ii-Separation Axioms in Topological Spaces

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***Abstract**. The aim of this paper is to study and investigate the ii-open sets in topological spaces and to obtain a relationship among b-open, pre-open, semi-open, α-open and β-open sets. Some new types of separation axioms such as ii-T_0, ii-T_1, ii-T_2, ii-R_0 and ii-R_1 axioms in topological spaces by using ii-open sets also are introduced. The relationships among ii-T_0, ii-T_1, ii-T_2 and some other separation axioms are investigated.*

Keywords: *ii-open and ii-closed sets; ii-continuous and ii-irresolute functions; ii-T_k (k = 0, 1, 2) and ii-R_k (k = 0, 1) spaces.*
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I. Introduction

In 1963, N. Levine [**5**] introduced the notion of semi-open sets which is a weaker form of open sets in topological spaces. In 1965, Njastad [**11**] introduced the notion of α-open sets. In 1975, Maheswari and Prasad [**6**] used semi-open sets to introduce the concepts of semi-T_0, semi-T_1 and semi-T_2 spaces. In 1980, Maheswari and Prasad [**7**] introduced the concept of α-T_2 space. In 1982, Mashhour [**8**] introduced the notion of pre-open sets and obtained their properties. In 1983, Monsef et al. [**1**] introduced and investigated the notion of β-open sets in topological spaces. In 1993, Maki et al. [**9**] introduced the concept of α-T_0 and α-T_0 spaces. In 1996, Andrijevic [**2**] introduced a new class of generalized open sets, called, b-open sets in topological spaces. This type of open sets were discussed by [**4**] under the name of γ-open sets. In 2006, Park [**12**] introduced the concept of b-T_2 spaces. In 2007, Caldas and Jafari [**3**] introduced and studied b-T_0 and b-T_1 spaces via b-open sets due to Andrijevic [**2**]. In 2019, Mohammed and Abdullah [**10**] introduced and investigated the notion of ii-open sets.

II. Preliminaries

Throughout this paper, spaces $(X, \Im)$, (Y, τ), and (Z, η) (or simply X, Y and Z) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of X, **Cl(A)** and **Int(A)** represents the closure of A and Interior of A respectively.

Definition 2.1. A subset A of a topological space $(X, \Im)$ is said to be

(i) pre-open set [**8**] if $A \subset Int(Cl(A))$;

(ii) semi-open set [**5**] if $A \subset Cl(Int(A))$;

(iii) α-open [**11**] if $A \subset Int(Cl(Int(A)))$;

(iv) β-open [**1**] if $A \subset Cl(Int(Cl(A)))$;

(v) b-open [**2**] (or γ-open [**4**]) if $A \subset In(Cl(A)) \cup Cl(Int(A))$.

The complement of the pre-open (resp. semi-open, α-open, β-open, b-open) set is called pre-closed (resp. semi-closed, α-closed, β-closed, b-closed).

Definition 2.1. A subset A of a topological space $(X, \Im)$ is said to be **ii-open** [**10**] set if there exists an open set $G \in \Im$, such that

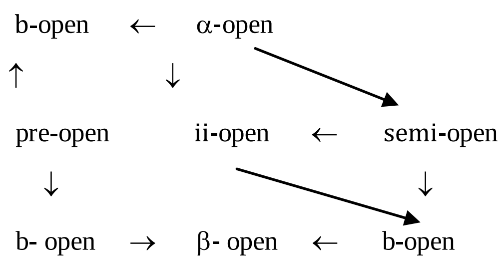
(i) $G \neq \phi$, X

(ii) $A \subset Cl(A \cap G)$

(iii) $Int(A) = G$.

The complement of the ii-open set is called ii-closed. We denote the family of all ii-open (resp. ii-closed) sets of a topological space by ii-O(X) (resp. ii-C(X)). The ii-closure of a subset A of X, is the intersection of all ii-closed sets containing A in X and is denoted by **ii-Cl(A)**. The ii-interior of a subset A of X is the union of all ii-open sets contained in A and is denoted by ii-Int(A).

Remark 2.2. For a subset of a space, we have following implications:



Where none of the implications is reversible as can be seen from the following examples:

Example 2.3. Let X = {a, b, c, d} and ℑ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. Then

(1) b-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(2) pre-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}.

(3) semi-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(4) α-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}.

(5) β-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(6) ii-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

Example 2.4. Let X = {a, b, c} and ℑ = {ϕ, {a}, {b}, {a, b}, X}. Then

(1) b-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}, {b, c}.

(2) pre-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}.

(3) semi-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {b, c}.

(4) α-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}.

(5) β-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}.

(6) ii-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {a, b}, {a, c}, {b, c}.

Example 2.5. Let X = {a, b, c} and ℑ = {ϕ, {a}, {b, c}, X}. Then

(1) b-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}.

(2) pre-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}.

(3) semi-open sets in (X, ℑ) are ϕ, X, {a}, {b, c}.

(4) α-open sets in (X, ℑ) are ϕ, X, {a}, {b, c}.

(5) β-open sets in (X, ℑ) are ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}.

(6) ii-open sets in (X, ℑ) are ϕ, X, {a}, {b, c}.

Example 2.6. Let X = {a, b, c, d} and ℑ = {ϕ, {a}, {b, c, d}, X}. Then

(1) b-open sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(2) pre-open sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(3) semi-open sets in (X, $\Im$) are ϕ, X, {a}, {b, c, d}.

(4) α-open sets in (X, $\Im$) are ϕ, X, {a}, {b, c, d}.

(5) β-open sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

(6) ii-open sets in (X, $\Im$) are ϕ, X, {a}, {b}, {c}, {d}, {b, c}, {b, d}, {c, d}, {b, c, d}.

Remark 2.7. The concepts of ii-open and pre-open sets are independent as shown in the above examples.

Remark 2.8. The concepts of ii-open and β-open sets are independent as shown in the above examples.

Definition 2.9. A space X is said to be:

(i) **b-T_0** [**3**] if for each pair of distinct points x and y in X, there exists a b-open set G containing x but not y or a b-open set H containing y but not x.

