z| - \frac{1}{|a_0|}\{(1-S)|a_0| + |a_1 - Sa_0| + |a_2 - a_1| + \cdots + |a_q - a_{q-1}| + |a_{q+1} - a_q| + \cdots + |a_p - a_{p-1}| + |a_{p+1} - a_p| + \cdots + |ta_n - a_{n-1}| + (t-1)|a_n| + |a_n|\}\Big]$$

$$\geq |a_0||z|^n \Big[|z| - \frac{1}{|a_0|}\{(1-S)|a_0| + a_1 - Sa_0 + a_2 - a_1 + \cdots + a_q - a_{q+1} + a_q - a_{q-1} + \ldots + a_{p+1} - a_p + a_{p-1} - a_p + \cdots + ta_n - a_{n-1} + (t-1)|a_n| + |a_n|\}\Big]$$

$$\geq |a_0||z|^n \Big[|z| - \frac{1}{|a_0|}\{(1-S)|a_0| - Sa_0 + 2a_q - 2a_p + ta_n + (t-1)|a_n| + |a_n|\}\Big]$$

$$\geq |a_0||z|^n \Big[|z| - \frac{1}{|a_0|}\{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)\}\Big] > 0$$

If $|z| - \frac{1}{|a_0|}\{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)\} > 0$

$$|z| > \frac{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)}{|a_0|}$$

This shows that all zero's of $R(z)$, whose modulus is greater than 1 lie in

$$|z| \leq \frac{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)}{|a_0|}$$

But those zero's of $R(z)$, whose modulus is less than or equal to 1. Already satisfy the above inequality, therefore it follows all zero's of $R(z)$ and hence $Q(z)$ lie in

$$|z| \leq \frac{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)}{|a_0|}$$

Since $P(z) = z^n Q\left(\frac{1}{z}\right)$, it follows that all zero's of $P(z)$ lie in

$$|z| \geq \frac{|a_0|}{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)}$$

Hence $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{|a_0| - S(a_0 + |a_0|) + 2(a_q - a_p) + t(a_n + |a_n|)}$$

Hence it completes proof.

Proof of theorem 2: Consider the Polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_{p+1} z^{p+1} + a_p z^p + \cdots + a_{q+1} z^{q+1} + a_q z^q + \ldots + a_1 z + a_0$$

Let $Q(z) = z^n p\left(\frac{1}{z}\right)$

$$= a_0 z^n + a_1 z^{n-1} + \cdots + a_q z^{n-q} + a_{q+1} z^{n-q-1} + \cdots + a_p z^{n-p} + a_{p+1} z^{n-p-1} + \cdots + a_{n-1} z + a_n$$

and$R(z) = (1-z)Q(z)$

$$= -a_0 z^{n+1} + (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{q-1} - a_q) z^{n-q+1} + (a_q - a_{q+1}) z^{n-q} + \ldots + (a_{p-1} - a_p) z^{n-p+1} + (a_p - a_{p+1}) z^{n-p} + \cdots + (a_{n-1} - a_n) z + a_n$$

$$= -a_0 z^{n+1} + (S + a_0 - a_1) z^n - Sz^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{q-1} - a_q) z^{n-q+1} + (a_q - a_{q+1}) z^{n-q} + \cdots + (a_{p-1} - a_p) z^{n-p+1} + (a_p - a_{p+1}) z^{n-p} + \cdots + (a_{n-1} - a_n + t) z - tz + a_n$$

$$|R(z)| = \left|-a_0 z^{n+1} + (S + a_0 - a_1)z^n - Sz^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{q-1} - a_q)z^{n-q+1} + (a_q - a_{q+1})z^{n-q} + \cdots + (a_{p-1} - a_p)z^{n-p+1} + (a_p - a_{p+1})z^{n-p} + \cdots + (a_{n-1} - a_n + t)z - tz + a_n\right|$$

$$|R(z)| \geq |a_0||z|^{n+1} - \left[|S + a_0 - a_1||z|^n + S|z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{q-1} - a_q||z|^{n-q+1} + |a_q - a_{q+1}||z|^{n-q} + \cdots + |a_{p-1} - a_p||z|^{n-p+1} + |a_p - a_{p+1}||z|^{n-p} + \cdots + |a_{n-1} - a_n + t||z| \quad + t|z| \quad + |a_n|\right]$$

$$|R(z)| \geq |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\left\{|S + a_0 - a_1| + S + |a_1 - a_2|\frac{1}{|z|} + \cdots + |a_{q-1} - a_q|\frac{1}{|z|^{q-1}} + |a_q - a_{q+1}|\frac{1}{|z|^q} + \cdots + |a_{p-1} - a_p|\frac{1}{|z|^{p-1}} + |a_p - a_{p+1}|\frac{1}{|z|^p} + \cdots + |a_{n-1} - a_n + t|\frac{1}{|z|^{n-1}} + t\frac{1}{|z|^{n-1}} + |a_n|\frac{1}{|z|^n}\right\}\right]$$

