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Controllability of Nonlocal Impulsive Differential Equations With Measure of Noncompactness

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Abstract

This paper is concerned with the controllability of impulsive differential equations with nonlocal conditions. First, we establish a property of measure of noncompactness in the space of piecewise continuous functions. Then, by using this property and Darbo-Sadovskii's fixed point theorem, we get the controllability of nonlocal impulsive differential equation under compactness conditions, Lipschitz conditions and mixed-type conditions respectively.

keywords: *Controllability, impulsive differential equations, nonlocal conditions, measure of non compactness, fixed point theorem.*

Introduction

Impulsive systems are described by the occurrence of an abrupt change in the state of the system, which arises at certain time instants over a negligible time period. The dynamical behaviour of systems with impulses is much more complex than the behaviour of dynamical systems without impulse effects. In these models, the investigated simulating processes and phenomena are subjected to certain perturbations whose duration is negligible in comparison with the total duration of the process. These processes tend to be more suitably modelled by impulsive differential equations, which allow for discontinuities in the evolution of the state. For more facts on the results and applications of impulsive differential systems, one can refer to the monographs of Bainov and Simenov [5], Lakshmikanthan et al. [3] and the papers of [4, 8, 19, 20, 21, 25, 28], where the numerous properties of their solutions are studied and detailed bibiliographies are given.

In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non-Newtonian fluid mechanics have mathematical models. Popular models essentially fall into two categories: the differential models and the integrodifferential models. A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups.

In general functional differential equations or evolution equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with heat-flow in materials with memory and many other physical phenomena.

The study of abstract nonlocal conditions was initiated by Byszewski [1], and the importance of the problem consists in the fact that it is more general and has better effect than the classical initial conditions $u(0) = u_0$. Therefore, it has been studied extensively under various conditions. Readers may refer to ([19, 20, 25, 27, 28]), where authors studied impulsive differential equations with nonlocal conditions. In particular, the measure of noncompactness has been used as an important tool to deal with some similar functional differential and integral equations; see [6, 9, 10].

Motivated by the fact that a dynamical system may evolve through an observable quantity rather than the state of the system, a general class of evolutionary equations is defined. This class includes standard ordinary and partial differential equations as well as functional differential equations of retarded and neutral type. In this way, the theory serves as a unifier of these classical problems. Included in this general formulation is a general theory for the evolution of temperature in a solid material. In the general case, temperature is transmitted as waves with a finite speed of propagation. Special cases include a theory of delayed diffusion. When physical problems are simulated, the model often takes the form of semilinear equations. Such problems in the control fluid flow can be modelled by a semilinear system in a Banach space. For actual flow, control problems are leading to this kind of model and the resulting model equation are discussed in [2]. Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behaviour so as to accomplish a desired goal. In order to implement this influence, practitioners build devices and their interaction with the object being controlled is the subject of control theory. In control theory, one of the most important qualitative aspects of a dynamical system is controllability. Controllability is an important property of a control system and the controllability property plays a crucial role in many control problems such as stabilization of unstable systems by feedback or optimal control. Roughly the concept of controllability denotes the ability to move a system around in its entire configuration space using only certain admissible manipulations.

The notion of controllability is of great importance in mathematical control theory. Many basic problems of control theory pole-assignment, structural engineering and optimal control may be solved under the assumption that the system is controllable. The concept of controllability plays an crucial role in both finite and infinite dimensional spaces, that is systems represented by ordinary differential equation and partial differential equation respectively. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [7, 14, 15, 16, 17, 18, 23].

In this paper, we discuss the controllability of the following impulsive differential equations with nonlocal conditions:

$$u'(t) = Au(t) + f\big(t, u(t)\big) + Bv(t),\ t \in J = [0, b], t \neq t_i\,,$$

$$u(0) = g(u), \tag{1.1}$$

$$\Delta u(t_i) = I_i\ \big(u(t_i)\big),\ i = 1,2, \dots \dots, s\,.$$

Where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ in a Banach space X, $B: U \subseteq X \to X$ is a bounded linear operator; the control function $v(.)$ is given in $L^2(J, U)$, with

U as a Banach space; $f,\ g$ are appropriate continuous functions to be specified later; $I_i : X \to X$ is a nonlinear map, $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, for all $i = 1,2,\ldots\ldots,s$, $0 = t_0 < t_1 < t_2 < \ldots\ldots\ldots < t_s < t_{s+1} = b$, where $u(t_i^-), u(t_i^+)$ denote the left and right limit of u at $t = t_i$, respectively.

From the viewpoint of theory and practical, it is natural for mathematics to combine impulsive conditions, nonlocal conditions, and controllability of the system. Recently, the controllability of nonlocal impulsive differential problem of type (1.1) has been discussed in the papers of Liu [31] and Ji et al. [10]. The main contributions are as follows:

1. The study of nonlocal controllability of impulsive differential equations via measure of non-compactness described in the form (1.1) is an untreated topic in the literature and this is an additional motivation for writing this paper.
2. We assume the nonlinear term only satisfies a weakly compactness condition and does not require the compactness of the semigroup.
3. We establish some sufficient conditions for the nonlocal controllability when the solution operators are only equicontinuous, by means of the Darbo's fixed point theorem via the noncompactness measure.
4. Our theorems guarantee the effectiveness of nonlocal controllability results under some weakly compactness conditions.
5. We emphasize that our methods avoid a technical error when the com- pactness of semigroup and other hypotheses are satisfied, the application of controllability results are only restricted to the finite dimensional space.

