Solution of Differential Equations of the First Order via Different Order of Runge-Kutta method with it's Application

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Abstract: *This analysis is related to Runge-Kutta (R-K) approach of upper order and an application to resolve the problem with initial value of ordinary differential equations (ODE) with 1st order. The second order derivation, fourth order and sixth order Runge-Kutta process were first performed. MATLAB code was subsequently applied to solve the problem numerically for different order R-K method and calculating error with the analytical value. Then we have given the graphical representation different order of R-K method as well as an analytical solution for various problems.*

Comparing approximate values from the different order of R-K method with the analytical values in tabular form and graphical representation, we can observe that the R-K method of 6th order gives better precision.

Keywords: *Runge-Kutta (R-K) method, ordinary differential equations, initial value problem*

I. Introduction

Differential equations (DE) are often used in the field of science and engineering for mathematical modeling In the shape of differential equations, many physical problems can be implemented. Solving differential equations that solve problems in most real-life situations is very complicated. In order to solve accurately different types of ODE, several different methods were suggested and used.

There are several well-known and used methods of numerical integration. To solve the ordinary differential equations (ODE), a numerical approximation is used. The best known and most commonly used method of incorporating algorithms from the R-K family.

The method's name R-K is given by the name of two German mathematicians, Wilhelm Kutta & Carl Runge. The methods were devised by Runge in 1894 and later extended by Kutta in 1901. Carl Runge has fabricated numerical solutions to differential equations which emerged in his atomic spectrum research.

It is less efficient to solve numerical differential equations using Euler's method and is not very useful in practical problems since it require a very small step length h for obtaining a reasonable accuracy. Of course one might argue that higher order terms in Taylor's expansion could also be considered for great accuracy. But then it needs obtaining higher order total derivatives of y(x). The rank of R-K methods is higher than previous methods in order to achieve greater solvent precision while simultaneously avoiding the need for derivatives of higher order The Runge-Kutta methods were designed to be more precise with the advantage of having only the operational values on the sub-interval at some selected points. With a definite set of rules, the function increments are determined for a given equation once. The first increase estimate is exactly the same as any other increase.

Our aim is to show the difference between the different order Runge-Kutta methods. And find out which one is giving better accuracy, especially the sixth order Runge-Kutta method.

II. Preliminaries

1. ***Definition of Differential Equations (DE):***

Differential Equations are equations consisting of derivatives with one or more variables.

The following is the general shape of the differential equation:

$$\mathbf{a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots\ldots\ldots\ldots + a_{n-1}(x)y^{(1)} + a_n(x)y = F(x)}$$

2. ***Differential Equations Classification***

Based on the number of variable's independency classification of differential equations into two groups:

(a) Ordinary Differential Equations (ODE)

(b) Partial Differential Equations (PDE)

2.1. ***Ordinary Differential Equations (ODE):***

"A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation."

$$\phi(x, y, y^{(1)}, y^{(2)}, -------- y^{(n-1)}, y^{(n)}) = 0$$

This is an ordinary differential equation in its general form

1.2. Partial Differential Equations (PDE)

"A differential equation involving partial derivative of one or more dependent variables with respect to two or more independent variable is called partial differential equation."

$$\psi(z, x_1, x_2, x_3, \ldots\ldots\ldots, x_n, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots\ldots\ldots \frac{\partial z}{\partial x_n}, \frac{\partial^2 z}{\partial x_1 \partial x_2}, \frac{\partial^2 z}{\partial x_2 \partial x_3} \ldots\ldots) = 0$$

This is the partial differential equation in its general form

3. Order of Differential Equations:

The peak derivative is the order of a differential equation

$$\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} = e^x$$

In the above equation, the highest derivative is 3. hence, the third order of DE is 3.

4. ***What's degree of Differential Equations?***

This is represented by the power of the derivative of the maximum order in the differential equation.

$$\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^4 = k^2\left(\frac{d^3y}{dx^3}\right)^2$$

This equation's order is 3 and the degree is 2 because the peak derivative is order 3 and The maximum derivative exponent is 2.

5. ***Linearity of DE's:***

It is possible to divide differential equations into two classes based on linearity.

(a) Linear Differential Equations.