For $|z| \geq 1$, so that $\frac{1}{|z|} \leq 1$, then we have

$$|R(z)| \geq |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\{S + a_0 - a_1 + S + a_1 - a_2 + \cdots + a_{q-1} - a_q + a_{q+1} - a_q + \cdots + a_p - a_{p-1} + a_p - a_{p+1} + \cdots + a_{n-1} - a_n + t + t + |a_n|\}\right]$$

$$\geq |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\left[(2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t\right]\right] > 0$$

If $|z| - \frac{1}{|a_0|}\left[2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t\right] > 0$

$$|z| > \frac{2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t}{|a_0|}$$

This shows that all zero's of $R(z)$, whose modulus is greater than 1 lie in

$$|z| \leq \frac{2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t}{|a_0|}$$

But those zero'sof (z) , whose modulus is less than or equal to 1. Already satisfy above inequality. Therefore it follows all zero's of $R(z)$ and hence $Q(z)$ lie in

$$|z| \leq \frac{2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t}{|a_0|}$$

Since $P(z) = Z^n Q\left(\frac{1}{z}\right)$, it follows that all zero's of $P(z)$ lie in

$$|z| \geq \frac{|a_0|}{2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t}$$

Hence $P(z)$ does not vanish in

$$|z| < \frac{|a_0|}{2S + a_0 + 2(a_p - a_q) - a_n + |a_n| + 2t}$$

Hence it completes Proof.

Proof of theorem 3:

Consider the Polynomial

$$F(z) = (1 - z)P(z)$$

$$= (1 - z)(a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_{p+1}z^{p+1} + a_p z^p + \ldots \quad + \cdots a_{q+1}z^{q+1} + a_q z^q + \cdots + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} - ka_n z^n + a_n z^n + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ \ldots + (a_{q+1} - a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + \rho a_0 z - a_0 z + a_0$$

$= -a_n z^{n+1}(z+k-1)+(ka_n - a_{n-1})z^n + (a_{n-1}-a_{n-2})z^{n-1} + \cdots + (a_{p+1}-a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_{q+1}-a_q)z^{q+1} + (a_q - a_{q-1})z^q + \cdots + (a_1 - \rho a_0)z + (\rho - 1)a_0 z + a_0$

This gives

$$|F(z)| \geq a_n|z|^n|z+k-1| - \left\{\begin{matrix} |ka_n - a_{n-1}||z|^n + |a_{n-1}-a_{n-2}||z|^{n-1} + \cdots + |a_{p+1}-a_p||z|^{p+1} + |a_p - a_{p-1}||z|^p \\ + \cdots |a_{q+1}-a_q||z|^{q+1} + |a_q - a_{q-1}||z|^q + |a_1 - \rho a_0||z| \\ +(\rho-1)|a_o||z| + |a_o| \end{matrix}\right\}$$

$$\geq |z|^n \Big[|a_n||z+k-1| - \Big\{|ka_n - a_{n-1}| + |a_{n-1}-a_{n-2}|\frac{1}{|z|} + \cdots + |a_{p+1}-a_p|\frac{1}{|z|^{n-p-1}} + |a_p - a_{p-1}|\frac{1}{|z|^{n-p}} + \cdots + |a_{q+1}-a_q|\frac{1}{|z|^{n-q-1}} + |a_q - a_{q-1}|\frac{1}{|z|^{n-q}} + \cdots + |a_1 - \rho a_0|\frac{1}{|z|^{n-1}} + (\rho-1)|a_0|\frac{1}{|z|^{n-1}} + |a_0|\frac{1}{|z|^n}\Big\}\Big]$$