The presentation of our work is as follows: Section 2 provides the definitions and preliminary results to be used in this article. In particular, we review some of the standard facts on evolution families, Hausdorff measure of noncompactness, and certain useful fixed point results. In Section 3, we focus our attention on controllability results for nonlinear systems using the measure of noncompactness and Darbo's fixed point theorem.

II. Preliminaries

Let $(X, \|.\|)$and $(U, \|.\|)$ be real Banach spaces. $T(t)$ is a strongly continuous semigroup on X, with generator A, is $A: D(A) \to X$. We denote by $C([0,b];X)$ the space of X-valued continuous functions on $[0,b]$ with the norm

$\|x\| = sup\{\|x(t)\|, t\epsilon[0,b]\}$. $L^1([0,b];X)$ is the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| dt$.

The semigroup $T(t)$ is said to be equicontinuous if $\{T(t)x : x \epsilon B\}$ is equicontinuous at $t > 0$ for any bounded subset $B \subset X$ (cf.27). Obviously, if $T(t)$ is a compact semigroup, it must be equicontinuous. The converse of the relation is not correct. Throughout this paper, we suppose that

(HA) The semigroup $T(t) : t \geq 0$ generated by A is equicontinuous. Moreover, there exists a positive number M such that $M = sup\ \sup_{0\leq t\leq b} \|T(t)\|$.

For the sake of simplicity, we put $J = [0,b]$; $J_0 = [0,t_1]$; $J_i = (t_i, t_{i+1}]$, $i = 1,2,\ldots\ldots s$. In order to define a mild solution of the problem (1.1), we introduce the set $([0,b];X) = \{u : [0,b] \to X : u$ is continuous at

$t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists, $i = 1,2,\ldots\ldots,s\}$. It is easy to verify that $PC\,([0,b];\, X)$ is a Banach space with the norm $\|u\|_{PC} = sup\,\{\|u(t)\|, t\,\epsilon\,[0,b]\}$.

Consider the infinite-dimensional linear control system

$$u'(t) = Au(t) + Bv(t),\; t\,\epsilon\, J = [0,b]\,,$$

$$u(0) = u_0, \qquad (2.1)$$

Where $v(t)\,\epsilon\, L^2(J,U)\,, : X \to X\,, B : U \to X$.

Let $B\,\epsilon\, L(U,X)$ and $b \geq 0$. The linear operator $W : L^2(J,U) \to X$ is defined by

$$W_v = \int_0^b T(b-s)Bv(s)ds$$

Such that

(i) W has an invertible operator W^{-1} which takes values in $L^2(J,U)/kerW$ and there exist positive constants M_1 and M_2 such that $\|B\| \leq M_1$ and $\|W^{-1}\| \leq M_2$.

(ii) There is $K_W\ \epsilon\ L^{-1}\,(J,R^+)$ such that, for every bounded set $Q \subset X$

$$\beta(W^{-1}Q)(t) \leq K_w(t)\beta(Q).$$

We define control formally as

$$v(t) = W^{-1}\left[u_1 - T(b)g(u) - \int_0^b T(b-s)f(s,u(s))ds - \sum_{0<t_i<t} T(t-t_i)I_i[u(t_i)](t)\right].$$

Definition 2.1.

A function $u\,\epsilon\, PC([0,b];X)$ is a mild solution of the problem (1.1) if

$$u(t) = T(t)g(u) + \int_0^t T(t-s)f(s,u(s))ds + \int_0^t T(t-s)Bv(s)ds \quad + \sum_{0<t_i<t} T(t-t_i)I_i[u(t_i)]\,,$$

for all $t\,\epsilon\,[0,b]$

Now, we introduce the Hausdorff measure of noncompactness (in short MNC) defined by $\beta(\Omega) = inf\{\epsilon > 0 : \Omega\ has\ a\ finite\ \epsilon - net\ in\ X\}$,

for each bounded subset Ω in a Banach space X.

Some basic properties of the Hausdroff measure of noncompactness $\beta(.)$ are given in the following lemma:

Lemma 2.2 ([6]). Let X be a real Banach space and $B, C \subseteq X$ be bounded. Then the following properties holds:

1. B is precompact if and only if $\beta(B) = 0$;

2. $\beta(B) = \beta(\bar{B}) = \beta(\mathrm{conv}B)$, where $\bar{B}$ and $convB$ means the closure of B and convex hull of B, respectively;

3. $\beta(B) \leq \beta(C)$, when $B \subseteq C$;

4. $\beta(B + C) \leq \beta(B) + \beta(C)$, where $B + C = \{x + y : x \epsilon B, y \epsilon C\}$;

5. $\beta(B \cup C) \leq \max\{\beta(B), \beta(C)\}$;

6. $\beta(\lambda B) \leq \|\lambda\|\beta(B)$, for any $\lambda \epsilon R$;

7. If the map $Q : D(Q) \subseteq X \rightarrow Z$ is Lipschitz continuous with constant k, then $\beta_Z(QB) \leq k\beta(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space.

The map $Q : D \subseteq X \rightarrow X$ is said to be β-condensing, if Q is continuous and bounded, and for any non compact bounded subset $B \subset D$, we have $\beta(QB) < \beta(B)$, where X is a Banach space.

Lemma 2.3 ([6], **Darbo-Sadovskii**). If $D \subset X$ is bounded, closed and convex; the continuous map $Q : D \rightarrow D$ is β-condensing; then Q has at least one fixed point in D.