(ii) **b-T_1** [**3**] if for each pair of distinct points x, y in X, there exist a b-open set G containing x but not y and a b-open set H containing y but not x.

(iii) **b-T_2** [**12**] if for each pair of distinct points x, y of X, there exist two disjoint b-open sets U and V containing x and y respectively.

Definition 2.10. A space X is said to be:

(i) **α-T_0** [**9**] if for each pair of distinct points x and y in X, there exists an α-open set G containing x but not y or an α-open set H containing y but not x.

(ii) **α-T_1** [**9**] if for each pair of distinct points x, y in X, there exist an α-open set G containing x but not y and an α-open set H containing y but not x.

(iii) **α-T_2** [**7**] if for each pair of distinct points x, y of X, there exist two of disjoint α-open sets U and V containing x and y respectively.

Definition 2.11. A space X is said to be:

(i) **semi-T_0** [**6**] if for each pair of distinct points x and y in X, there exists a semi-open set G containing x but not y or a semi-open set H containing y but not x.

(ii) **semi-T_1** [**6**] if for each pair of distinct points x, y in X, there exist a semi-open set G containing x but not y and a semi-open set H containing y but not x.

(iii) **semi-T_2** [**6**] if for each pair of distinct points x, y of X, there exist two disjoint semi-open sets U and V containing x and y respectively.

Theorem 2.12. (i) Every open set is ii-open.

(ii) Every α-open set is ii-open.

(iii) Every semi-open set is ii-open.

(iv) Every ii-open set is b-open.

Definition 2.13. Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is said to be **ii-continuous** if the inverse image of every open set in Y is ii-open in X.

Definition 2.14. Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is said to be **ii-closed** if the image of every closed set in X is ii-closed in Y.

Definition 2.15. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **ii-irresolute** if the inverse image of every ii-closed set in Y is ii-closed in X.

Definition 2.16. Let X be a topological space. A subset $N \subset X$ is called an **ii-neighbourhood** (briefly **ii-nhd**) of a point $x \in X$ if there exists an ii-open set G such that $x \in G \subset N$.

III. ii-T_0 Spaces

In this section, we define ii-T_0 space and study some of their properties via some other weaker forms of open sets.

Definition 3.1. A topological space $(X, \Im)$ is said to be **ii-T_0** if for each pair of distinct points x, y in X, there exists an ii-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

Theorem 3.2. (i) Every semi-T_0 space is ii-T_0.

(ii) Every α-T_0 space is ii-T_0.

(iii) Every ii-T_0 space is b-T_0.

Proof. (i) Let X be a semi-T_0 space. Let x and y be any two distinct points in X. Since X is semi-T_0, there exists a semi-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. By **Theorem 2.12 (iii)**, U is an ii-open set such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Thus X is ii-T_0.

(ii) Since every α-open set is ii-open and so, by the **Theorem 2.12 (ii)**, every α-T_0 space is ii-T_0.

(iii) Since every ii-open set is b-open and so, by the **Theorem 2.12 (iv)**, every ii-T_0 space is b-T_0.

Theorem 3.3. Every topological space is ii-T_0.

Proof. Since every open set is ii-open and so, by the **Theorem 2.12 (i)**, every T_0 space is ii-T_0.

Theorem 3.4. A topological space $(X, \Im)$ is ii-T_0 if and only if for each pair of distinct points x, y of X, ii-Cl({x}) $\neq$ ii-Cl({y}).

Proof. **Necessity**. Let $(X, \Im)$ be an ii-T_0 space and x, y be any two distinct points of X. There exists an ii-open set U containing x or y, say x but not y. Then $X - U$ is an ii-closed set which does not contain x but contains y. Since ii-bCl({y}) is the smallest ii-closed set containing y, ii-Cl({y}) $\subset X - U$ and therefore $x \notin$ ii-Cl({y}). Consequently ii-Cl({x}) $\neq$ ii-Cl({y}).

Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and ii-Cl({x}) $\neq$ ii-Cl({y}). Let z be a point of X such that $z \in$ ii-Cl({x}) but $z \notin$ ii-Cl({y}). We claim that $x \notin$ ii-Cl({y}). For, if $x \in$ ii-Cl({y}) then ii-Cl({x}) $\subset$ ii-Cl({y}). This contradicts the fact that $z \notin$ ii-Cl({y}). Consequently x belongs to the ii-open set $X -$ ii-Cl({y}) to which y does not belong.

Theorem 3.5. Every subspace of an ii-T_0 space is ii-T_0.

Proof. Let (Y,τ) be a subspace of a topological space $(X, \Im)$, where τ is the relative topology of $\Im$ on Y. Let x, y be two distinct points of Y. As $Y \subset X$, x and y are also distinct points of X and there exists an ii-open set G such that $x \in G$ but $y \notin G$, since X is ii-T_0. Then $G \cap Y$ is an ii-open set in (Y, τ) which contains x but does not contain y. Hence (Y, τ) is an ii-T_0 space.

IV. ii-T_1 Spaces

In this section, we define ii-T_1 space and study some of their properties via some other weaker forms of open sets.

Definition 4.1. A topological space $(X, \Im)$ is said to be **ii-T_1** if for each pair of distinct points x, y in X, there exist two ii-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Theorem 4.2. (i) Every semi-T_1 space is ii-T_1.

(ii) Every α-T_1 space is ii-T_1.

(iii) Every ii-T_1 space is b-T_1.

Proof. (i). Suppose X is a semi-T_1 space. Let x and y be two distinct points in X. Since X is semi-T_1**,** there exist semi-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. By **Theorem 2.12 (iii)**, every semi-open set is ii-open, so U and V are ii-open sets such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. Hence X is ii-T_1**.**

(ii). Since every α-open set is ii-open and so, by the **Theorem 2.12 (ii)**, every α-T_1 space is ii-T_1.

(iii) Since every ii-open set is b-open and so, by the **Theorem 2.12 (iv)**, every ii-T_1 space is b-T_1.

Theorem 4.3. Let $f : X \to Y$ be an ii-irresolute, injective map. If Y is ii-T_1, then X is ii-T_1.