(b) Non-linear Differential Equations.

5.1. Linear DE's:

A linear differential equation is one where any dependent variables and their derivatives appear to the first power only.

$$y'' - 3y' + y = 0$$

5.2 ***Non-linear equations with differentials***

In the context of all their dependent variables, a differential equation; which is not linear, is usually said to be non-linear.

$$xy'' + x(y')^2 + y = 0$$

6. Initial Value Problems:

To solve a D.E, integration constants must arise. To find the values of integration constants, additional equations are required. Such additional equations are called initial conditions. Any differential equation includes initial conditions are called initial value problems

For example- $xy'' - y' + xy = e^x \ ; \ \ y(1) = 1, \ y'(2) = -1$

III. Runge-Kutta Method

There are many different schemes for numerically resolving ordinary differential equations. One of the standard workhorses called the Runge-Kutta method for solving ODE. In R-K method, with a definite set of rules, the function increments are determined for a given equation once. The first increase estimate is exactly the same as any other increase. The improved values of independent and dependent variables are to be substituted in a set of recursive formulae.

In order to obtain an approximate solution with numerical value of differential equation of the 1st order, we will use the formulae of Runge-Kutta method

$y' = f(x,y)$ With the initial condition $f(x_0) = y_0$

1. ***Runge-Kutta Method of 2nd Order:***

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1) \qquad\qquad y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

.2. R-K method of Order 4:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

3. R-K method of 6th Order

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$k_3 = hf(x_n + h/2, y_n + \{3k_1 + k_2\}/8)$$

$$k_4 = hf(x_n + 2h/3, y_n + \{8k_1 + 2k_2 + 8k_3\}/27)$$

$$k_5 = hf(x_n + (7-(21)^{1/2})h/14, y_n + \{3(3(21)^{1/2} - 7)k_1 - 8(7-(21)^{1/2})k_2 + 48(7-(21)^{1/2})k_3 - 3(21-(21)^{1/2})k_4\}/392)$$

$$k_6 = hf(x_n + (7+(21)^{1/2})h/14, y_n + \{-5(231+51(21)^{1/2})k_1 - 40(7+(21)^{1/2})k_2 - 320(21)^{1/2}k_3 + 3(21+121(21)^{1/2})k_4$$
$$+ 392(6+(21)^{1/2})k_5\}/1960$$

$$k_7 = hf(x_n + h, y_n + \{15(22+7(21)^{1/2})k_1 + 120k_2 + 40(7(21)^{1/2} - 5)k_3 - 63(3(21)^{1/2} - 2)k_4 - 14(49+9(21)^{1/2})k_5$$
$$+ 70(7-(21)^{1/2})k_6\}/180$$

IV. Applications of Runge-Kutta Method

Comparison of Different Order R-K methods with analytical solution is in table along with graphical representation.

1. **Problem:** A cup of coffee at 190^0F is left in the room of 70^0F. At time $t = 0$, the coffee is cooling at 15^0F per minute. What will be the temperature at time $t = 60$ minute?

OR

The law of cooling of Newton is a differential equation that predicts the cooling of a warm body placed in a cold environment. If the equation is given by

$$\frac{dT}{dt} = -0.125(T - 70)$$

The object's temperature is denoted by T. At time $t = 0$, the temperature of the body is 190^0F. Find the temperature at $t = 60$ using Runge-Kutta method for step size $h = 15$ min, $h = 10$ min and $h = 4$ min.

Solution:

Using different step size for this particular problem we will compare the analytical value with different order Runge-Kutta method.

Analytical Solution of this problem is:

$$T(t) = 120e^{-0.125t} + 70$$

Now, for step size h=15 the comparison chart is given below.

Table: 1

't' value	Analytical	RK-2	RK-4	RK-6
15	8.840260e+01	1.759375e+02	1.058997e+02	9.449356e+01
30	7.282213e+01	1.635229e+02	8.073988e+01	7.499946e+01
45	7.043279e+01	1.525632e+02	7.321298e+01	7.102045e+01
60	7.006637e+01	1.428879e+02	7.096121e+01	7.020829e+01

Error Analysis for step size h=15:

Table: 2

't' value	RK-2 Error (%)	RK-4 Error (%)	RK-6 Error (%)
15	8.753490e-01	1.749706e-01	6.090968e-02
30	9.070082e-01	7.917749e-02	2.177326e-02
45	8.213044e-01	2.780196e-02	5.876665e-03
60	7.282148e-01	8.948382e-03	1.419179e-03

Now, for step size h=10 the comparison chart is given below.