In order to remove the strong restriction on the coefficient in Darbo-Sadovskii's fixed point theorem, Sun and Zhang [29] generalized the definition of a β-condensing operator. At first, we give some notation. Let $D \subset X$ be closed and convex, the map $Q : D \rightarrow D$ and $x_0 \epsilon D$. For every $B \subset D$, set

$$Q^{(1,x_0)}(B) = Q(B), \quad Q^{(1,x_0)}(B) = Q\ (\overline{conv}\{Q^{(n-1,\ x_0)}B,\ x_0\}),$$

where $\overline{conv}$ means the closure convex hull, $n = 2,3, \dots \dots \dots$

Definition 2.4. Let $D \subset X$ be closed and convex. The map $Q: D \rightarrow D$ is said to be β-convex-power condensing if Q is continuous, bounded and there exist $x_0 \epsilon D,\ n_0 \epsilon N$ such that for every nonprecompact bounded subset $B \subset D$, we have

$$B(Q^{(n_0, x_0)}(B)) < \beta(B).$$

Obviously, if $n_0 = 1$, then a β-convex-power condensing operator is β-condensing. Thus, the convex-power condensing operator is a generalization of the condensing operator. Now, we give the fixed point theorem about the convex-power condensing operator.

Lemma 2.5 ([29]**).** If $D \subset X$ is bounded, closed and convex, the map $Q : D \rightarrow D$ is β-convex-power condensing, then Q has atleast one fixed point in D.

Now, we give an important property of the Hausdorff MNC in $PC([0, b]; X)$, which is an extension to the property of MNC in $C([0, b]: X)$ and makes us to deal with the impulsive differential equations.

Lemma 2.6 ([6]**).** If $W \subseteq C([O, b]; X)$ is bounded, then $\beta(W(t)) \leq \beta(W)$ for all $t \epsilon [0, b]$, where $W(t) = \{u(t) : u \epsilon W\} \subseteq X$. Furthermore, if W is equicontinuous on $[0, b]$, then $\beta(W(t))$ is continuous on $[0, b]$ and $\beta(W) = sup\ \{\beta(W(t)),\ t \epsilon [0, b]\}$.

By applying Lemma 2.6, we shall extend the result to the space $PC([0,b];X)$.

Lemma 2.7. If $W \subseteq PC([0,b];X)$ is bounded, then $\beta(W(t)) \leq \beta(W)$ for all $t \epsilon [0,b]$, where $W(t) = \{u(t); u \epsilon W\} \subseteq X$. Furthermore, suppose the following conditions are satisfied:

1. W is equicontinuous on $J_0 = [0,t_1]$ and each $J_i = (t_i,t_{i+1}], i = 1,\ldots\ldots,s$;

2. W is equicontinuous at $t = t_i^+, i = 1,\ldots\ldots,s$.

Then $Sup_{t\epsilon[0,b]}\beta(W(t)) = \beta(W)$.

Proof. For arbitrary $\epsilon > 0$, there exists $W_i \subseteq PC([0,b];X), 1 \leq i \leq n$, such that

$$\mathrm{W} = \cup_{i=1}^{n} \mathrm{W_i} \text{ and } \mathrm{diam}(\mathrm{W_i}) \leq 2\beta(\mathrm{W}) + 2\epsilon, \ \mathrm{i} = 1,2,\ldots.\mathrm{n},$$

where $diam(.)$ denotes the diameter of a bounded set. Now, we have $\mathrm{W(t)} = \cup_{i=1}^{n} \mathrm{W_i}\,(\mathrm{t})$ for each $\mathrm{t} \epsilon [\mathrm{a},\mathrm{b}]$, and

$$\|\mathrm{x(t) - y(t)}\| \leq \|\mathrm{x - y}\| \leq \mathrm{diam}(\mathrm{W_i})$$

for $x,y \epsilon W_i$. From the above two inequalities, it follows that

$$2\beta(W(t)) \leq diam(W_i(t)) \leq diam(W_i) \leq 2\beta(W) + 2\varepsilon.$$

By the arbitrariness of ε, we get that $\beta(W(t)) \leq \beta(W)$ for every $t \epsilon [0,b]$. Therefore, we have $sup_{t\epsilon[0,b]}\beta(W(t)) \leq \beta(W)$.

Next, if the conditions (1) and (2) of Lemma 2.7 are satisfied, it remains to prove that $\beta(W) \leq sup_{t\epsilon[0,b]}\beta(W(t))$. We denote $W|_{\overline{J_i}}$ by the restriction of W on $\overline{J_i} = [t_i,\ t_{i+1}]$, $i = 0,1,\ldots\ldots\ldots s$ That is, for $x \epsilon W|_{\overline{J_i}}$, define that

$$x(t) = \begin{cases} x(t), & t_i < t \leq t_{i+1}, \\ x\left(t_i^+\right), & t = t_i. \end{cases}$$

and obviously $W|_{\overline{J_i}}$ is equicontinuous on $\overline{J_i}$ due to the condition (1) and (2) of Lemma 2.7. Then from Lemma 2.6, we have that

$$\beta(W|_{\overline{J_i}}) = \min_{t\epsilon\overline{J_i}}\beta(W|_{\overline{J_i}}(t)).$$