Proof. Assume that Y is ii-T_1. Let $x, y \in Y$ with $x \neq y$. Then there exists a pair of ii-open sets U, V of Y such that $f(x) \in U$, $f(y) \in V$ and $f(x) \notin V$, $f(y) \notin U$. Then $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$ and $y \in f^{-1}(V)$, $x \notin f^{-1}(V)$. Since f is ii-irresolute, X is ii-T_1.

Theorem 4.4. A topological space $(X, \Im)$ is ii-T_1 if and only if the singletons are ii-closed sets.

Proof. Let $(X, \Im)$ be ii-T_1 and x be any point of X. Suppose $y \in X - \{x\}$, then $x \neq y$ and so there exists an ii-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subset X - \{x\}$, that is $X - \{x\} = \cup \{U : y \in X - \{x\}\}$ which is ii-open.

Conversely, suppose {p} is ii-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X - \{x\}$. Hence $X - \{x\}$ is an ii-open set contains y but not x. Similarly $X - \{y\}$ is an ii-open set contains x but not y. Accordingly, X is an ii-T_1 space.

Theorem 4.5. Let $f : X \to Y$ be bijective.

(i) If f is ii-continuous and $(Y, \Im_2)$ is T_1, then $(X, \Im_1)$ is ii-T_1.

(ii) If f is ii-open and $(X, \Im_1)$ is ii-T_1, then $(Y, \Im_2)$ is ii-T_1.

Proof. Let $f : (X, \Im_1) \to (Y, \Im_2)$ be bijective.

(i) Suppose $f : (X, \Im_1) \to (Y, \Im_2)$ is ii-continuous and $(Y, \Im_2)$ is T_1. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is bijective, $y_1 = f(x_1) \neq f(x_2) = y_2$ for some $y_1, y_2 \in Y$. Since $(Y, \Im_2)$ is T_1, there exist open sets G sand H such that $y_1 \in G$ but $y_2 \notin G$ and $y_2 \in H$ but $y_1 \notin H$. Since f is bijective, $x_1 = f^{-1}(y_1) \in f^{-1}(G)$ but $x_2 = f^{-1}(y_2) \notin f^{-1}(G)$ and $x_2 = f^{-1}(y_2) \in f^{-1}(H)$ but $x_1 = f^{-1}(y_1) \notin f^{-1}(H)$. Since f is ii-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are ii-open sets in $(X, \Im_1)$. It follows that $(X, \Im_1)$ is ii-T_1. This proves (i).

(ii) Suppose f is ii-open and $(X, \Im_1)$ is ii-T_1. Let $y_1 \neq y_2 \in Y$. Since f is bijective, there exist x_1, x_2 in X, such that $y_1 = f(x_1)$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since $(X, \Im_1)$ is ii-T_1, there exist ii-open sets G sand H in X such that $x_1 \in G$ but $x_2 \notin G$ and $x_2 \in H$ but $x_1 \notin H$. Since f is ii-open, f(G) and f(H) are ii-open in Y such that $y_1 = f(x_1) \in f(G)$ and $y_2 = f(x_2) \in f(H)$. Again since f is bijective, $y_2 = f(x_2) \notin f(G)$ and $y_1 = f(x_1) \notin f(H)$. Thus $(Y, \Im_2)$ is ii-T_1. This proves (ii).

Definition 4.6. A topological space $(X, \Im)$ is said to be **ii-symmetric** if for x and y in X, $x \in$ ii-Cl({y}) implies $y \in$ ii-Cl({x}).

Theorem 4.7. If $(X, \Im)$ is a topological space, then the following are equivalent:

(i) $(X, \Im)$ is an ii-symmetric space.

(ii) {x} is ii-closed, for each $x \in X$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\{x\} \subset U \in$ ii-O(X), but ii-Cl({x}) $\not\subset$ U. Then ii-Cl({x}) $\cap (X - U) \neq \phi$. Now, we take $y \in$ ii-Cl({x}) $\cap (X - U)$, then by hypothesis $x \in$ ii-Cl({y}) $\subset X - U$ and $x \notin U$, which is a contradiction. Therefore {x} is ii-closed, for each $x \in X$.

(ii) $\Rightarrow$ (i). Assume that $x \in$ ii-Cl({y}), but $y \notin$ ii-Cl({x}). Then $\{y\} \subset X -$ ii-Cl({x}) and hence ii-Cl({y}) $\subset X -$ ii-Cl({x}). Therefore $x \in X -$ ii-Cl({x}), which is a contradiction and hence $y \in$ ii-Cl({x}).

Corollary 4.8. If a topological space $(X, \Im)$ is an ii-T_1 space, then it is ii-symmetric.

Proof. In an ii-T_1 space, every singleton is ii-closed (**Theorem 4.4**) and therefore is by **Theorem 4.7**, $(X, \Im)$ is ii-symmetric.

Corollary 4.9. If a topological space $(X, \Im)$ is ii-symmetric and ii-T_0, then $(X, \Im)$ is ii-T_1.

Proof. Let $x \neq y$ and as $(X, \Im)$ is ii-T_0, we may assume that $x \in U \subset X - \{y\}$ for some $U \in$ ii-O(X). Then $x \notin$ ii-Cl($\{y\}$) and hence $y \notin$ ii-Cl($\{x\}$). There exists an ii-open set V such that $y \in V \subset X - \{x\}$ and thus $(X, \Im)$ is an ii-T_1 space.

V. ii-T_2 Spaces

In this section, we introduce ii-T_2 space and investigate some of their basic properties via some other weaker forms of open sets.

Definition 5.1. A space X is said to be ii-T_2 if for every pair of distinct points x and y in X, there exist disjoint ii-open sets U and V of X containing x and y respectively.

