Characterization of Einstein-Finsler Space With Special **(α; β)**-Metric

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Abstract *-In theory of relativity Einstein-Finsler metrics are very useful to study geometric structure of space-time and to build applications. In order to characterize Einstein-Finsler (α, β)-metrics, it is necessary to compute the Riemann curvature and the Ricci curvature. In this paper, we consider the special (α, β)-metric and obtained the Riemann curvature. Then we characterize the Einstein criterion for that metric, when β is a constant killing form. Further, we proved that the metric is Riemannian.*

Keywords — (α, β)*-metrics, Riemannian curvature, Ricci curvature, Einstein Finsler space.*

I. INTRODUCTION

In Finsler geometry Einstein metrics form an major focus to study the applications in general relativity. Einstein metrics are solutions of Einstein Field equations in general relativity. In Finsler space, Riemannian curvature $R_y: T_xM \to T_xM$ is given by $R_y(u) = R^i_k(y)u^k \frac{\partial x}{\partial x^i}$. By this curvature, Ricci scalar defined by $Ric(x, y) = R^i_k$. A Finsler metric is Einstein if the Ricci scalar is of the form $Ric = c(x)F^2(x, y)$for some function $c$on manifold M, i.e., the Ricci scalar is a function of x alone [3]. If Ricci tensor vanishes then the manifold is called Ricci flat, which represents vacuum solution to Einstein field equations in relativity [4].

A Finsler space is a manifold M together with positively homogeneous metric function $L(x, y)$. (α, β)-metrics are the special class of Finsler metrics which having a major role in formulating applications in Einstein theory of relativity, Mechanics, Biology, control theory, etc., [1,2,11]. C. Robles invented Randers Einstein metrics in 2003, she derived the necessary and sufficient conditions for Randers metrics to be Einstein. In [12], authors proved the Einstein Schur type lemma for (α, β)-metrics. In [7], Cheng, Shen and Tian, proved (α, β)-metric is Ricci flat. In [14], authors classified the projectively related Einstein Finsler metrics over compact manifold.

In [12], Razaei, Razavi and Sadeghazadeh, consider the (α, β)-metrics such as generalized Kropina metric, Matsumoto metric with β a constant killing form and obtained the necessary and sufficient conditions to be Einstein metrics. In 2012 [10], Rafie and Rezaei proved that the second Schur type lemma for Finsler-Matsumoto metric. Then, Narasimhamurthy, Ajith and Mallikarjun, worked on some Einstein-Finsler special (α, β)-metrics, with β a constant killing form and proved that the space is Ricci flat.

In this paper we study the Einstein criterion for Finsler spaces with special (α, β)-metrics. We consider the special (α, β)-metric $L = \alpha + \beta - \frac{\beta^2}{\alpha}$, where α is the Riemannian metric, β is a constant killing form. Then we find the Riemannian curvature for that metric and we obtained the necessary and sufficient condition for them to be Einstein metric, when β as a constant killing form. Further, we occur at the conclusion lemma that the above mentioned metric is Ricci flat. In the entire paper we use the Einstein convention.

II. PRELIMINARIES

Let $F^n(M, L)$ be a Finsler space, where M be an n-dimensional connected C^∞-manifold and L is a Finsler metric defined on the manifold M as a function $L: TM \to [0, 1)$, where $TM = \cup_{x \in M} T_xM$ is the tangent bundle and T_xM in the tangent space at $x \in M$ and L satisfies the following properties: (i)Regular: L is C^∞ in $TM\backslash 0$, (ii)Positive homogeneous: $L(x,\ \lambda y) = \lambda L(x,\ y)$, where λ is any scalar, $x \in M,\ y \in T_x M$, (iii)Positive definiteness: For any tangent vector $y \in T_x M$, the y-Hessian matrix $g_{ij}(x,\ y) = \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite [10].

