# Finding Some General Exact Solutions to the Non-Linear Monge-Ampere Equations (PDEs)

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***Abstract** - We look at general exact solutions for non-linear Monge-Ampere partial differential equations (PDEs). They are nonlinear elliptic PDEs which typically model surfaces with curvature from differential geometry, optics and other applications of this used by physicists as well as applied and industrial mathematicians.*

**Keywords —** *Monge, Ampere, partial, differential equations, PDEs, differential geometry, elliptic, solution, optics, math, applied, calculus*

We start off with the simplest case for a non-homogeneous Monge-Ampere PDE, using a constant A.

**Definition 1**

$$(u_{xy})^2 - u_{xx}u_{yy} = A, A = a^2 > 0$$

$$u = \pm\frac{\sqrt{A}}{C_2}x(C_1x + C_2y) + \phi(C_1x + C_2y) + C_3x + C_4y$$

**Example 1**

$$(u_{xy})^2 - u_{xx}u_{yy} = 9, 9 = 3^2 > 0$$

$$u = \pm 3C_2x(C_1x + C_2y) + \phi(C_1x + C_2y) + C_3x + C_4y$$

Next, we look at an arbitrary function f

**Definition 2**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(x)$$

$$u = \pm\int\sqrt{f(x)}dx + \phi(x) + C_1y$$

**Example 2.1**

$$(u_{xy})^2 - u_{xx}u_{yy} = 4e^{6x}$$

$$u = \pm\int\sqrt{4e^{6x}}dx + \phi(x) + C_1y$$

$$u = \pm\frac{2\sqrt{e^{6x}}}{3} + C_2 + \phi(x) + C_1y$$

We now take an arbitrary functional and multiply it by the independent variable y.

**Definition 3**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(x)y$$

$$u = C_1y^3 - \frac{1}{6C_1}\int_a^x (x-t)f(t)dt + C_2x + C_3y + C_4$$

**Example 3.1**

$$(u_{xy})^2 - u_{xx}u_{yy} = ln(x)y$$

$$u = C_1y^3 - \frac{1}{6C_1}\int_0^x (x-t)ln(t) + C_2x + C_3y + C_4$$

$$u = C_1y^3 - (\frac{1}{12C_1}x(x-2t)ln(t)) + C_2x + C_3y + C_4$$

**Example 3.2**

$$(u_{xy})^2 - u_{xx}u_{yy} = 3x^3y$$

$$u = C_1y^3 - \frac{1}{6C_1}\int_0^x (x-t)3t^3dt + C_2x + C_3y + C_4$$

$$u = C_1y^3 - \frac{1}{4C_1}t^3x(x-2t) + C_2x + C_3y + C_4$$

Now, the independent variable y is squared times the arbitrary function f in the following.

**Definition 4**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(x)y^2$$

$$u = C_1y^4 - \frac{1}{12C_1}\int_a^x (x-t)f(t)dt + C_2x + C_3y + C_4$$

**Example 4.1**

$$(u_{xy})^2 - u_{xx}u_{yy} = \frac{1}{x-t}y^2$$

$$u = C_1y^4 - \frac{1}{12C_1}\int_a^x (x-t)\frac{1}{x-t}dt + C_2x + C_3y + C_4$$

$$u = C_1y^4 - \frac{1}{12C_1}\int_a^x dt + C_2x + C_3y + C_4$$

$$u = C_1y^4 - \frac{1}{12C_1}\int_0^x dt + C_2x + C_3y + C_4$$

$$u = C_1 y^4 - \frac{1}{12C_1}x + C_2 x + C_3 y + C_4$$

Now, the independent variable y is to the nth power in the following.

**Definition 5**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(x)y^k$$

$$u = \frac{C_1 y^{k+2}}{(k+1)(k+2)} - \frac{1}{C_1}\int_a^x (x-t)f(t)dt + C_2 x + C_3 y + C_4$$

**Example 5.1**

$$(u_{xy})^2 - u_{xx}u_{yy} = x^2 y^5$$

$$u = \frac{C_1 y^7}{42} - \frac{1}{C_1}\int_a^x (x-t)t^2 dt + C_2 x + C_3 y + C_4$$

$$u = \frac{C_1 y^7}{42} - \frac{1}{C_1}[\frac{x^4}{3} - \frac{x^4}{4}] + C_2 x + C_3 y + C_4$$

**Example 5.2**

$$(u_{xy})^2 - u_{xx}u_{yy} = \frac{y^5}{x-t}$$

$$u = \frac{C_1 y^7}{42} - \frac{1}{C_1}\int_0^x dt + C_2 x + C_3 y + C_4$$

$$u = \frac{C_1 y^7}{42} - \frac{x}{C_1} + C_2 x + C_3 y + C_4$$

The following M-A Equations equal a function in the form of Ax to the n power times y to the k power.

**Definition 6**

$$(u_{xy})^2 - u_{xx}u_{yy} = Ax^n y^k$$

$$u = \frac{C_1 x^{(n+2)}}{(n+1)(n+2)} - \frac{Ay^{k+2}}{C_1(k+1)(k+2)} + C_2 y + C_3 x + C_4$$

**Example 6**

$$(u_{xy})^2 - u_{xx}u_{yy} = 25x^3 y^4$$

$$u = \frac{C_1 x^5}{20} - \frac{25y^6}{30C_1} + C_2 y + C_3 x + C_4$$

Next, we have the following definition and example that multiplies an arbitrary function f(x) with an exponential function to the power of lamda times y.