Remark 5.2. For a space, we have following implications:

$$\begin{array}{ccccc} T_2 & \rightarrow & T_1 & \rightarrow & T_0 \\ \downarrow & & \downarrow & & \downarrow \\ \alpha\text{-}T_2 & \rightarrow & \alpha\text{-}T_1 & \rightarrow & \alpha\text{-}T_0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{semi-}T_2 & \rightarrow & \text{semi-}T_1 & \rightarrow & \text{semi-}T_0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{ii-}T_2 & \rightarrow & \text{ii-}T_1 & \rightarrow & \text{ii-}T_0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{b-}T_2 & \rightarrow & \text{b-}T_1 & \rightarrow & \text{b-}T_0 \end{array}$$

Where none of the implications are reversible as can be seen from the following examples.

Example 5.3.Consider the space $(X, \Im)$, where $X = \{a, b, c, d\}$ and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Clearly $(X, \Im)$ is ii-T_1 as well as b-T_1. But it is neither T_1 nor α-T_1.

Example 5.4. Consider the space $(X, \Im)$, where $X = \{a, b, c, d\}$ and $\Im = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Then $(X,\Im)$ is b-T_1 but not ii-T_1. But it is neither semi-T_1 nor α-T_1.

Example 5.5. Consider the space $(X, \Im)$, where $X = \{a, b, c\}$ and $\Im = \{\phi, \{a, b\}, X\}$. Then $(X,\Im)$ is b-T_2 but not ii-T_2. But it is neither semi-T_2 nor α-T_2.

Example 5.6. Consider the space $(X, \Im)$, where $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b, c\}, X\}$. Then $(X,\Im)$ is b-T_2 but not ii-T_2. But it is neither semi-T_2 nor α-T_2.

Example 5.7. Consider the space $(X, \Im)$, where $X = \{a, b, c, d\}$ and $\Im = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then $(X,\Im)$ is semi-T_2 as well as ii-T_2. But it is not α-T_2.

Example 5.8. Consider the space $(X, \Im)$, where $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{b, c\}, X\}$. Then $(X,\Im)$ is b-T_0. But it is neither ii-T_0 nor α-T_0.

Theorem 5.9. (i) Every semi-T_2 space is ii-T_2.

(ii) Every α-T_2 space is ii-T_2.

(iii) Every ii-T_2 space is b-T_2.

Proof. (i) Let X be a semi-T_2 space. Let x and y be two distinct points in X. Since X is semi-T_2, there exist disjoint semi-open sets U and V such that $x \in U$ and $y \in V$. By **Theorem 2.12 (iii)**, U and V are disjoint ii-open sets such that $x \in U$ and $y \in V$. Hence X is ii-T_2.

(ii) Suppose X is α-T_2 space. Let x and y be two distinct points in X. Since X is α-T_2, there exist disjoint α-open sets U and V such that $x \in U$ and $y \in V$. By **Theorem 2.12 (ii)**, U and V are disjoint ii-open sets such that $x \in U$ and $y \in V$. Hence X is ii-T_2.

(iii) Suppose X is ii-T_2 space. Let x and y be two distinct points in X. Since X is ii-T_2, there exist disjoint ii-open sets U and V such that $x \in U$ and $y \in V$. By **Theorem 2.12 (iv)**, U and V are disjoint b-open sets such that $x \in U$ and $y \in V$. Hence X is b-T_2.

Theorem 5.10. Every ii-T_2 space is ii-T_1.

Proof. Let X be an ii-T_2 space. Let x and y be two distinct points of X. Since X is ii-T_2, there exist disjoint ii-open sets U and V such that $x \in U$ and $y \in V$. Since U and V are disjoint, so $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. Hence X is ii-T_1.

Theorem 5.11. For a topological space X, the following are equivalent:

(i) X is an ii-T_2 space.

(ii) Let $x \in X$. Then for each $x \neq y$, there exists an ii-open set U such that $x \in U$ and $y \notin$ ii-Cl(U).

(iii) For each $x \in X$, $\cap$ {ii-Cl(U) : $U \in$ ii-O(X) and $x \in U$} = {x}.

Proof. (i) $\Rightarrow$ (ii). Suppose X is an ii-T_2 space. Then for each $x \neq y$, there exist disjoint ii-open sets U and V such that $x \in U$ and $y \in V$. Since V is ii-open, V^c is ii-closed and $U \subset V^c$. This implies that ii-Cl(U) $\subset V^c$. Since $y \notin V^c$, $y \notin$ ii-Cl(U).

(ii) $\Rightarrow$ (iii). If $y \neq x$, then there exists an ii-open set U such that $x \in U$ and $y \notin$ ii-Cl(U). Therefore $y \notin \cap$ {ii-Cl(U) : $U \in$ ii-O(X) and $x \in U$}. Therefore $\cap$ {ii-Cl(U) : $U \in$ ii-O(X) and $x \in U$} = {x}. This proves (iii).

(iii) $\Rightarrow$ (i). Let $y \neq x$ in X. Then $y \notin \{x\} = \cap$ {ii-Cl(U) : $U \in$ ii-O(X) and $x \in U$}. This implies that there exists an ii-open set U such that $x \in U$ and $y \notin$ ii-Cl(U). Let V = $(\text{ii-Cl(U)})^c$. Then V is ii-open and $y \in V$. Now $U \cap V = U \cap (\text{ii-Cl(U)})^c \subset U \cap (U)^c = \phi$. Therefore, X is ii-$T_2$ space.

Theorem 5.12. Let $f : X \to Y$ be a bijection.

(i) If f is ii-open and X is T_2, then Y is ii-T_2.

(ii) If f is ii-continuous and Y is T_2, then X is ii-T_2.

Proof. Let $f : X \to Y$ be a bijection.

(i) Suppose F is ii-open and X is T_2. Let $y_1 \neq y_2 \in Y$. Since f is a bijection. There exist x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$ with $x_1 \neq x_2$. Since X is T_2, there exist disjoint open sets U and V in X such that $x_1 \in U$ and $x_2 \in V$. Since f is ii-open, f(U) and f (V) are ii-open in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Again since f is a bijection, f(U) and f(V) are disjoint in Y. Thus Y is ii-T_2.

(ii) Suppose $f : X \to Y$ is ii-continuous and Y is T_2. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is one-one, $y_1 \neq y_2$. Since Y is T_2, there exist disjoint open sets U and V containing y_1 and y_2 respectively. Since f is ii-continuous bijective, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint ii-open sets containing x_1 and x_2 respectively. Thus X is ii-T_2.