Table: 3

't' value	Analytical	RK-2	RK-4	RK-6
10	1.043806e+02	1.337500e+02	1.068945e+02	1.047435e+02
20	7.985020e+01	1.038672e+02	8.134339e+01	8.005924e+01
30	7.282213e+01	8.799194e+01	7.348757e+01	7.291244e+01
40	7.080855e+01	7.955822e+01	7.107227e+01	7.084324e+01
50	7.023165e+01	7.507780e+01	7.032967e+01	7.024414e+01
60	7.006637e+01	7.269758e+01	7.010136e+01	7.007069e+01

Error Analysis for step size h=10:

Table: 4

‘t’ value	RK-2 Error (%)	RK-4 Error (%)	RK-6 Error (%)
10	2.936942e-01	2.513956e-02	3.628902e-03
20	2.401699e-01	1.493187e-02	2.090370e-03
30	1.516981e-01	6.654450e-03	9.031009e-04
40	8.749666e-02	2.637166e-03	3.468174e-04
50	4.846150e-02	9.801973e-04	1.248650e-04
60	2.631213e-02	3.498968e-04	4.315745e-05

Now, for step size h=4 the comparison chart is given below.

Table: 5

‘t’ value	Analytical	RK-2	RK-4	RK-6
4	1.427837e+02	145.0000e+00	1.428125e+02	1.427843e+02
8	1.141455e+02	1.168750e+02	1.141805e+02	1.141463e+02
12	9.677562e+01	9.929688e+01	9.680744e+01	9.677629e+01
16	8.624023e+01	8.831055e+01	8.626597e+01	8.624078e+01
20	7.985020e+01	8.144409e+01	7.986972e+01	7.985061e+01
24	7.597445e+01	7.715256e+01	7.598866e+01	7.597475e+01
28	7.362369e+01	7.447035e+01	7.363374e+01	7.362390e+01
32	7.219788e+01	7.279397e+01	7.220485e+01	7.219802e+01
36	7.133308e+01	7.174623e+01	7.133784e+01	7.133318e+01
40	7.080855e+01	7.109139e+01	7.081176e+01	7.080862e+01
44	7.049041e+01	7.068212e+01	7.049255e+01	7.049046e+01
48	7.029745e+01	7.042633e+01	7.029887e+01	7.029748e+01
52	7.018041e+01	7.026645e+01	7.018134e+01	7.018043e+01
56	7.010943e+01	7.016653e+01	7.011003e+01	7.010944e+01
60	7.006637e+01	7.010408e+01	7.006677e+01	7.006638e+01

Error Analysis for step size h=4:

Table: 6

't' value	RK-2 Error (%)	RK-4 Error (%)	RK-6 Error (%)
4	2.216321e-02	2.882083e-04	6.090289e-06
8	2.729467e-02	3.496836e-04	7.387925e-06
12	2.521256e-02	3.182037e-04	6.721533e-06
16	2.070313e-02	2.573847e-04	5.435777e-06
20	1.593892e-02	1.951783e-04	4.121224e-06
24	1.178109e-02	1.420861e-04	2.999591e-06
28	8.466623e-03	1.005627e-04	2.122577e-06
32	5.960911e-03	6.972167e-05	1.471329e-06
36	4.131502e-03	4.758380e-05	1.003961e-06
40	2.828400e-03	3.207417e-05	6.765955e-07
44	1.917085e-03	2.140360e-05	4.514154e-07
48	1.288754e-03	1.416492e-05	2.986892e-07
52	8.604082e-04	9.309259e-06	1.962620e-07
56	5.710762e-04	6.081891e-06	1.281963e-07
60	3.771328e-04	3.953126e-06	8.330925e-08