Moreover, we define the map

$$A : PC([0,b];X) \to C([0,t_1];X) \times C([t_1,t_2];X) \times \ldots.\times C([t_s,b];X)$$

by $\quad x \to (x_0,x_1,\ldots,x_s), where\ x \epsilon PC([0,b];X), x_i = x|\overline{J_i}\ \|(x_0,x_1,\ldots,x_s)\|$

$$= \max_{0\leq i\leq s}\|xi\|.$$

As A is an isometric mapping, noticing the equicontinuity of $W|_{\overline{J_i}}$ on $\overline{J_i}$ we have that

$$\beta(W) = \beta(W|_{\bar{J_0}} \times W|_{\bar{J_1}} \ldots . \times W|_{\bar{J_s}} \leq \max_i \beta(W|_{\bar{J_i}}) = \max_i \sup_{t\in\bar{J_i}} \beta(W|_{\bar{J_i}}(t)).$$

And from the fact that $\sup_{t\in\bar{J_i}} \beta(W|_{\bar{J_i}}(t) \leq \sup_{t\in[0,b]} \beta(W(t))$, for each $i = 0, \ldots., s$, we get that

$\beta(W) \leq \sup\limits_{t\in[0,b]} \beta(W(T))$. This completes the proof. ☐

Lemma 2.8 ([6]). If $W \subset C([0\,,b]; X)$ is bounded and equicontinuous, then $\beta(W(t))$ is continuous and

$$\beta\left(\int_0^t W(s)ds\right) \leq \int_0^t \beta\big(W(s)\big)ds,$$

for all $t \epsilon [0,b]$, where $\int_0^t W(s)ds = \left\{\int_0^t x(s)ds : x \in W\right\}$.

Lemma 2.9. If the hypothesis (HA) is satisfied, i.e., $\{T(t) : t \geq 0\}$ is equicontinuous and $\eta \epsilon\ L^1([0,b];\ R^+)$, then the set $\left\{\int_0^t T(t - s)u(s)ds : \|u(s)\| \leq \eta(s)\ for\ a.e. s\ \epsilon\ [0,b]\right\}$ is equicontinuous for $t\ \epsilon\ [0,b]$.

Proof. We let $0 \leq t < t + h \leq b$ and have that

$$\left\|\int_0^{t+h} T(t+h-s)u(s)ds - \int_0^t T(t-s)u(s)ds\right\|$$

$$\leq \left\|\int_0^t T(t+h-s)u(s)ds - \int_0^t T(t-s)u(s)ds\right\| + \int_t^{t+h} \|T(t+h-s)u(s)\|ds \tag{2.2}$$

If $t = 0$, then the right hand side of (2.1) can be made small when h is small and independent of u. If $t > 0$, then we can find a small $\varepsilon > 0$ with $t - \varepsilon > 0$. Then it follows from (2.1) that

$$\begin{aligned}&\left\|\int_0^t T(t+h-s)u(s)ds - \int_0^t T(t-s)u(s)ds\right\| \\ &\qquad \leq \left\|T(h+\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)u(s)ds - T(\in)\int_0^{t-\in} T(t-\epsilon-s)u(s)ds\right\| \\ &\qquad + \left\|\int_{t-\epsilon}^t T(t+h-s)u(s)ds\right\| + \left\|\int_{t-\epsilon}^t T(t-s)u(s)ds\right\| \end{aligned} \tag{2.3}$$

Here, as $T(t)$ is equicontinuous for $t > 0$, thus

$$\left\|[T(h+\epsilon) - T(\epsilon)]\int_0^{t-\epsilon} T(t-\epsilon-s)u(s)ds\right\| \to 0, \quad \text{as h} \to 0,$$

uniformly for u.

Then from (2.1),(2.2), and the absolute continuity of integrals, we get that $\{\int_0^{t-\epsilon} T(t-s)u(s)ds, \|u(s)\| \leq \eta(s)\,for\ a.e.\ s \in [0,b]\}$ is equicontinuous for $t \in [0,b]$.

Lemma 2.10 ([30]). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,b];R^+)$. Assume that there exist μ, $\eta \in L^1([0,b];R^+)$, satisfying $\sup_{n\geq 1}\|f_n(t)\| \leq \mu(t)$ and $\beta(\{f_n(t)\}_{n=1}^{\infty}) \leq \eta(t)\ a.e.\ t \in [0,b]$, then for all $t \in [0,b]$, we have

$$\beta\left(\left\{\int_0^t T(t-s)f_n(s)ds : n \geq 1\right\}\right) \leq 2M_1\int_0^t \eta(s)ds.$$

III. Main Result

In this section we give the existence results for the problem (1.1) under different conditions on g and I_i when the semigroup is not compact and f is not compact or Lipschitz continuous, by using Lemma 2.7 and the generalized β-condensing operator. More precisely, Theorem 3.1 is concerned with the case that compactness conditions are satisfied. Theorem 3.2 deals with the case that Lipschitz conditions are satisfied. And mixed-type conditions are considered in Theorem 3.3 and Theorem 3.4.

Let r be a finite positive constant, and set $B_r = \{x \epsilon X : \|x\| \leq r\}$, $W_r = \{u \epsilon PC([0,b];X) : u(t) \epsilon B_r,\ t \epsilon [0,b]\}$. We define the solution map $G : PC([0,b];X) \to PC([0,b];X)$ by

$$(Gu)(t) = T(t)g(u) + \int_0^t T(t-s)f\big(s,u(s)\big)ds + \int_0^t T(t-s)Bv(s)ds$$

$$+ \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)) \qquad (3.1)$$

With

$$(G_1u)(t) = T(t)g(u),$$

$$(G_2u)(t) = \int_0^t T(t-s)f\big(s,u(s)\big)ds\,,$$

$$(G_3u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))$$

for all $t \in [0,b]$. It is easy to see that u is the mild solution of the problem (1.1) if and only if u is a fixed point of the map G.