Theorem 5.13. A topological space $(X, \Im)$ is ii-T_2 if and only if the intersection of all ii-closed, ii-neighbourhoods of each point of the space is reduced to that point.

Proof. Let $(X, \Im)$ be ii-T_2 and $x \in X$. Then for each $y \neq x$ in X, there exist disjoint ii-open sets U and V such that $x \in U$, $y \in V$. Now $U \cap V = \phi$ implies $x \in U \subset V^c$. Therefore V^c is an ii-neighbourhood of x. Since V is ii-open, V^c is ii-closed and ii-neighbourhood of x to which y does not belong. That is there is an ii-closed, iineigbourhoods of x which does not contain y. So we get the intersection of all ii-closed, ii-neighbourhood of x is {x}.

Conversely, let x, y $\in$ X such that x $\neq$ y in X. Then by assumption, there exist an ii-closed, ii-neighbourgood V of x such that y $\notin$ V. Now there exists an ii-open set U such that x $\in$ U $\subset$ V. Thus U and V^c are disjoint ii-open sets containing x and y respectively. Thus (X, $\Im$) is ii-T_2.

Theorem 5.14. If f : X $\to$ Y be bijective, ii-irresolute map and X is ii-T_2, then (X, $\Im_2$) is ii-T_2.

Proof. Suppose f : (X, $\Im_1$) $\to$ (Y, $\Im_2$) is bijective. f is ii-irresolute, and (Y, $\Im_2$) is ii-T_2. Let x_1, x_2 $\in$ X with $x_1 \neq x_2$. Since f is bijective, $y_1 = f(x_1) \neq f(x_2) = y_2$ for some y_1, y_2 $\in$ Y. Since (Y, $\Im_2$) is ii-T_2, there exist disjoint ii-open sets G and H such that $y_1 \in$ G and $y_2 \in$ H. Again since f is bijective, $x_1 = f^{-1}(y_1) \in f^{-1}(G)$ and $x_2 = f^{-1}(y) \in f^{-1}(H)$. Since f is ii-irresolute, $f^{-1}(G)$ and $f^{-1}(H)$ are ii-open sets in (X, $\Im_1$). Also f is bijective, G $\cap$ H = ϕ implies that $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f(\phi) = \phi$. It follows that (X, $\Im_2$) is ii-T_2.

VI. ii-R_k Space (k = 0, 1)

In this section, new classes of topological spaces called, ii-R_0 and ii-R_1 spaces are introduced.

Definition 6.1. A topological space (X, $\Im$) is said to be **ii-R_0** if U is an ii-open set and x $\in$ U then ii-Cl({x}) $\subset$ U.

Theorem 6.2. For a topological space (X, $\Im$) the following properties are equivalent:

(i) (X, $\Im$) is ii-R_0.

(ii) For any F $\in$ ii-C(X), x $\notin$ F implies F $\subset$ U and x $\notin$ U for some U $\in$ ii-O(X).

(iii) For any F $\in$ ii-C(X), x $\notin$ F implies F $\cap$ ii-Cl({x}) = ϕ.

(iv) For any distinct points x and y of X, either ii-Cl({x}) = ii-Cl({y}) or ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ.

Proof. (i) $\Rightarrow$ (ii). Let F $\in$ ii-C(X) and x $\notin$ F. Then by (i), ii-Cl({x}) $\subset$ X – F. Set U = X – ii-Cl({x}), then U is an ii-open set such that F $\subset$ U and x $\notin$ U.

(ii) $\Rightarrow$ (iii). Let F $\in$ ii-C(X) and x $\notin$ F. There exists U $\in$ ii-O(X) such that F $\subset$ U and x $\notin$ U. Since U $\in$ ii-O(X), U $\cap$ ii-Cl({x}) = ϕ and F $\cap$ ii-Cl({x}) = ϕ.

(iii) $\Rightarrow$ (iv). Suppose that ii-Cl({x}) $\neq$ ii-Cl({y}) for distinct points x, y $\in$ X. There exists z $\in$ ii-Cl({x}) such that z $\notin$ ii-Cl({y}) (or z $\in$ ii-Cl({y}) such that z $\notin$ ii-Cl({x})). There exists V $\in$ ii-O(X) such that y $\notin$ V and z $\in$ V; hence x $\in$ V. Therefore, we have x $\notin$ ii-Cl({y}). By (iii), we obtain ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ.

(iv) $\Rightarrow$ (i). Let V $\in$ ii-O(X) and x $\in$ V. For each y $\notin$ V, x $\neq$ y and x $\notin$ ii-Cl({y}). This shows that ii-Cl({x}) $\neq$ ii-Cl({y}). By (iv), ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ for each y $\in$ X – V and hence ii-Cl({x}) $\cap$ ($\cup_{y \in X - V}$ ii-Cl({y})) = ϕ. On other hand, since V $\in$ ii-O(X) and y $\in$ X – V, we have ii-Cl({y}) $\subset$ X – V and hence X – V = $\cup_{y \in X - V}$ ii-Cl({y}). Therefore, we obtain (X – V) $\cap$ ii-Cl({x}) = ϕ and ii-Cl({x}) $\subset$ V. This shows that (X, $\Im$) is an ii-R_0 space.

Theorem 6.3. If a topological space (X, $\Im$) is ii-T_0 and an ii-R_0 space then it is ii-T_1.

Proof. Let x and y be any distinct points of X. Since X is ii-T_0, there exists an ii-open set U such that x $\in$ U and y $\notin$ U. As x $\in$ U implies that ii-Cl({x}) $\subset$ U. Since y $\notin$ U, so y $\notin$ ii-Cl({x}). Hence y $\in$ V = X – ii-Cl({x}) and it is clear that x $\notin$ V. Hence it follows that there exist ii-open sets U and V containing x and y respectively, such that y $\notin$ U and x $\notin$ V. This implies that X is ii-T_1.