Matsumoto introduced the class of $(\alpha,\ \beta)$-metrics [9]. An $(\alpha,\ \beta)$-metric is a scalar function L on TM defined by $L = \alpha\,\varphi(s)$, where $s = \frac{\beta}{\alpha}$. Here $\varphi = \varphi(s)$ is a C^∞ on $(-b_0,\ b_0)$ with certain regularity, α is a Riemannian metric and β ia a one form on M. Denote the Levi-Civita connection of α by ∇. Recall some geometric quantities of $(\alpha;\ \beta)$-metric:

$$r_{ij} = \nabla_j b_i + \nabla_i b_j;\ s_{ij} = \nabla_j b_i - \nabla_i b_j,$$
$$r^i_j = a^{ik} r_{kj}, \qquad r_{00} = r_{ij} y^i y^j, \qquad r = r_{i,j} b^i b^j, \tag{2.1}$$
$$s^i_j = a^i_k s^k_j;\ s_j = b^i s_{ij};\ s_0 = s_i y^i;\ B = b^i b_j.$$

For a Finsler metric geodesic spray is defined by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,\ y)\frac{\partial}{\partial y^i}$, for which spray coefficients G^i is given by,

$$G^i(x,y) = \frac{1}{4} g^{ij}(x.\ y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\} y^j y^k, \tag{2.2}$$

where $(g^{ij})$is the inverse matrix of (g_{ij}). For the Berwald connection the coefficients $G^i_j,\ G^i_{jk}$ of spray G^i defined as,

$$G^i = \frac{\partial G^i}{\partial y^j}. \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

In Finsler geometry, Riemannian curvature tensor R_y is the function $R_y = R^i_k(y) dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_x M \to T_x M$ is defined as,

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^i \partial y^k} y^j + 2G^j \frac{\partial G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k} \tag{2.3}$$

If $L = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric, then $R = R^i_{jkl}(x) y^j y^l$, where $R^i_{jkl}(x)$ denote the coefficients of Riemannian curvature tensor. Thus, R_y is called Riemannian curvature in Finsler geometry. With respect to the Riemannian curvature, Ricci scalar function for the Finsler metric defined by $\rho = \frac{1}{L^2} R^i_i$, which is positive homogeneous function of degree 0 in y. It shows that $\rho(x,\ y)$ depends on the direction of the flag pole y, but not its length. Then the Ricci tensor given by,

$$Ric_{ij} = \left\{\frac{1}{2} R^i_j\right\}_{y^i y^j}. \tag{2.4}$$

Suppose the Ricci tensor on a manifold becomes zero, then such manifold called as Ricci-flat [3].

The Ricci tensor plays major role Finsler geometry to study the Einstein criterion for Finsler spaces. A Finsler metric becomes Einstein metric if the Ricci scalar function is a function of x-alone. i.e.,

$$Ric_{ij} = \rho(x) g_{ij}. \tag{2.5}$$

Let (*M; L*) be an *n*-dimensional Finsler space equipped with an (*α; β*)-metric *L*, where $\alpha(y) = \sqrt{a_{ij}y^iy^j}$, $\beta(y) = b_i(x)y^i$. In [8] M. Matsumoto, proved that G^i of (*α; β*)-metric space are given by,

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.6}$$

where,

$$B^i = \left(\frac{E}{\alpha}\right)y^i = \left(\alpha\frac{L_\beta}{L_\alpha}\right)s^i_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)C\left\{\left(\frac{y^i}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right)b^i\right\}, \tag{2.7}$$

$$E = (\beta L_\beta/L)C; \;\; C = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})};$$

$$b^i = a^{ir}b_r; \;\; b^2 = b^r b_r; \;\; \gamma^2 = b^2\alpha^2 - \beta^2;$$

$$r_{ij} = \frac{1}{2}\left(b_{i/j} + b_{\frac{j}{i}}\right); \;\; s_{ij} = \frac{1}{2}\left(b_{i/j} - b_{j/i}\right);$$

$$s_ij = a^{ih}s_{hj}, \;\; s_j = b_i s^i_j.$$

Where "**|**"in the above formula stands for the *h*-covariant derivation with respect to the Riemannian connection in the space (M, α), and the matrix (a^{ij}) denotes the inverse of matrix (a_{ij}). The functions γ^i_{jk} stand for the Christoffel symbols in the space (*M; α*). Now (2.3) is re-written as,