2. Graphical Representation

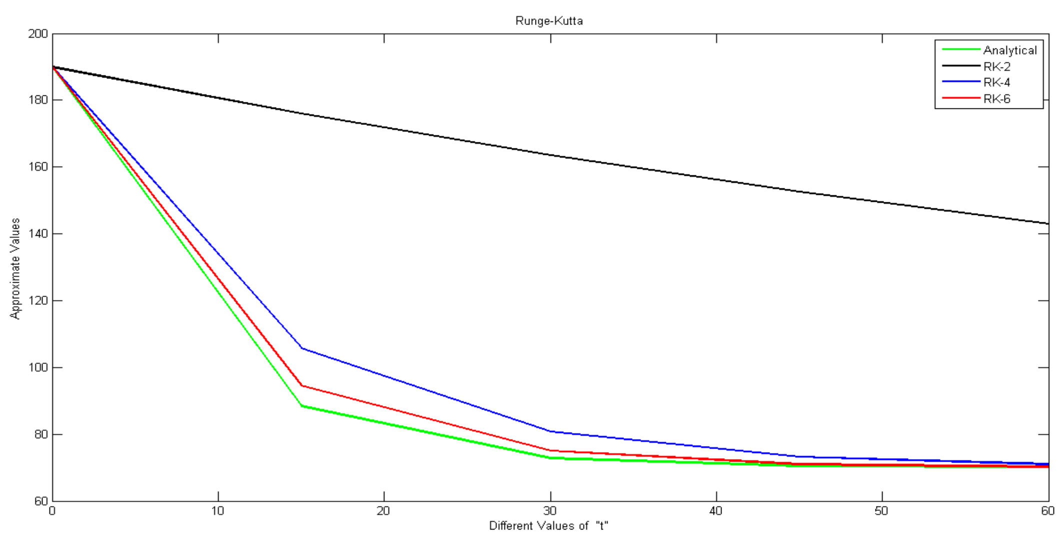

Fig: 1. (length size h=15)

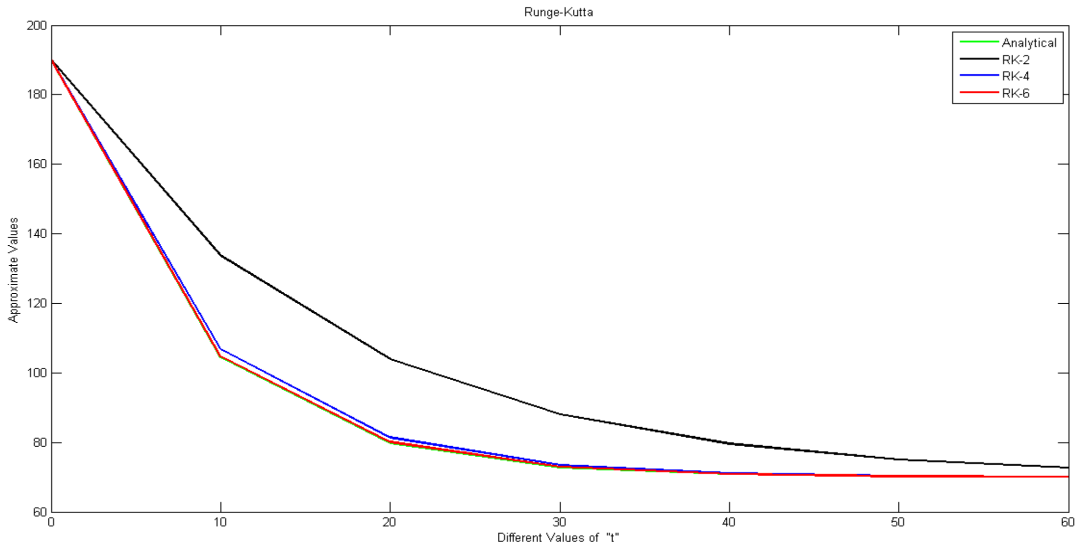

Fig: 2. (length size h=10)