Theorem 6.4. For a topological space (X, $\Im$) the following properties are equivalent:

(i) (X, $\Im$) is ii-R_0.

(ii) x $\in$ ii-Cl({y}) if and only if y $\in$ ii-Cl({x}), for any points x and y in X.

Proof. (i) $\Rightarrow$ (ii). Assume that X is ii-R_0. Let $x \in$ ii-Cl({y}) and V be any ii-open set such that $y \in V$. Now, by hypothesis, $x \in V$. Therefore, every ii-open set which contain y contains x also. Hence $y \in$ ii-Cl({x}).

(ii) $\Rightarrow$ (i). Let U be an ii-open set and $x \in U$. If $y \notin U$, then $x \notin$ ii-Cl({y}) and hence $y \notin$ ii-Cl({x}). This implies that ii-Cl({x}) $\subset$ U. Hence (X, $\Im$) is ii-R_0.

From **Definition 4.6** and **Theorem 6.4**, the notions of ii-symmetric and ii-R_0 are equivalent.

Definition 6.5. Let A be a subset of a topological space (X, $\Im$). The **ii-kernel** of A, denoted by ii-ker(A) is defined to be the set

$$\text{ii-ker}(A) = \cap \{U \in \text{ii-O}(X) : A \subset U\}.$$

Theorem 6.6. Let (X, $\Im$) be a topological space and $x \in X$. Then $y \in$ ii-ker({x}) if and only if $x \in$ ii-Cl({y}).

Proof. Suppose that $y \notin$ ii-ker({x}). Then there exists an ii-open set V containing x such that $y \notin V$. Therefore, we have $x \notin$ ii-Cl({y}). The proof of the converse case can be done similarly.

Theorem 6.7. Let (X, $\Im$) be a topological space and A be a subset of X. Then, ii-ker(A) = {$x \in X$: ii-Cl({x}) $\cap$ A $\neq \phi$}.

Proof. Let $x \in$ ii-ker(A) and suppose ii-Cl({x}) $\cap$ A = ϕ. Hence $x \notin$ X – ii-Cl({x}) which is an ii-open set containing A. This is impossible, since $x \in$ ii-ker(A). Consequently, ii-Cl({x}) $\cap$ A $\neq \phi$. Next, let $x \in X$ such that ii-Cl({x}) $\cap$ A $\neq \phi$ and suppose that $x \notin$ ii-ker(A). Then, there exists an ii-open set V containing A and $x \notin$ V. Let $y \in$ ii-Cl({x}) $\cap$ A. Hence, V is an ii-neighbourhood of y which does not contain x. By this contradiction $x \in$ ii-ker(A) and the claim.

Definition 6.8. A subset A of a topological space X is called an **ii-difference set** (briefly, **ii-D-set**) if there are U, V $\in$ ii-O(X) such that U $\neq$ X and A = U – V.

Theorem 6.9. The following properties hold for the subsets A, B of a topological space (X, $\Im$):

(i) A $\subset$ ii-ker(A).

(ii) A $\subset$ B implies that ii-ker(A) $\subset$ ii-ker(B).

(iii) If A is ii-open in (X, $\Im$), then A = ii-ker(A).

(iv) ii-ker(ii-ker(A)) = ii-ker(A).

Proof. (i), (ii) and (iii) are immediate consequences of **Definition 6.5**. To prove (iv), first observe that by (i) and (ii), we have ii-ker(A) $\subset$ ii-ker(ii-ker(A)). If $x \notin$ ii-ker(A), then there exists U $\in$ ii-O(X) such that A $\subset$ U and $x \notin$ U. Hence ii-ker(A) $\subset$ U, and so we have $x \notin$ ii-ker(ii-ker(A)). Thus ii-ker(ii-ker(A)) = ii-ker(A).

Proposition 6.10. If a singleton {x} is an ii-D-set of (X, $\Im$), then ii-ker({x}) $\neq$ X.

Proof. Since {x} is an ii-D-set of (X, $\Im$), then there exist two subsets U_1, U_2 $\in$ ii-O(X) such that {x} = $U_1 - U_2$, {x} $\subset U_1$ and $U_1 \neq X$. Thus, we have that ii-ker({x}) $\subset U_1 \neq X$ and so ii-ker({x}) $\neq$ X.

Theorem 6.11. The following statements are equivalent for any points x and y in a topological space (X, $\Im$):

(i) ii-ker({x}) $\neq$ ii-ker({y}).

(ii) ii-Cl({x}) $\neq$ ii-Cl({y}).

Proof. (i) $\Rightarrow$ (ii). Suppose that ii-ker({x}) $\neq$ ii-ker({y}), then there exists a point z in X such that $z \in$ ii-ker({x}) and $z \notin$ ii-ker({y}). From $z \in$ ii-ker({x}) it follows that {x} $\cap$ ii-Cl({z}) $\neq \phi$ which implies $x \in$ ii-Cl({z}). By $z \notin$ ii-ker({y}), we have {y} $\cap$ ii-Cl({z}) = ϕ. Since $x \in$ ii-Cl({z}), ii-Cl({x}) $\subset$ ii-Cl({z}) and {y} $\cap$ ii-Cl({x}) = ϕ. Therefore, it follows that ii-Cl({x}) $\neq$ ii-Cl({y}). Now ii-ker({x}) $\neq$ ii-ker({y}) implies that ii-Cl({x}) $\neq$ ii-Cl({y}).

(ii) $\Rightarrow$ (i). Suppose that ii-Cl({x}) $\neq$ ii-Cl({y}). Then there exists a point z in X such that z $\in$ ii-Cl({x}) and z $\notin$ ii-Cl({y}). Then, there exists an ii-open set containing z and therefore x but not y, namely, y $\notin$ ii-ker({x}) and thus ii-ker({x}) $\neq$ ii-ker({y}).

Theorem 6.12. Let (X, $\Im$) be a topological space. Then $\cap$ {ii-Cl({x}) : x $\in$ X} = ϕ if and only if ii-ker({x}) $\neq$ X for every x $\in$ X.