$$B^i = (\tilde{p}r_{00} + \tilde{q}_0 s_0)y^i + \tilde{r}s^i_0 + (\tilde{s}_0 r_{00} + \tilde{t}s_0)b^i \tag{2.8}$$

where

$$\tilde{p} = \frac{\beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})}{2L(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})} \tag{2.9}$$

$$\tilde{q} = -\frac{\alpha\beta L_\beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})}{L L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})} \tag{2.10}$$

$$\tilde{r} = \frac{\alpha L_\beta}{L_\alpha} \tag{2.11}$$

$$\tilde{s}_0 = \frac{\alpha^3 L_{\alpha\alpha}}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})} \tag{2.12}$$

$$\tilde{t} = -\frac{\alpha^4 L_{\alpha\alpha}L_\beta}{L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})} \tag{2.13}$$

Substituting (2.4) in (2.2) and (2.1), we obtain Berwalds formula for Riemannian curvature tensor as follows:

$$K^i_k(y) = \bar{K}^i_k + \left\{2B^i_{|k} - y^j\left(B^i_{|j}\right)_{y^k} - (B^i)_{y^j}(B^j)_{y^k} + 2B^j(B^i)_{y^jy^k}\right\} \tag{2.14}$$

The 1-form β is said to be Killing (closed) 1-form if $r_{ij} = 0$ ($s_{ij} = 0$ respectively). β is said to be a constant Killing form if it is Killing and has constant length with respect to α, equivalently $r_{ij} = 0; s_i = 0$.

III. RIEMANNIAN CURVATURE OF FINSLER SPACE WITH SPECIAL (*A; B*)-METRICS

In this section, we consider Finsler space with special (α; β)-metric $L = \alpha + \beta - \frac{\beta^2}{\alpha}$, then we derive the Riemannian curvature. For this metric partial derivatives with respect to both α and β respectively given by,

$$L_\alpha = 1 + \frac{\beta^2}{\alpha^2}; \qquad L_\beta = 1 - \frac{2\beta}{\alpha} \tag{3.1}$$

Now by equation (2.11) we have,

$$\tilde{r} = \frac{(\alpha - 2\beta)\alpha^2}{\alpha^2 + \beta^2} \tag{3.2}$$

Suppose that β is a constant Killing form, then by substituting (3.2) in (2.8), we get

$$B^i = \frac{\alpha^3 - 2\alpha^2\beta}{\alpha^2 + \beta^2} s_0^i \tag{3.3}$$

Now, by Covariant and contravariant differentiation of (3.3), we obtained that,

$$B^i_{\{.j\}} = \frac{C_1 y_j}{(\alpha^2+\beta^2)^2} s_0^i - \frac{C_2 b_j}{(\alpha^2+\beta^2)^2} s_0^i + \alpha^3 - \frac{2\alpha^2\beta}{\alpha^2+\beta^2} s_j^i \tag{3.4}$$

$$B^i_{\{|j\}} = \frac{C_3 b_{0|j}}{(\alpha^2+\beta^2)^2} s_0^i + \frac{\alpha^3 - 2\alpha^2\beta}{\alpha^2+\beta^2} s^i_{0|j} \tag{3.5}$$

where

$$B^i_{:j} = B^i_{y^j}$$

$$C_1 = \frac{1}{2}\alpha^3 + \frac{3}{2}\alpha\beta^2 - 2\beta^3$$

$$C_2 = 2\alpha^4 + 2\alpha^3\beta - 2\alpha^2\beta^2$$

$$C_3 = -2\alpha^3(\alpha + \beta)$$

From (3.4), we have

$$B^i B^i_{.j.i} = 0, \tag{3.6}$$

$$B^i_{.j} B^j_{.i} = \frac{\alpha^5 + 3\alpha^3\beta^2 - 4\alpha^2\beta^3}{2(\alpha^2+\beta^2)^4} s_0^i s_{i0} + \frac{(\alpha^3 - 2\alpha^2\beta)^2}{\left(\alpha^2+\beta^2\right)^2} s^{ij} s_{ij}. \tag{3.7}$$

