Some New Status Indices of Graphs

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Abstract: *In this study, we propose the first and second hyper status indices, sum connectivity status index, product connectivity status index, atom bond connectivity status index, geometric-arithmetic status index, general first and second status indices of a graph and determine their values for some standard graphs, wheel graphs, friendship graphs.*

Keywords: *hyper status indices, sum connectivity status index, product connectivity status index, atom bond connectivity status index, geometric-arithmetic index, graphs.*

Mathematics Subject Classification: 05*C*05, 05*C*07, 05*C*12, 05*C*35.

1. Introduction

Let *G* be a finite, simple, connected graph with vertex set *V*(*G*) and edge set *E*(*G*). The edge connecting the vertices *u* and *v* will be denoted by *uv*. The degree $d_G(v)$ of a vertex *v* is the number of vertices adjacent to *v*. The distance *d*(*u, v*) between any two vertices *u* and *v* is the length of shortest path connecting *u* and *v*. The status $\sigma(u)$ of a vertex *u* in G is the sum of distances of all other vertices from *u* in *G*. We refer [1] for undefined terms and notations from graph theory.

A topological index or graph index is a numerical parameter mathematically derived from the graph structure. It is a graph invariant. The graph indices have their applications in various disciplines of Science and Technology, see [2, 3]. Some of the graph indices may be found in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

The first and second status connectivity indices of a graph *G* are introduced by Ramane et al. in [15], defined as

$$S_1(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)], \qquad S_2(G) = \sum_{uv \in E(G)} \sigma(u)\sigma(v).$$

We now introduce the first and second hyper status indices of a graph *G*, defined as

$$HS_1(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]^2, \qquad HS_2(G) = \sum_{uv \in E(G)} [\sigma(u)\sigma(v)]^2.$$

We also introduce the connectivity status indices as follows:

The sum connectivity status index of a graph *G* is defined as

$$SS(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma(u) + \sigma(v)}}.$$

The product connectivity status index of a graph *G* is defined as

$$PS(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma(u)\sigma(v)}}.$$

The reciprocal product connectivity status index of a graph *G* is defined as

$$RPS(G) = \sum_{uv \in E(G)} \sqrt{\sigma(u)\sigma(v)}.$$

The general first and second status indices of a graph *G* are defined as

$$S_1^a(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)]^a, \qquad (1)$$

$$S_2^a(G) = \sum_{uv \in E(G)} [\sigma(u)\sigma(v)]^a \qquad (2)$$

where *a* is a real number.

Some of the research works on status indices can be found in [16, 17, 18, 19, 20].

In this paper, we introduce some new status indices of a graph. Furthermore, the values of these newly proposed indices for some standard graphs are provided.

2. Results for Some Graphs

2.1. Complete graph K_n on n vertices.

Theorem 1. The general first status index of a complete graph K_n is

$$S_1^a(K_n) = \frac{n(n-1)}{2}[2(n-1)]^a. \tag{3}$$

Proof: If K_n is a complete graph with n vertices, then $d_{K_n}(u) = n-1$ and □(u) = n – 1 for any vertex u in K_n. Thus

$$S_1^a(K_n) = \sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^a = [(n-1)+(n-1)]^a \frac{n(n-1)}{2}$$
$$= \frac{n(n-1)}{2}[2(n-1)]^a.$$

We obtain the following results by using Theorem 1.

Corollary 1.1. If K_n is a complete graph, then

i) $S_1(K_n) = n(n-1)^2$.

ii) $HS_1(K_n) = 2n(n-1)^3$.

iii) $SS(K_n) = \dfrac{n\sqrt{n-1}}{2\sqrt{2}}$.

Proof: Put a =1, 2, –½ in equation (3), we get the desired results.

Theorem 2. The general second status index of a complete graph K_n is

$$S_2^a(K_n) = \frac{1}{2}n(n-1)(n-1)^{2a}. \tag{4}$$

Proof: If K_n is a complete graph with n vertices, then $d_G(u)$ = n – 1, □(u) = n – 1 for any vertex u in K_n. Therefore

$$S_2^a(K_n) = \sum_{uv\in E(G)} [\sigma(u)\sigma(v)]^a = [(n-1)(n-1)]^a \frac{n(n-1)}{2}$$
$$= \frac{1}{2}n(n-1)(n-1)^{2a}.$$

We establish the following results by using Theorem 2.

Corollary 2.1. If K_n is a complete graph with n vertices, then

i) $S_2(K_n) = \dfrac{1}{2}n(n-1)^3$. ii) $HS_2(K_n) = \dfrac{1}{2}n(n-1)^5$.

iii) $PS_2(K_n) = \dfrac{1}{2}n$. iv) $RPS(K_n) = \dfrac{1}{2}n(n-1)^2$.