Proof. **Necessity**. Suppose that $\cap$ {ii-Cl({x}) : x $\in$ X} = ϕ. Assume that there is a point y in X such that ii-ker({y}) = X. Let x be any point of X. Then x $\in$ V for every ii-open set V containing y and hence y $\in$ ii-Cl({x}) for any x $\in$ X. This implies that y $\in$ $\cap${ii-Cl({x}) : x $\in$ X}. But this is a contradiction.

Sufficiency. Assume that ii-ker({x}) $\neq$ X for every x $\in$ X. If there exists a point y in X such that y $\in$ $\cap$ {ii-Cl({x}) : x $\in$ X}, then every ii-open set containing y must contain every point of X. This implies that the space X is the unique ii-open set containing y. Hence ii-ker({y}) = X which is a contradiction. Therefore, $\cap$ {ii-Cl({x}) : x $\in$ X} = ϕ.

Theorem 6.13. A topological space (X, $\Im$) is ii-R_0 if and only if for every x and y in X, ii-Cl({x}) $\neq$ ii-Cl({y}) implies ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ.

Proof. **Necessity**. Suppose that (X, $\Im$) is ii-R_0 and x, y $\in$ X such that ii-Cl({x}) $\neq$ ii-Cl({y}). Then, there exists z $\in$ ii-Cl({x}) such that z $\notin$ ii-Cl({y}) (or z $\in$ ii-Cl({y}) such that z $\notin$ ii-Cl({x})). There exists V $\in$ ii-O(X) such that y $\notin$ V and z $\in$ V, hence x $\in$ V. Therefore, we have x $\notin$ ii-Cl({y}). Thus x $\in$ [X – ii-Cl({y})] $\in$ ii-O(X), which implies ii-Cl({x}) $\subset$ [X – ii-Cl({y})] and ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ.

Sufficiency. Let V $\in$ ii-O(X) and let x $\in$ V. We still show that ii-Cl({x}) $\subset$ V. Let y $\notin$ V, that is y $\in$ X – V. Then x $\neq$ y and x $\notin$ ii-Cl({y}). This shows that ii-Cl({x}) $\neq$ ii-Cl({y}). By assumption, ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ. Hence y $\notin$ ii-Cl({x}) and therefore ii-Cl({x}) $\subset$ V.

Theorem 6.14. A topological space (X, $\Im$) is ii-R_0 if and only if for any points x and y in X, ii-ker({x}) $\neq$ ii-ker({y}) implies ii-ker({x}) $\cap$ ii-ker({y}) = ϕ.

Proof. Suppose that (X, $\Im$) is an ii-R_0 space. Thus by **Theorem 6.11**, for any points x and y in X if ii-ker({x}) $\neq$ ii-ker({y}) then ii-Cl({x}) $\neq$ ii-Cl({y}). Now we prove that ii-ker({x}) $\cap$ ii-ker({y}) = ϕ. Assume that z $\in$ ii-ker({x}) $\cap$ ii-ker({y}). By z $\in$ ii-ker({x}) and **Theorem 6.6**, it follows that x $\in$ ii-Cl({z}). Since x $\in$ ii-Cl({x}), by **Theorem 6.2**, ii-Cl({x}) = ii-Cl({z}). Similarly, we have ii-Cl({y}) = ii-Cl({z}) = ii-Cl({x}). This is a contradiction. Therefore, we have ii-ker({x}) $\cap$ ii-ker({y}) = ϕ.

Conversely, let (X, $\Im$) be a topological space such that for any points x and y in X, ii-ker({x}) $\neq$ ii-ker({y}) implies ii-ker({x}) $\cap$ ii-ker({y}) = ϕ. If ii-Cl({x}) $\neq$ ii-Cl({y}), then by **Theorem 6.11**, ii-ker({x}) $\neq$ ii-ker({y}). Hence, ii-ker({x}) $\cap$ ii-ker({y}) = ϕ which implies ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ. Because z $\in$ ii-Cl({x}) implies that x $\in$ ii-ker({z}) and therefore ii-ker({x}) $\cap$ ii-ker({z}) $\neq$ ϕ. By hypothesis, we have ii-ker({x}) = ii-ker({z}). Then z $\in$ ii-Cl({x}) $\cap$ ii-Cl({y}) implies that ii-ker({x}) = ii-ker({z}) = ii-ker({y}). This is a contradiction. Therefore, ii-Cl({x}) $\cap$ ii-Cl({y}) = ϕ and by **Theorem 6.2**, (X, $\Im$) is an ii-R_0 space.

Theorem 6.15. For a topological space (X, $\Im$) the following properties are equivalent:

(i) (X, $\Im$) is an ii-R_0 space.

(ii) For any non-empty set A and G $\in$ ii-O(X) such that A $\cap$ G $\neq$ ϕ, there exists F $\in$ ii-C(X) such that A $\cap$ F $\neq$ ϕ and F $\subset$ G.

(iii) For any G $\in$ ii-O(X), we have G = $\cup$ {F $\in$ ii-C(X) : F $\subset$ G}.

(iv) For any F $\in$ ii-C(X), we have F = $\cap$ {G $\in$ ii-O(X) : F $\subset$ G}.

(v) For every x $\in$ X, ii-Cl({x}) $\subset$ ii-ker({x}).

Proof. (i) $\Rightarrow$ (ii). Let A be a non-empty subset of X and G $\in$ ii-O(X) such that A $\cap$ G $\neq \phi$. There exists x $\in$ A $\cap$ G. Since x $\in$ G $\in$ ii-O(X), ii-Cl({x}) $\subset$ G. Set F = ii-Cl({x}), then F $\in$ ii-C(X), F $\subset$ G and A $\cap$ F $\neq \phi$.