We list the following hypotheses:

(Hf) $f : [0,b] \times X \to X$ satisfies the following conditions:

(i) $f(t,.) : X \to X$ is continuous for $a.e.\ t \epsilon [0,b]$ and $f(.,x) : [0,b] \to X$ is measurable for all $x \epsilon X$. Moreover, for any $r > 0$, there exists a function $\rho_r \in L^1([0,b],R)$ such that

$$\|f(t,x)\| \leq \rho_r(t)$$

for $a.e.\ t \epsilon [0,b]$ and $x \epsilon B_r$.

(ii) there exists a constant $L > 0$ such that for any bounded set $D \subset X$,

$$\beta(f(t,D)) \leq L\beta(D) \qquad (3.2)$$

for $a.e.\ t \in [0,b]$.

(Hg1) $g: PC([0,b];\ X) \to X$ is continuous and compact.

(HI1) $\mathrm{I_i} : X \to X$ is continuous and compact for $i = 1,\dots,s$.

Theorem 3.1. Assume that the hypotheses (HA), (Hf), (Hgl), (HI1) are satisfied, then the nonlocal impulsive problem (1.1) has at least one mild solution $[0,b]$ provided that there exists a constant r > 0 such that

$$M\left[\sup_{u\in W_r}\|g(u)\| + \|\rho_r\|_{L^1} + M_1M_2\sqrt{b}\|V\|_{L^2} + sup_{u\in W_r}\sum_{i=1}^{s}\|\mathrm{I}_i[\mathrm{u}(\mathrm{t_i})]\|\right] \le r \qquad (3.3)$$

Proof. We will prove that the solution map G has a fixed point by using the fixed point theorem about the β-convex-power condensing operator.

Firstly, we prove that the map G is continuous on $PC([0,b];X)$. For this purpose, let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $PC([0,b];X)$ with

$$\lim_{n\to\infty} u_n = u$$

in $PC([0,b];X)$. By the continuity of f with respect to the second argument, we deduce that for each $s \in [0,b]$, $f(s,u_n(s))$ converges to $f(s,u(s))$ in X. And we have,

$$\|\mathrm{Gu_n} - \mathrm{Gu}\| = \Bigg\|[T(t)g(u_n) + \int_0^t T(t-s)f(s,u_n(s))ds + \int_0^t T(t-s)Bv_n(s)ds$$

$$+ \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))] - [T(t)g(u) + \int_0^t T(t-s)f(s,u(s))ds$$

$$+ \int_0^t T(t-s)Bv(s)ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))]\Bigg\|$$

$$\le M\|g(u_n) - g(u)\| + M\int_0^t\|f(s,u_n(s)) - f(s,u(s))\|\,ds + MM_1\|v_n - v\|_{L^2}$$

Where

$$\|v_n - v\| \le MM_2\left\{\int_0^b \|f(s,u_n(s)) - f(s,u(s))\|\,ds + \sum_{i=1}^{s}\|I_i(u_n(t_i)) - I_i(u(t_i))\|\right\}$$

Then, by the continuity of g, I_i and using the dominated convergence theorem, we get $\lim_{n\to\infty} Gu_n = Gu$ in $PC([0,b];X) \Longrightarrow G$ is continuous on $PC([0,b];X)$.

Secondly, we claim that $GW_r \subseteq W_r$. In fact, for any $u \in W_r \subset PC([0,b];X)$, from (3.1) and (3.3), we have

$$\|(Gu)(t)\| = \|(G_1u)(t) + (G_2u)(t) + (G_3u)(t) + (G_4u)(t)\|$$

$$\leq \|T(t)g(u)\| + \left\|\int_0^t T(t-s)f(s,u(s))ds\right\| + \left\|\int_0^t T(t-s)Bv(s)ds\right\|$$
$$+\left\|\sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))\right\|$$

$$\leq M\{\|g(u)\| + \|\rho_r\|_{L^1} + MM_2\sqrt{b}\|v\|_{L^2} + \sum_{i=1}^{s} I_i(u(t_i))\}$$

Where

$$\|v\|_{L^2} \leq M_2\left\{\|u_1\| + M\|g(u)\| + M\int_0^b \|f(s,u(s))\|ds + M\sum_{i=1}^{s} I_i(u(t_i))\right\}$$

Hence, $\|(Gu)(t)\| \leq r$ for each $t \,\epsilon\, [0,b]$.

It implies that

$$GW_r \subseteq W_r$$

Now, we show that GW_r is equicontinuous on $J_0 = [0,t_i]$, $J_i = [t_i,t_{i+1}]$ and is also equicontin-uous at $t = t_i^+$, $i = 1,\ldots,s$. Indeed, we only need to prove that GW_r is equicontinuous on $[t_1,t_2]$ as the cases for other subintervals are the same.

For $u \,\epsilon\, W_r$, $t_1 \leq s < t \leq t_2$, we have, using the semigroup property,

$$\|T(t)g(u) - T(s)g(u)\| \leq M\|[T(t-s) - T(0)]g(u)\|$$

Thus G_1W_r is equicontinuous on $[t_1,t_2]$ due to the compactness of g and the strong continuity of $T(.)$. The same idea can be used to prove the equicontinuity of G_4W_r on $[t_1,t_2]$. i.e., for $u \,\epsilon\, W_r$, $t_1 \leq s < t \leq t_2$, we have

$$\|T(t-t_1)I_1(u(t_1)) - T(s-t_1)I_1(u(t_1))\| \leq M\|[T(t-s) - T(0)]I_1(u(t_1))\|$$

which implies the equicontinuity of G_4W_r on $[t_1,t_2]$. due to the compactness of I_1 and the strong continuity of $T(.)$.