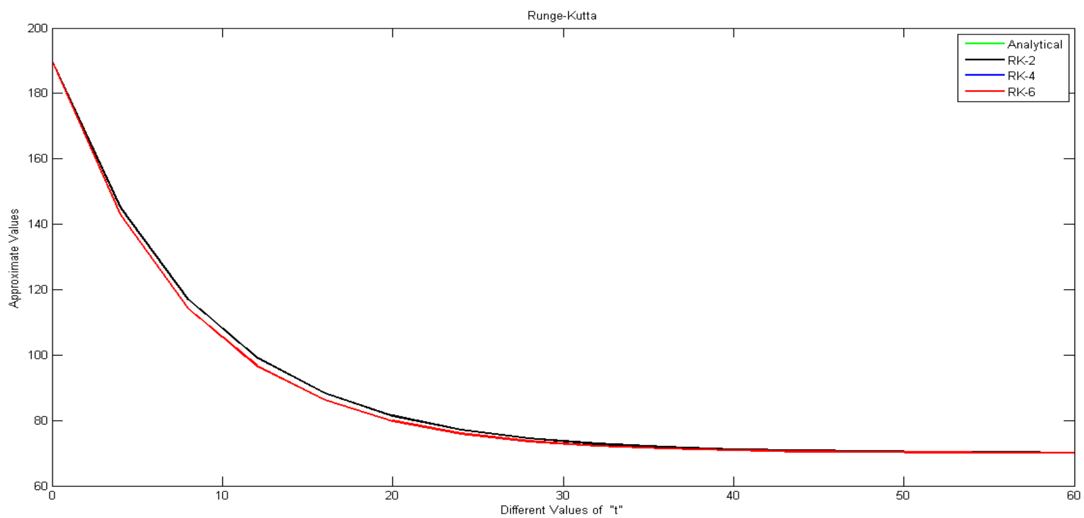

Fig: 3. (length size h=4)

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3. MATLAB Code:

```
function Runge_Kutta()
close all

dx = 0.2;
x0 = 0;
y0=50;
Expected_value=5;

N=(Expected_value-x0)/dx;
```

```
%%%Analytical solution
x(1) =  x0;
yE(1) = y0;

            for  n = 1:N
             x(n+1) = x(n)+dx;
   yE(n+1)= 10.714 + 39.286*exp(-3.5*X(n+1));
   %%fprintf('%d\n', yE(n))
end

   plot(X,yE,'g-','LineWidth',2)

   hold on


   title('Runge-Kutta');
   xlabel('Different Values of  "t"');
   ylabel('Approximate Values');
   grid on


%%% RK-2
x2(1)=x0;
y2(1)=y0;

for n = 1:N

   x2 (n+1) = x2(n) + dx;

   k1=dx * f(x2 (n), y2(n));
   k2=dx*f(x2(n)+dx,y2(n)+k1);

   y2(n+1)=y2(n)+ 0.5*(k1+k2);
   %%%fprintf('%d %d\n',x2(n+1),y2(n+1))
end

   plot(x2,y2,'k-','LineWidth',2)


%%% Rk-4
x4(1)=x0;
  y4(1)=y0;

  for n=1:N

     x4(n+1)=x4(n)+dx;

     k1=dx*f(x4(n),y4(n));
     k2=dx*f(x4(n)+dx/2,y4(n)+k1/2);
     k3=dx*f(x4(n)+dx/2,y4(n)+k2/2);
     k4=dx*f(x4(n)+dx,y4(n)+k3);

     y4(n+1) = y4(n) +(k1+2*k2+2*k3+k4)/6;
     %%%fprintf('%d %d\n',x4(n+1),y4(n+1))
```

```
  end

  plot (x4,y4,'b-','LineWidth',2)


%%%Rk-6
   x6(1)=x0;
   y6(1)=y0;

   for n=1:N

       x6(n+1)=x6(n)+dx;

       k1=dx*f(x6(n),y6(n));
       k2=dx*f(x6(n)+dx,y6(n)+k1);
       k3=dx*f(x6(n)+dx/2,y6(n)+(3*k1+k2)/8);
       k4=dx*f(x6(n)+2*dx/3,y6(n)+(8*k1+2*k2+8*k3)/27);
          a1=x6(n)+(7-21^0.5)*dx/14;
          a2=3*(3*21^0.5-7)*k1;
          a3=8*(7-21^0.5)*k2;
          a4=48*(7-21^0.5)*k3;
          a5=3*(21-21^0.5)*k4;
       k5=dx*f(a1,y6(n)+(a2-a3+a4-a5)/392);
          b1=x6(n)+(7+21^0.5)*dx/14;
          b2=-5*(231+51*21^0.5)*k1;
          b3=40*(7+21^0.5)*k2;
          b4=320*21^0.5*k3;
          b5=3*(21+121*21^0.5)*k4;
          b6=392*(6+21^0.5)*k5;
       k6=dx*f(b1,y6(n)+(b2-b3-b4+b5+b6)/1960);
          c1=x6(n)+dx;
          c2=15*(22+7*21^0.5)*k1;
          c3=120*k2;
          c4=40*(7*21^0.5-5)*k3;
          c5=63*(3*21^0.5-2)*k4;
          c6=14*(49+9*21^0.5)*k5;
          c7=70*(7-21^0.5)*k6;
       k7=dx*f(c1,y6(n)+(c2+c3+c4-c5-c6+c7)/180);

      y6(n+1)= y6(n)+(9*k1+64*k3+49*k5+49*k6+9*k7)/180;

   end

   plot(x6,y6,'r-','LineWidth',2)

   %%%fprintf(' x Value, Analytical, RK-2, RK-4, RK-6\n')

     %%%Error Calculation
   for n=1:N

       e2 (n+1) = abs ((yE(n+1)-y2(n+1))/100);
       e4 (n+1) = abs ((yE(n+1)-y4(n+1))/100);
       e6 (n+1) = abs ((yE(n+1)-y6(n+1))/100);
   end

       for n =1:N
```

```
fprintf('%d %d %d %d\n',x2(n+1), e2(n+1), e4(n+1), e6(n+1));
   end
 legend('Analytical','RK-2','RK-4','RK-6')

   function ydot = f(x,y)
          ydot= 37.5 - 3.5*y;
```