For $|z| \geq 1$,therefore $\frac{1}{|z|} \leq 1$, we have

$$|F(z)| \geq |z|^n\big[|a_n||z+k-1| - \{|ka_n - a_{n-1}| + |a_{n-1}-a_{n-2}| + \cdots + |a_{p+1}-a_p| + |a_p - a_{p-1}| + \cdots + |a_{q+1}-a_q| + |a_q - a_{q-1}| + \cdots + |a_1 - \rho a_0| + (\rho-1)|a_0| + |a_0|\}\big]$$

$$|F(z)| \geq |z|^n\big[|a_n||z+k-1| - \{|ka_n - a_{n-1}| + |a_{n-1}-a_{n-2}| + \cdots + |a_{p+1}-a_p| + |a_p - a_{p-1}| + \cdots + |a_{q+1}-a_q| + |a_q - a_{q-1}| + \cdots + |a_1 - \rho a_0| + (1-\rho)|a_0| + |a_0|\}\big]$$

$$\geq |z|^n\big[|a_n||z+k-1| - \{(k|a_n| - |a_{n-1}|)cos\alpha + (k|a_n| + |a_{n-1}|)sin\alpha + \cdots + (|a_{p+1}| - |a_p|)cos\alpha + (|a_{p+1}| + |a_p|)sin\alpha + (|a_p| - |a_{p-1}|)cos\alpha + (|a_p| + |a_{p-1}|)sin\alpha + \cdots + (|a_q| - |a_{q+1}|)cos\alpha + (|a_q| + |a_{q+1}|)sin\alpha + (|a_q| - |a_{q-1}|)cos\alpha + (|a_q| + |a_{q-1}|)sin\alpha + \cdots + (|a_1| - \rho|a_0|)cos\alpha + (|a_1| + \rho|a_0|)sin\alpha + (1-\rho)|a_0| + |a_0|\}\big]$$

$$|F(z)| \geq |z|^n\Big[|a_n||z+k-1| - \Big\{k|a_n|(cos\alpha + sin\alpha) - |a_p|cos\alpha - |a_p|cos\alpha + |a_q|cos\alpha + |a_q|cos\alpha - \rho|a_0|cos\alpha + (1-\rho)|a_0| + |a_0| + \rho|a_0|sin\alpha + 2sin\alpha\sum_{i=0}^{n-1}|a_i|\Big\}\Big]$$

$$\geq |z|^n\big[|a_n||z+k-1| - \{k|a_n|(cos\alpha + sin\alpha) + 2cos\alpha(|a_q| - |a_p|) + 2|a_0| - \rho|a_0|(cos\alpha - sin\alpha + 1) + S\}\big] > 0$$

Where $S = 2sin\alpha\sum_{i=0}^{n-1}|a_i|$

If $\left[|a_n||z+k-1| - \left\{\begin{matrix} k|a_n|(cos\alpha + sin\alpha) + 2cos\alpha(|a_q| - |a_p|) + 2|a_0| \\ -\rho|a_0|(cos\alpha - sin\alpha + 1) + S \end{matrix}\right\}\right] > 0$

i. e $|a_n||z+k-1| > \left\{\begin{matrix} k|a_n|(cos\alpha + sin\alpha) + 2cos\alpha(|a_q| - |a_p|) + 2|a_0| \\ -\rho|a_0|(cos\alpha - sin\alpha + 1) + S \end{matrix}\right\}$

i.e$|z+k-1| > \frac{k|a_n|(cos\alpha + sin\alpha) + 2cos\alpha(|a_q| - |a_p|) + 2|a_0| - \rho|a_0|(cos\alpha - sin\alpha + 1) + S}{|a_n|}$

This shows that all zero's of $F(z)$, whose modulus is greater than 1 lie in

$$|z+k-1| \leq \frac{k|a_n|(cos\alpha + sin\alpha) + 2cos\alpha\left(|a_q| - |a_p|\right) + 2|a_0| - \rho|a_0|(cos\alpha - sin\alpha + 1) + S}{|a_n|}$$

Since the zero'sof $F(z)$, whose modulus is less than or equal to 1 already satisfy the above inequality and Since thezero's of $P(z)$ are also the zero's of $F(z)$, it follows that all zero's of $P(z)$ lie in

$$|z+k-1| \leq \frac{k|a_n|(cos\alpha + sin\alpha) + 2cos\alpha\left(|a_q| - |a_p|\right) + 2|a_0| - \rho|a_0|(cos\alpha - sin\alpha + 1) + S}{|a_n|}$$

Hence the proof of theorem 3.

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