Moreover, from Lemma 2.9, we have that G_2W_r is equicontinuous on $[0,b]$. Therefore, we have that the function in $GW_r = (G_1 + G_2 + G_3 + G_4)W_r$ are equicontinuous on each $[t_i,t_{i+1}]$, $i = 0,1,\ldots,s$.

Set $W = \overline{conv}\, G(W_r)$, where $\overline{conv}$ means the closure of convex hull. It is easy to verify that G maps W into itself and W is equicontinuous on each $J_i = [t_i,t_{i+1}]$, $i = 0,1,\ldots,s$. Now, we show that G: W → W is a convex-power condensing operator. Take $x_0 \,\epsilon\, W$, we shall prove that there exists a positive integral n_0 such that

$$\beta(G^{(n_0,x_0)}(D)) < \beta(D)$$

for every nonprecompact bounded subset $D \subset W$.

From Lemma 2.2 and Lemma 2.8, noticing the compactness of g and have I_i, we have

$$\beta((G^{(1,x_0)})D(t)) = \beta((GD)(t))$$

$$\leq \beta(T(t)g(D)) + \beta\left(\int_0^t T(t-s)f(s,D(s))ds\right)$$

$$+\beta\left(\int_0^t T(t-s)Bv(s)ds\right)+\beta\big(\textstyle\sum_{0<t_i<t} T(t-t_i)I_i\big(D(t_i)\big)\big)$$

$$\leq \int_0^t \beta\left(T(t-s)f\big(s,D(s)\big)\right)ds+\int_0^t \beta\big(T(t-s)Bv(s)\big)ds$$

$$\leq M\left\{\int_0^t \beta\left(f\big(s,D(s)\big)\right)ds+\int_0^t \beta\big(Bv(s)\big)ds\right\}$$

$$\leq M\left\{\int_0^t L\beta(D)ds+M_1\int_0^t \beta\big(v(s)\big)ds\right\}$$

$$\leq ML\left\{t+MM_1b^{3/2}K_w(s)\right\}\beta(D)$$

For each $t \epsilon [0,b]$, where $\beta(v(s)) \leq K_w(s)MLb\beta(D)$.

Further,

$$\beta\big((G^{(2,\ x_0)}D)(t)\big) = \beta\big((G\overline{conv}\{G^{(1,\ x_0)}D,\ x_0\ \})\ (t)\big)$$

$$\leq \beta(T(t)\ g\big(\overline{conv}\{G^{(1,\ x_0)}D(s),x_0\ \}\big))$$

$$+\beta\left(\int_0^t(T(t-s)f(s,\ \overline{conv}\{G^{(1,x_0)}D(s),x_0(s)\ \})ds\right)$$

$$+\beta\left(\int_0^t\big(T(t-s)Bv(s)\big)ds\right)$$

$$+\beta\big(\textstyle\sum_{0<t_i<t} T(t-t_i)I_i[\ \overline{conv}\{G^{(1,x_0)}D(t_i),x_0(t_i)\}]\big)$$

$$\leq \beta\left(\int_0^t(T(t-s)f(s,\ \overline{conv}\{G^{(1,x_0)}D(s),x_0(s)\ \})ds\right)$$

$$+\beta\left(\int_0^t\big(T(t-s)Bv(s)\big)ds\right)$$

$$\leq M\int_0^t \beta\big(f(s,\ \overline{conv}\{G^{(1,x_0)}D(s),x_0(s)\ \}\big)ds$$

$$+MM_1\int_0^t \beta\big(v(s)\big)ds$$

$$\leq M\int_0^t L\beta(\ \overline{conv}\{G^{(1,\ x_0)}D(s),x_0(s)\ \}\big)ds+MM_1\int_0^t \beta\big(v(s)\big)ds$$

$$\leq ML\int_0^t \beta(G^{(1,x_0)}D(s))ds+MM_1\int_0^t \beta\big(v(s)\big)ds$$

$$\leq ML\int_0^t MLs\beta(D)ds+MM_1\int_0^t \beta\big(v(s)\big)ds$$

$$\leq M^2L^2\,\beta(D)\int_0^t tsds+MM_1\int_0^t \beta\big(v(s)\big)ds$$

$$\leq M^2L^2\,\beta(D)\frac{t^2}{2!}+MM_1\sqrt{b}\left\{\frac{M^2L^2b^2}{2!}\beta(D)\right\}$$

$$\leq \frac{M^2L^2}{2!}\beta(D)\left\{t^2+MM_1b^{5/2}\right\}$$

for $t \in [0,b]$, where $\beta(v(s)) \le \frac{M^2L^2\sqrt{b}b^2}{2!}\beta(D)$.