V. Conclusion

We addressed the different order of R-K methods for solving the ODE in this study. We have used MATLAB code to calculate the analytical values and the values of different order R-K method. We have also shown the error in between analytical value and different order R-K methods in the table. For the R-K method, accuracy level increases with the order. That is the method Runge-Kutta of 6^{th} order is more precise whereas the order two is less accurate. Nevertheless, the method Runge-Kutta of order four approaches is mostly used as it is exact, stable and easy programming. The accuracy of the approximate value also depends on the step size.

From the present study we can conclude this followings-

I. The R-K sixth order method is laborious to derive and execute but giving better approximation than second as well as fourth-order R-K method.

II. When we use different step length of the same ordinary differential equation, from there we got better approximation and approximately exact results if we choose very small step length.

III. Eventually, we conclude that the Runge-Kutta of order six approach is the relatively better way of solving the differential equation with the initial value problem

References

[1] Richard L. Burden & J. Douglas Faires (2018-2019). “Numerical Analysis”. Brooks/Cole cengage learning.

[2] Sastry S. S (2006). “Introductory Methods of Numerical Analysis”. Prentice Hall of India Private Limited.

[3] John H. Mathews & Kurtis K. Fink (1999). “Numerical Methods Using Matlab”, Prentice-Hall Inc.

[4] Luther, H. (1968). “An Explicit Sixth-Order Runge-Kutta Formula”. Mathematics of Computation, 22(102), p.434

[5] Hossain S, Bhattacharjee P.k &, Hossain E. (2018) “Numerical Analysis”. Titas Publications.

[6] Mahtab Uddin, “Study on different numerical methods for solving differential equations”.

[7] Islam, M. (2015). A Comparative Study on Numerical Solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE) with Euler and Runge Kutta Methods. American Journal of Computational Mathematics, 05(03), pp.393-404.

[8] GM Fahad Bin Mostafa, Mohammad R . Islam. “Study of Runge-Kutta Method of Higher orders and its Applications”.

[9] Mathworld.wolfram.com. (2019). Wolfram MathWorld: The Web's Most Extensive Mathematics Resource. [online] Available at: http://mathworld.wolfram.com/ [Accessed 25 Jun. 2019].

[10] Wikipedia.org. (2019). Wikipedia. [online] Available at: https://www.wikipedia.org/ [Accessed 25 Jun. 2019].