Proof: Put a = 1, 2, –½, ½ in equation (4), we get the desired results.

2.2. Cycle C_n on n vertices

Theorem 3. Let C_n be a cycle on n vertices. Then

$$S_1^a(C_n) = n\left(\frac{n^2}{2}\right)^a, \quad \text{if } n \text{ is even,} \tag{5}$$

$$= n\left(\frac{n^2-1}{2}\right)^a, \quad \text{if } n \text{ is odd.} \tag{6}$$

Proof: If C_n is a cycle with n vertices, then $d_{C_n}(u) = 2$ for every vertex u in C_n.

Case 1. Suppose n is even. Then $\sigma(u) = \dfrac{n^2}{4}$ for any vertex u in C_n. Therefore

$$S_1^a(C_n) = \sum_{uv\in E(C_n)} [\sigma(u)+\sigma(v)]^a = \left(\frac{n^2}{4}+\frac{n^2}{4}\right)^a n = n\left(\frac{n^2}{2}\right)^a.$$

Case 2. Suppose n is odd. Then $\sigma(u)=\frac{n^2-1}{4}$ for any vertex u in C_n. Thus

$$S_1^a(C_n)=\sum_{uv\in E(C_n)}[\sigma(u)+\sigma(v)]^a=\left(\frac{n^2-1}{4}+\frac{n^2-1}{4}\right)^a n=n\left(\frac{n^2-1}{2}\right)^a.$$

Corollary 3.1. Let C_n be a cycle on n vertices. Then

(i) $S_1(C_n)=\frac{1}{2}n^3,$ if n is even,

$=\frac{1}{2}n(n^2-1),$ if n is odd.

(ii) $HS_1(C_n)=\frac{1}{4}n^5,$ if n is even,

$=\frac{1}{4}n(n^2-1)^2,$ if n is odd.

(iii) $SS(C_n)=\sqrt{2},$ if n is even,

$=n\sqrt{\frac{2}{n^2-1}},$ if n is odd.

Proof: Put a = 1, 2, –½, ½ in equations (5), (6), we get the desired results.

Theorem 4. Let C_n be a cycle on n vertices. Then

$$S_2^a(C_n)=n\left(\frac{n^4}{16}\right)^a, \quad \text{if } n \text{ is even,} \tag{7}$$

$$=n\left(\frac{(n^2-1)^2}{16}\right)^a, \quad \text{if } n \text{ is odd.} \tag{8}$$

Proof: If C_n is a cycle with n vertices, then $d_{C_n}(u)=2$ for every vertex u in C_n.

Case 1. Suppose n is even. Then $\sigma(u)=\frac{n^2}{4}$ for any vertex u in C_n. Hence

$$S_2^a(C_n)=\sum_{uv\in E(C_n)}[\sigma(u)\sigma(v)]^a=\left(\frac{n^2}{4}\times\frac{n^2}{4}\right)^a n=n\left(\frac{n^4}{16}\right)^a.$$

Case 2. Suppose n is odd. Then $\sigma(u)=\frac{n^2-1}{4}$ for any vertex u in C_n. Thus

$$S_2^a(C_n)=\sum_{uv\in E(C_n)}[\sigma(u)\sigma(v)]^a=\left(\frac{n^2-1}{4}\times\frac{n^2-1}{4}\right)^a n=n\left(\frac{(n^2-1)^2}{16}\right)^a.$$

Corollary 4.1. Let C_n be a cycle on n vertices. Then

(i) $S_2(C_n)=\frac{1}{16}n^5,$ if n is even,

$=\frac{1}{16}n(n^2-1)^2,$ if n is odd.

(ii) $HS_2(C_n)=\frac{1}{256}n^9,$ if n is even,

$=\frac{1}{256}n(n^2-1)^4,$ if n is odd.

(iii) $PS(C_n)=\frac{4}{n},$ if n is even,

$$=\frac{4n}{n^2-1},\qquad \text{if } n \text{ is odd.}$$

(iv) $$RPS(C_n)=\frac{1}{4}n^3,\qquad \text{if } n \text{ is even,}$$

$$=\frac{1}{4}n(n^2-1),\qquad \text{if } n \text{ is odd.}$$

Proof: Put a = 1, 2, –½, ½ in equations (7), (8), we get the desired results.

2.3. Complete bipartite graph $K_{p,q}$

Theorem 5. The general first status index of a complete bipartite graph $K_{p,q}$ is

$$S_1^a(K_{p,q})=pq[3(p+q)-4]^a. \tag{9}$$

Proof: Let $K_{p,q}$ be a complete bipartite graph with $p+q$ vertices and pq edges. The vertex set of $K_{p,q}$ can be partitioned into two independent sets V_1 and V_2 such that for every edge uv in $K_{p,q}$, $u\in V_1$ and $v\in V_2$. Thus $d(u) = q$, $d(v) = p$. Then

$\sigma(u) = q + 2(p – 1)$ and $\sigma(v) = p + 2(q – 1)$.