We can continue this iterative procedure and get that

$$\beta((G^{(n,\ x_0)}D)(t)) \le \frac{M^nL^n}{n!}\beta(D)\{b^n + MM_1b^n\sqrt{b}\}$$

$$\le \frac{M^nL^nb^n}{n!}\beta(D)\{1 + MM_1\sqrt{b}\},\ \text{for } t \in [0,b].$$

As $G^{(n,\ x_0)}(D)$ is equicontinuous on each $[t_i, t_{i+1}]$, by Lemma 2.7, we have that

$$B\left(G^{(n,\ x_0)}D\right) = \sup_{t\in[0,b]} \beta((G^{(n,\ x_0)}D)(t)) \le \frac{M^nL^nb^n}{n!}\beta(D)\{1 + MM_1\sqrt{b}\}$$

By the fact that $\frac{M^nL^nb^n}{n!} \to 0$ as $n \to \infty$, we know that there exists a large enough positive integral n_0 such that

$$\frac{M^{n_0}L^{n_0}b^{n_0}}{n_0!} < 1$$

which implies that $G : W \to W$ is a convex-power condensing operator. From lemma 2.5, G has at least one fixed point in W, which is just a mild solution of the non local impulsive problem (1.1). This completes the proof of Theorem 3.1. □

Remark 3.2. By using the method of the measure of noncompactness, we require f to satisfy some proper conditions of MNC, but do not require the compactness of a semigroup $T(t)$. Note that if f *is* compact or Lipschitz continuous, then the condition (Hf) (ii) is satisfied. And our work improves many previous results, where they need the compactness of $T(t)$ of f, or the Lipschitz continuity of f. In the proof, Lemma 2.7 plays an important role for the impulsive differential equations, which provides us with the way to calculate the measure of noncompactness in $PC([0,b];\ X)$. The use of noncompact measure in functional differential and integral equations can also be seen in [18-20, 22].

Remark 3.3. When we apply Darbo-Sadovskii's fixed point theorem to get the fixed point of a map, a strong inequality is needed to guarantee its condensing property. By using the β-convex-power condensing operator developed by sun et al. [29], we do not impose any restrictions on the coefficient L. This generalized condensing operator also can be seen in Liu et al. [31], where nonlinear Volterra integral equations are discussed. In the following, by using Lemma 2.7 and Darbo-Sadovskii's fixed point theorem, we give the existence results of the problem (1.1) under Lipschitz conditions and mixed-type conditions respectively.

We give the following hypothesis:

(Hg2) $g : PC([0,b];X) \to X$ is Lipscitz continuous with the Lipscitz constant k.

(HI2) $I_i : X \to X$ is Lipscitz continuous with the Lipscitz constant k_i; that is,

$$\|I_i(x) - I_i(y)\| \le k_i\|x - y\|,$$

for $x, y \epsilon X,\quad i = 1,2,\ldots\ldots,s$.

Theorem 3.4. Assume that the hypotheses (HA), (Hf), (Hg2), (HI2) are satisfied, then the nonlocal impulsive problem (1.1) has at least one mild solution on $[0,b]$ provided that

$$M\left(k+Lb+MM+1Lb^{3/2}K_w(s)+\sum_{i=1}^{s}k_i\right)<1 \tag{3.4}$$

and (3.3) are satisfied.

Proof. From the proof of Theorem 3.1, we have that the solution operator G is continuous and maps W_r into itself. It remains to show that G is β-condensing in W_r.

By the conditions (Hg2) and (HI2), we get that $G_1 + G_4 : W_r \to PC([0,b];X)$ is Lipschitz continuous with the Lipschitz constant $M(k + \sum_{i=1}^{s} k_i)$. Infact, for $u, w \epsilon W_r$, we have

$$\begin{aligned}\left\|(G_1 + G_4)u - (G_1 + G_4)w\right\|_{PC} &= \sup_{t\in[0,b]}\|T(t)(g(u)-g(w)\| \\ &\quad + \sum_{0<t_i<t}\left\|T(t-t_i)(I_i(u(t_i)) - I_i(w(t_i))\right\| \\ &\le M\left\{\|g(u)-g(w)\| + \sum_{i=1}^{s}\left\|(I_i(u(t_i)) - I_i(w(t_i))\right\|\right\} \\ &\le M\{k+\sum_{i=1}^{s}k_i\}\|u-w\|_{PC}.\end{aligned}$$

Thus from Lemma 2.2(7), we obtain that

$$\beta((G_1 + G_4)W_r) \le M(k+\sum_{i=1}^{s}k_i)\beta(W_r) \tag{3.5}$$

For the operator $(G_2u)(t) = \int_0^t T(t-s)f(s,u(s))ds$, from lemma 2.6, lemma 2.8 and lemma 2.9, We have

$$\begin{aligned}\beta(G_2W_r) &= \sup_{t\in[0,b]}\beta((G_2W_r)(t)) \\ &\le \sup_{t\in[0,b]}\int_0^t \beta(T(t-s)f(s,W_r(s))ds \\ &\le \sup_{t\in[0,b]} M\int_0^t L\beta(W_r(s))ds \\ &\le MLb\beta(W_r)\end{aligned} \tag{3.6}$$

For the operator $(G_3u)(t) = \int_0^t T(t-s)Bv(s)ds$

$$\begin{aligned}\beta(G_3W_r) &= \sup_{t\in[0,b]}\beta((G_3W_r)(t)) \\ &\le \sup_{t\in[0,b]}\int_0^t \beta(T(t-s)Bv(s))ds \\ &\le \sup_{t\in[0,b]} MM_1\int_0^t \beta(v(s))ds \\ &\le M^2M_1Lb^{3/2}K_w(s)\beta(W_r)\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we have

$$\beta(GWr) \le \beta((G_1 + G_4)W_r) + \beta(G_2W_r) + \beta(G_3W_r)$$

$$\leq M\left\{k + \sum_{i=1}^{s} k_i + Lb + MM_1 Lb^{3/2} K_w(s)\right\} \beta(W_r)$$

From the condition (3.4), $M\left(k + Lb + MM_1 Lb^{3/2} K_w(s) + \sum_{i=1}^{s} k_i\right) < 1$ the solution map G is β-condensing in W_r.