**Definition 7**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(x)e^{\lambda y}$$

$$u = C_1\int_a^x (x-t)f(t)dt + C_2x - \frac{1}{C_1\lambda^2}e^{\lambda y} + C_3y + C_4$$

**Example 7.1**

$$(u_{xy})^2 - u_{xx}u_{yy} = 3x^2e^{3y}$$

$$u = C_1\int_0^x (x-t)3t^2dt + C_2x - \frac{1}{9C_1}e^{3y} + C_3y + C_4$$

$$u = 3C_1[\frac{x^4}{3} - \frac{x^4}{4}] + C_2x - \frac{1}{9C_1}e^{3y} + C_3y + C_4$$

Now, we have two arbitrary functions f(x) times g(y) plus some arbitrary constant squared.

**Definition 8**

$$(u_{xy})^2 - u_xu_y = f(x)g(y) + A^2$$

$$u = C_1\int_0^x (x-t)f(t)dt - \frac{1}{C_1}\int_0^y (y-\xi)g(\xi)d\xi \pm Axy + C_2x + C_3y + C_4$$

**Example 8**

$$(u_{xy})^2 - u_xu_y = x^2y^3 + 9$$

$$u = C_1\int_0^x (x-t)t^2dt - \frac{1}{C_1}\int_0^y (y-\xi)\xi^3d\xi \pm 3xy + C_2x + C_3y + C_4$$

$$u = C_1[\frac{x^4}{3} - \frac{x^4}{4}] - \frac{1}{C_1}[\frac{y^5}{4} - \frac{y^5}{5}] \pm 3xy + C_2x + C_3y + C_4$$

Now, we have M-A Equation that equals some arbitrary function f that has the form of ax+by:

**Definition 9**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(ax + by)$$

$$u = \pm\frac{x}{b}\int\sqrt{f(z)}dz + \phi(z) + C_1x + C_2y$$

**Example 9**

$$(u_{xy})^2 - u_{xx}u_{yy} = (3x+5y)^2$$

$$u = \pm\frac{x}{5}\frac{z\sqrt{z^2}}{2} + C_3 + \phi(z) + C_1x + C_2y$$

$$u = \pm\frac{xz\sqrt{z^2}}{10} + C_3 + \phi(z) + C_1x + C_2y$$

$$z = 3x + 5y$$

$$u = \pm\frac{x(3x+5y)\sqrt{(3x+5y)^2}}{10} + C_3 + \phi(3x+5y) + C_1x + C_2y$$

For the rest of this article, we look at other variations of some arbitrary function in varying forms.

**Definition 10**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(x^2+y^2)$$

$$u = C_1\phi \pm \int \sqrt{C_2 - 2F(r) - C_1^2r^{-2}}dr + C_3$$

$$F(r) = \int rf(r)dr$$

$$r = \sqrt{x^2+y^2}, tan\phi = y/x$$

**Example 10**

$$(u_{xy})^2 - u_{xx}u_{yy} = 4(x^2+y^2)$$

$$F(r) = \int r4r^2dr = r^4 + C$$

$$u = C_1\phi \pm \int \sqrt{C_2 - 2F(r) - C_1^2r^{-2}}dr + C_3$$

$$u = C_1\phi \pm \int \sqrt{C_2 - 2(r^4 + C_4) - C_1^2r^{-2}}dr + C_3$$

**Definition 11**

$$(u_{xy})^2 - u_{xx}u_{yy} = \frac{1}{x^4}f(\frac{y}{x})$$

$$u = x\phi(\frac{y}{x}) \pm \int \sqrt{f(z)}dz + C, z = y/x$$

**Example 11**

$$(u_{xy})^2 - u_{xx}u_{yy} = \frac{1}{x^4}(\frac{y}{x})^3$$
$$u = x\phi(\frac{y}{x}) \pm \int \sqrt{z^3}dz + C, z = y/x$$

$$u = x\phi(z) \pm \frac{2z\sqrt{z^3}}{5} + C_2 + C_1$$

$$u = x\phi(\frac{y}{x}) \pm \frac{2y\sqrt{\frac{y}{x}^3}}{5x} + C_2 + C_1$$

**Definition 12**

$$(u_{xy})^2 - u_{xx}u_{yy} = f(ax - by^2)$$

$$u = \pm \int \sqrt{F(z) - C_1}dz + C_2x + C_3y + C_4$$

$$F(z) = \frac{1}{a^2b}\int f(z)dz, z = ax - by^2$$

**Example 12**

$$(u_{xy})^2 - u_{xx}u_{yy} = 3(5x - 7y^2)$$

$$z = 5x - 7y^2, f(z) = 3z$$

$$F(z) = \frac{1}{5^27}\int 3zdz = \frac{3}{350}z^2 + C$$

$$u = \pm \int \sqrt{F(z) - C_1}dz + C_2x + C_3y + C_4$$

$$u = \pm \int \sqrt{\frac{3}{350}z^2 + C_5 - C_1}dz + C_2x + C_3y + C_4$$

$$u = \pm\frac{1}{10}z\sqrt{25C_5 - C_1 + \frac{3z^2}{14}} - 5\sqrt{\frac{7}{6}}(C_1 - C_5)ln(\sqrt{9z^2 - 1050(C_1 - C_5)} + 3z) + C_6 + C_2x + C_3y + C_4$$

**AUTHOR BIO**

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