And differentiate (3.5) with respect to yi and transvecting by yj, we get

$$y^j \left(B^i_{|j}\right)_{.i} = 0 \tag{3.8}$$

Finally by substituting (3.4) to (3.8) in Berwald's formula (2.14), we obtain,

$$R^i_i = \bar{R}^i_i + \left\{2B^i_{|i} - y^j\left(B^i_{|j}\right)_{.i} - B^i_{.j}B^j_{.i} + 2B^j\left(B^i\right)_{y^i y^j}\right\}$$

$$= \bar{R}^i_i + \frac{2(\alpha^3 - 2\alpha^2\beta)}{\alpha^2+\beta^2} s^i_{0|j} - \frac{\alpha^5 + 3\alpha^3\beta^2 - 4\alpha^2\beta^3}{2(\alpha^2+\beta^2)^4} s_0^i s_{i0} - \frac{(\alpha^2 - 2\alpha^2\beta)^2}{(\alpha^2+\beta^2)^2} s^{ij} s_{ij} \tag{3.9}$$

Where $\bar{R}^i_i$ is the Riemannian curvature of the Finsler space, thus we state the following;

Theorem 3.1. *The Riemannian curvature of the Finsler space with special $(\alpha;\beta)$-metric $L = \alpha + \beta - \frac{\beta^2}{\alpha}$, with β as constant Killing form, is given in the equation (3.9).*

IV. EINSTEIN CONDITION FOR FINSLER SPACE WITH SPECIAL (A; B)-METRICS

In this section we consider the Finsler space with special (α; β)-metric $L = \alpha + \beta - \frac{\beta^2}{\alpha}$ and characterize the Einstein criterion. A Finsler metric $L = L(x, y)$ on an n-dimensional manifold M is called an Einstein metric if the Ricci scalar satisfies the following condition,

$$Ric = (n-1)\lambda L^2, \tag{4.1}$$

where $\lambda = \lambda(x)$ is a scalar function on M. L is Ricci constant if λ is constant[2-4]. Now, we suppose the Ricci scalar of the mentioned (α; β)-metric is the function of x alone, i.e., L is Einstein, then we have $L^2 Ric(x) = R^i_i$, so we can derive the necessary and sufficient conditions for this to be Einstein.

From(3.9), we have,

$$0 = \overline{Ric}_{00} + \frac{2(\alpha^3 - 2\alpha^2\beta)}{\alpha^2+\beta^2} s^i_{0/j} - \frac{\alpha^5 + 3\alpha^3\beta^2 - 4\alpha^2\beta^3}{2(\alpha^2+\beta^2)^4} s_0^i s_{0i}$$

$$-\frac{(\alpha^2-2\alpha^2\beta)^2}{(\alpha^2+\beta^2)^2}s^{ij}s_{ij}-\left(\alpha+\beta-\frac{\beta^2}{\alpha}\right)^2 Ric(x) \tag{4.2}$$

Multiplying (4.3) by 2α2(α2 + β2)4 removes y from the denominators and after simplification we obtained as follows:

$$Rat+\alpha\,Irrat=0,$$

where

$$\begin{aligned}Rat &= (2\alpha^{10}+8\alpha^8\beta^2+12\alpha^6\beta^4+8\alpha^4\beta^6+2\alpha^2\beta^8)Ric_{00}+4\alpha^4\beta^3 s_0^i s_{0i}\\ &\quad-(8\alpha^{10}\beta+24\alpha^8\beta^3+24\alpha^6\beta^5+8\alpha^4\beta^7)s_{0/i}-\left(2\alpha^{12}+12\alpha^1 0^{\beta^2}+18\alpha^8\beta^4+8\alpha^6\beta^6\right)s^{ij}s_{ij}\\ &\quad-2(\alpha^{12}+5\alpha^{10}\beta^2+\alpha^8\beta^4+2\alpha^6\beta^6+3\alpha^4\beta^8+3\alpha^2\beta^{10}+\beta^{12})Ric(x)\\ Irrat &= (4\alpha^{10}+12\alpha^8\beta^2+12\alpha^6\beta^4+4\alpha^4\beta^6)s^i_{0ji}\\ &\quad-(\alpha^6+3\alpha^4\beta^2)s_0^i s_{0i}+(8\alpha^{10}\beta+16\alpha^8\beta^3+8\alpha^6\beta^5)s^{ij}s_{ij}\\ &\quad-2(2\alpha^{10}\beta+6\alpha^8\beta^3+4\alpha^6\beta^5-4\alpha^4\beta^7-6\alpha^2\beta^9-2\alpha\beta^{11})Ric(x.)\end{aligned} \tag{4.3}$$