By Darbo-sadovskii's fixed point theorem, G has a fixed point in W_r. Which is just a mild solution of the nonlocal impulsive problem (1.1). This completes the proof of Theorem 3.4. □

Among the previous works on nonlocal impulsive differential equations, few are concerned with the mixed-type conditions. Here, by using Lemma 2.7, we can also deal with the mixed-type conditions in a similar way.

Theorem 3.5. Assume that the hypotheses (HA), (Hf), (Hg1), (HI2) are satisfied, then the nonlocal impulsive problem (1.1) has at least one mild solution on $[0, b]$ provided that

$$M\left(1 + MM_1\sqrt{b}K_w\right)\left(Lb + \sum_{i=1}^{s} k_i\right) < 1 \qquad (3.8)$$

and (3.3) are satisfied.

Proof. We will also use Darbo-Sadovskii's fixed point theorem to obtain a fixed point of the solution operator G. From the proof of Theorem 3.1, we have that G is continuous and maps W_r into itself.

subsequently, we show that G is β-condensing in W_r. From the compactness of g and the strong continuity of $T(.)$, we get that $T(.)g(u) : u \in W_r$ is equicontinuous on $[0, b]$. Then by Lemma 2.6, we have that

$$\beta(G_1 W_r) = \sup_{t\in[0,b]} \beta\big((G_1 W_r)(t)\big) = \sup_{t\in[0,b]} \beta(T(t)g(W_r)) = 0 \qquad (3.9)$$

On the other hand, for $u, w \in W_r$, we have

$$\|G_4 u - G_4 w\| = \sup_{t\in[0,b]} \left\|\sum_{0<t_i<t} T(t - t_i)\big(I_i(u(t_i)) - I_i(w(t_i))\big)\right\|$$

$$\leq M \sum_{i=1}^{s} \left\|I_i(u(t_i)) - I_i(w(t_i))\right\|$$

$$\leq M \sum_{i=1}^{s} k_i \|u - w\|_{PC}$$

Then by Lemma 2.2(7), we obtain that

$$\beta(G_4 W_r) \leq M \sum_{I=1}^{s} k_i\, \beta(W_r) \qquad (3.10)$$

and

$$\beta(G_3 W_r) \leq M^2 M_1 L\sqrt{b}K_w\left(Lb + \sum_{I=1}^{s} k_i\right) \beta(W_r) \qquad (3.11)$$

Combining (3.6),(3.9),(3.10) and (3.11), we get that

$$\beta(GW_r) \leq \beta(G_1 W_r) + \beta(G_2 W_r) + \beta(G_3 W_r) + \beta(G_4 W_r)$$

$$\leq M\left(1 + MM_1\sqrt{b}K_w\right)\left(Lb + \sum_{I=1}^{s} k_i\right) \beta(W_r)$$

From the condition (3.8), the map G is β-condensing in W_r. So, G has a fixed point in W_r due to Darbo-Sadovskii's fixed point theorem, which is just a mild solution of the nonlocal impulsive problem (1.1). This completes the proof of Theorem 3. □

Theorem 3.6. Assume that the hypotheses (HA), (Hf), (Hg2), (HI1) are satisfied, then the nonlocal impulsive problem (1.1) has atleast one mild solution on $[0, b]$ provided that

$$M\big(1 + MM_1\sqrt{b}K_w\big)\,(k + Lb\big) < 1 \tag{3.12}$$

and (3.3) are satisfied.

Proof. From the proof of Theorem 3.1, we have that the solution operator G is continuous and maps W_r into itself. In the following, we shall show that G is β-condensing in W_r.

By the Lipschitz continuity of g, we have that for $u, w \in W_r$,

$$\|G_1u - G_1w\|_{PC} = \sup_{t\in[0,b]} \|T(t)(g(u) - g(w))\| \leq Mk\|u - w\|_{PC}$$

Then by Lemma 2.2(7), we obtain that

$$\beta(G_1W_r) \leq Mk\beta(W_r) \tag{3.13}$$

Similar to the discussion in Theorem 3.1, from the compactness of I_i and the strong continuity $T(.)$, we get that G_4W_r is equicontinuous on each $\overline{J_i} = [t_i, t_{i+1}], i = 0,1, \ldots, s$. Then by Lemma 2.7, we have that

$$\beta(G_4W_r) = \sup_{t\in[0,b]} \beta((G_4W_r))(t)) \leq \textstyle\sum_{i=1}^{s} \beta(T(t - t_i)\, I_i(W_r(t_i))) = 0 \tag{3.14}$$

and

$$\beta(G_3W_r) = \sup_{t\in[0,b]} \beta((G_3W_r)(t)) \leq M^2M_1K_w\sqrt{b}(k + Lb)\beta(W_r) \tag{3.15}$$

Combining (3.6), (3.13), (3.14) and (3.15), we have that

$$\beta(GW_r) \leq \beta(G_1W_r) + \beta(G_2W_r) + \beta(G_3W_r) + \beta(G_4W_r)$$

$$\leq M\big(1 + MM_1\sqrt{b}K_w)\,(k + Lb\big)\beta(W_r)$$

From condition (3.12), the map G is β-condensing in W_r. So, G has a fixed point in W_r. due

to Darbo-Sadovskii's fixed point theorem, which is just a mild solution of the nonlocal impulsive

problem (1.1). This completes the proof of Theorem 3.6. □

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