Therefore

$$S_1^a(K_{p,q})=\sum_{uv\in E(K_{p,q})}[\sigma(u)+\sigma(v)]^a=[q+2(p-1)+p+2(q-1)]^a\,pq$$

$$=pq[3(p+q)-4]^a.$$

We obtain the following results by using Theorem 1.

Corollary 5.1. If $K_{p,q}$ is a complete bipartite graph, then

i) $S_1(K_{p,q}) = 3pq(p + q) – 4pq$.

ii) $HS_1(K_{p,q}) = pq[3(p + q) – 4]^2$.

iii) $$SS(K_{p,q})=\frac{pq}{\sqrt{3(p+q)-4}}.$$

Proof: Put a =1, 2, –½ in equation (9), we get the desired results.

Theorem 6. The general second status index of a complete bipartite graph $K_{p,q}$ is

$$S_2^a(K_{p,q})=[(p+2q-2)(q+2p-2)]^a\,pq. \tag{10}$$

Proof: Let V_1 and V_2 be two independent vertex partitions of $K_{p,q}$ such that $u\in V_1$ and $v\in V_2$. Hence $d(u) = q$ and $d(v) = p$. The graph $K_{p,q}$ has $p+q$ vertices and pq edges. Then

$\sigma(u) = q + 2(p – 1)$ and $\sigma(v) = p + 2(q – 1)$.

Thus

$$S_2^a(K_{p,q})=\sum_{uv\in E(K_{p,q})}[\sigma(u)\sigma(v)]^a=[(q+2p-2)(p+2q-2)]^a\,pq.$$

We obtain the following results by using Theorem 6.

Corollary 6.1. If $K_{p,q}$ is a complete bipartite graph, then

i) $S_2(K_{p,q}) = pq(p + 2q – 2)(q + 2p – 2)$.

ii) $HS_2(K_{p,q}) = pq(p + 2q – 2)^2 (q + 2p – 2)^2$.

iii) $$PS(K_{p,q})=\frac{pq}{\sqrt{(p+2q-2)(q+2p-2)}}.$$

iv) $$RPS(K_{p,q})=pq\sqrt{(p+2q-2)(q+2p-2)}.$$

Proof: Put a =1, 2, –½, ½ in equation (10), we obtain the desired results.

3. Results for wheel graphs

A wheel W_n is the join of C_n and K_1. Clearly W_n has n+1 vertices and $2n$ edges. A graph W_4 is depicted in Figure 1.

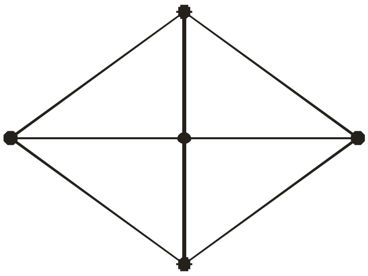

Figure 1. Wheel graph W_4

Let W_n be a wheel with n+1 vertices and $2n$ edges. In W_n, there are two types of edges as follows:

$E_1 = \{uv$ □ $E(W_n) \mid d_{W_n}(u) = d_{W_n}(v) = 3\}$, $|E_1| = n$.

$E_2 = \{uv$ □ $E(W_n) \mid d_{W_n}(u) = 3,\ d_{W_n}(v) = n\}$, $|E_2| = n$.

Thus there are two types of status edges as given in Table 1.

□(u), □(v) \ uv □ $E(W_n)$	$(2n-3, 2n-3)$	$(n, 2n-3)$
Number of edges	n	n

Table 1. Status edge partition of W_n

Theorem 7. The general first status index of a wheel graph W_n is

$$S_1^a(W_n) = (4n-6)^a\, n + (3n-3)^a\, n. \qquad (11)$$

Proof: By using equation (1) and Table 1, we deduce

$$S_1^a(W_n) = \sum_{uv \in E(W_n)} [\sigma(u) + \sigma(v)]^a$$

$$= (2n-3+2n-3)^a\, n + (n+2n-3)^a\, n = (4n-6)^a\, n + (3n-3)^a\, n.$$

Corollary 7.1. If W_n is a wheel graph with n+1 vertices and $2n$ edges, then

i) $S_1(W_n) = 7n^2 - 9n$.

ii) $HS_1(W_n) = 25n^3 - 66n^2 + 45n$.

iii) $SS(W_n) = \dfrac{n}{\sqrt{4n-6}} + \dfrac{n}{\sqrt{3n-3}}$.

Proof: Put a =1, 2, –½ in equation (11), we get the desired results.