Here Rat and Irrat are polynomials of degree 12 and 10 in y respectively. According to the above we state the Einstein criterion as follows,

Lemma 4.1. A Finsler space with special (α; β)-metric $L=\alpha+\beta-\frac{\beta^2}{\alpha}$ with constant Killing form β is Einstein if and only if both $Rat = 0$ and $Irrat = 0$ hold.

Proof: Let *Rat* = *P*(*y*) and *Irrat* = *Q*(*y*). We know that *α* can never be polynomial in *y*. Otherwise, the quadratic $\alpha^2 = a_{ij}(x)y^i y^j$ would have been factored into linear term. It's zero set would then consist of a hyperplane, contradicting the positive definiteness of a_{ij}. Now, suppose the polynomial *Rat* is not zero. Then the above equation would imply that it is the product of polynomial *Irrat* with a non-polynomial factor *α*, this is not possible. So *Rat* must must vanish and, since *α* is positive at all *y* 6= 0, we see that *Irrat* also must be zero. Hence the proof.

Now if *Rat* = 0, then we have

$$0=\alpha^2 C_1 + C_2 \tag{4.4}$$

Where C_1 and C_2 are as follows:

$$\begin{aligned}C_1 &= (2\alpha^8+8\alpha^6\beta^2+12\alpha^4\beta^4+8\alpha^2\beta^6+2\beta^8)\overline{Ric_{00}}\\ &\quad-4\alpha^2\beta^3 s_0^i s_{0i}-(8\alpha^8\beta+24\alpha^6\beta^3+24\alpha^4\beta^5+8\alpha^2\beta^7)s^i_{0|j}\\ &\quad-(2\alpha^{10}+12\alpha^8\beta^2+18\alpha^6\beta^4+8\alpha^4\beta^6)s^{ij}s_{ij}\\ &\quad-2(\alpha^{10}+5\alpha^8\beta^2+\alpha^6\beta^4+2\alpha^4\beta^6+3\alpha^2\beta^8+3\beta^{10})Ric(x)\\ C_2 &= -2\beta^{12}Ric(x).\end{aligned}$$

Thus, by (4.4) we conclude that α^2 divides C_2 and so $\beta = 0$. Then the Finsler metric is Riemannian. Thus we state that,

Theorem 4.2. *An Einstein Finsler metric* $L=\alpha+\beta-\frac{\beta^2}{\alpha}$ *with constant killing form β. Then L is Einstein if and only if L is Riemannian Einstein metric, i.e., Ricci flat.*

V. CONCLUSION

The Einstein metrics comprise a major focus in differential geometry and mainly connect with gravitation in general relativity. In particular, Einstein metric are solutions to Einstein field equations in general relativity containing the Ricci-flat metric. Einstein Finsler metric which represent a non Riemannian stage for the extensions of metric gravity provide an interesting source of geometric issues and the (α; β)-metric is an important class of Finsler metric appearing frequently in the study of applications in Physics.

In this paper we consider a special (α; β)-metric such as $L=\alpha+\beta-\frac{\beta^2}{\alpha}$. For this (α; β)-metric, we obtain Riemannian curvature. Further we find the necessary and sufficient conditions for this (α; β)-metric to be Einstein metric, when β is a constant Killing form. Finally we prove that the above mentioned Einstein metric must be Riemannian or Ricci flat.

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