Theorem 8. The general second status index of a wheel graph W_n is

$$S_2^a(W_n) = (2n-3)^{2a}\, n \times (2n^2 - 3n)^a\, n. \qquad (12)$$

Proof: By using equation (2) and by using Table 1, we derive

$$S_2^a(W_n) = \sum_{uv \in E(W_n)} [\sigma(u)\sigma(v)]^a$$

$$= [(2n-3)(2n-3)]^a\, n + [n(2n-3)]^a\, n = (2n-3)^{2a}\, n + (2n^2 - 3n)^a\, n.$$

From Theorem 8, we establish the following results.

Corollary 8.1. If W_n is a wheel graph with n+1 vertices and $2n$ edges, then

i) $S_2(W_n) = 6n^3 - 15n^2 + 9n$.

ii) $HS_2(W_n) = (2n-3)^4 n + (2n^2 - 3n)^2 n$.

iii) $PS(W_n) = \dfrac{n}{2n-3} + \dfrac{n}{\sqrt{n(2n-3)}}$.

iv) $RPS(W_n) = n(2n-3) + n\sqrt{n(2n-3)}$.

Proof: Put a =1, 2, –½, ½ in equation (12), we obtain the desired results.

4. Results for friendship graphs

A friendship graph F_n is the graph obtained by taking $n \geq 2$ copies of C_3 with a vertex in common. A graph F_4 is shown in Figure 2.

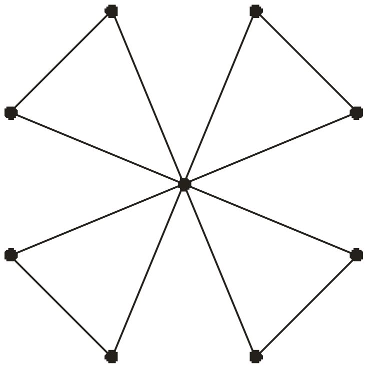

Figure 2. A friendship graph F_4

A friendship graph F_n has $2n+1$ vertices and $3n$ edges. There are two types of edges as follows:

$E_1 = \{uv \in E(F_n) \mid d_{F_n}(u) = d_{F_n}(v) = 2\}$, $\quad |E_1| = n$.

$E_2 = \{uv \in E(F_n) \mid d_{F_n}(u) = 2,\ d_{F_n}(v) = 2n\}$, $\quad |E_2| = 2n$.

Hence there are two types of status edges as given in Table 2.

$\sigma(u), \sigma(v) \setminus uv \in E(F_n)$	$(4n-2, 4n-2)$	$(2n, 4n-2)$
Number of edges	n	$2n$

Table 2. Status edge partition of F_n

Theorem 9. The general first status index of a friendship graph F_n is

$$S_1^a(F_n) = (8n-4)^a n + (6n-2)^a 2n. \tag{13}$$

Proof: By using equation (1) and Table 2, we obtain

$$S_1^a(F_n) = \sum_{uv \in E(F_n)} [\sigma(u) + \sigma(v)]^a$$

$$= (4n-2+4n-2)^a n + (2n+4n-2)^a 2n = (8n-4)^a n + (6n-2)^a 2n.$$

Corollary 9.1. Let F_n be a friendship graph with $2n+1$ vertices and $3n$ edges. Then

i) $S_1(F_n) = 20n^2 - 8n$.

ii) $HS_1(F_n) = 100n^3 - 88n^2 + 20n$.

iii) $SS(F_n) = \dfrac{n}{2\sqrt{2n-1}} + \dfrac{2n}{\sqrt{6n-2}}$.

Proof: Put $a = 1, 2, -½$ in equation (13), we get the desired results.

Theorem 10. The general second status index of a friendship graph F_n is

$$S_2^a(F_n) = (4n-2)^{2a} n + (8n^2 - 4n)^a 2n. \tag{14}$$

Proof: From equation (2) and by using Table 2, we have

$$S_2^a(F_n) = \sum_{uv \in E(F_n)} [\sigma(u)\sigma(v)]^a$$

$$= [(4n-2)(4n-2)]^a n + [2n(4n-2)]^a 2n = (4n-2)^{2a} n + (8n^2 - 4n)^a 2n.$$

We obtain the following results by using Theorem 10.

Corollary 10.1. Let F_n be a friendship graph with $2n+1$ vertices and $3n$ edges. Then

i) $S_2(F_n) = 32n^3 - 24n^2 + 4n$.

ii) $HS_2(F_n) = (4n-2)^4 n + (8n^2 - 4n)^2 2n$.

iii) $PS(F_n) = \frac{n}{4n-2} + \frac{n}{\sqrt{n(2n-1)}}.$

iv) $RPS(F_n) = n(4n-2) + 4n\sqrt{n(2n-1)}.$

Proof: Put *a* =1, 2, –½, ½ in equation (14), we obtain the desired results.

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