(ii) $\Rightarrow$ (iii). Let G $\in$ ii-O(X), then G $\supset \cup${F $\in$ ii-C(X) : F $\subset$ G}. Let x be any point of G. There exists F $\in$ ii-C(X) such that x $\in$ F and F $\subset$ G. Therefore, we have x $\in$ F $\subset \cup$ {F $\in$ ii-C(X) : F $\subset$ G} and hence G = $\cup$ {F $\in$ ii-C(X) : F $\subset$ G}.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (v). Let x be any point of X and y $\notin$ ii-ker({x}). There exists V $\in$ ii-O(X) such that x $\in$ V and y $\notin$ V, hence ii-Cl({y}) $\cap$ V = ϕ. By (iv), ($\cap$ {G $\in$ ii-O(X) : ii-Cl({y}) $\subset$ G}) $\cap$ V = ϕ and there exists G $\in$ ii-O(X) such that x $\notin$ G and ii-Cl({y}) $\subset$ G. Therefore ii-Cl({x}) $\cap$ G = ϕ and y $\notin$ ii-Cl({x}). Consequently, we obtain ii-Cl({x}) $\subset$ ii-ker({x}).

(v) $\Rightarrow$ (i). Let G $\in$ ii-O(X) and x $\in$ G. Let y $\in$ ii-ker({x}), then x $\in$ ii-Cl({y}) and y $\in$ G. This implies that ii-ker({x}) $\subset$ G. Therefore, we obtain x $\in$ ii-Cl({x}) $\subset$ ii-ker({x}) $\subset$ G. This shows that (X, $\Im$) is an ii-R_0 space.

Corollary 6.16. For a topological space (X, $\Im$) the following properties are equivalent:

(i) (X, τ) is an ii-R_0 space.

(ii) ii-Cl({x}) = ii-ker({x}) for all x $\in$ X.

Proof. (i) $\Rightarrow$ (ii). Suppose that (X, $\Im$) is an ii-R_0 space. By **Theorem 6.15**, ii-Cl({x}) $\subset$ ii-ker({x}) for each x $\in$ X. Let y $\in$ ii-ker({x}), then x $\in$ ii-Cl({y}) and by **Theorem 6.2**, ii-Cl({x}) = ii-Cl({y}). Therefore, y $\in$ ii-Cl({x}) and hence ii-ker({x}) $\subset$ ii-Cl({x}). This shows that ii-Cl({x}) = ii-ker({x}).

(ii) $\Rightarrow$ (i). Follows from **Theorem 6.15**.

Theorem 6.17. For a topological space (X, $\Im$) the following properties are equivalent:

(i) (X, $\Im$) is an ii-R_0 space.

(ii) If F is ii-closed, then F = ii-ker(F).

(iii) If F is ii-closed and x $\in$ F, then ii-ker({x}) $\subset$ F.

(iv) If x $\in$ X, then ii-ker({x}) $\subset$ ii-Cl({x}).

Proof. (i) $\Rightarrow$ (ii). Let F be an ii-closed and x $\notin$ F. Thus (X – F) is an ii-open set containing x. Since (X, $\Im$) is ii-R_0, ii-Cl({x}) $\subset$ (X – F). Thus ii-Cl({x}) $\cap$ F = ϕ and by **Theorem 6.7**, x $\notin$ ii-ker(F). Therefore ii-ker(F) = F.

(ii) $\Rightarrow$ (iii). In general, A $\subset$ B implies ii-ker(A) $\subset$ ii-ker(B). Therefore, it follows from (ii), that ii-ker({x}) $\subset$ ii-ker(F) = F.

(iii) $\Rightarrow$ (iv). Since x $\in$ ii-Cl({x}) and ii-Cl({x}) is ii-closed, by (iii), ii-ker({x}) $\subset$ ii-Cl({x}).

(iv) $\Rightarrow$ (i). We show the implication by using **Theorem 6.4**. Let x $\in$ ii-Cl({y}). Then by **Theorem 6.6**, y $\in$ ii-ker({x}). Since x $\in$ ii-Cl({x}) and ii-Cl({x}) is ii-closed, by (iv), we obtain y $\in$ ii-ker({x}) $\subset$ ii-Cl({x}). Therefore x $\in$ ii-Cl({y}) implies y $\in$ ii-Cl({x}). The converse is obvious and (X, $\Im$) is ii-R_0.

Definition 6.18. A topological space (X, $\Im$) is said to be **ii-R_1** if for x, y in X with ii-Cl({x}) $\neq$ ii-Cl({y}), there exist disjoint ii-open sets U and V such that ii-Cl({x}) $\subset$ U and ii-Cl({y}) $\subset$ V.

Theorem 6.19. A topological space (X, $\Im$) is ii-R_1 if it is ii-T_2.

Proof. Let x and y be any points of X such that ii-Cl({x}) $\neq$ ii-Cl({y}). By **Theorem 5.10**, every ii-T_2 space is ii-T_1. Therefore, by **Theorem 4.4**, ii-Cl({x}) = {x}, ii-Cl({y}) = {y} and hence {x} $\neq$ {y}. Since (X, $\Im$) is ii-T_2, there exist disjoint ii-open sets U and V such that ii-Cl({x}) = {x} $\subset$ U and ii-Cl({y}) = {y} $\subset$ V. This shows that (X, $\Im$) is ii-R_1.

Theorem 6.20. If a topological space $(X, \Im)$ is ii-symmetric, then the following are equivalent:

(i) $(X, \Im)$ is ii-T_2.

(ii) $(X, \Im)$ is ii-R_1 and ii-T_1.

(iii) $(X, \Im)$ is ii-R_1 and ii-T_0.

Proof. Straightforward.

Theorem 6.21. For a topological space $(X, \Im)$ the following statements are equivalent:

(i) $(X, \Im)$ is ii-R_1.

(ii) If $x, y \in X$ such that ii-Cl($\{x\}$) $\neq$ ii-Cl($\{y\}$), then there exist ii-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. Obvious.

Theorem 6.22. If $(X, \Im)$ is ii-R_1, then $(X, \Im)$ is ii-R_0.

Proof. Let U be an ii-open set such that $x \in U$. If $y \notin U$, since $x \notin$ ii-Cl($\{y\}$), we have ii-Cl($\{x\}$) $\neq$ ii-Cl($\{y\}$). So, there exists an ii-open set V such that ii-Cl($\{y\}$) $\subset$ V and $x \notin V$, which implies $y \notin$ ii-Cl($\{x\}$). Hence ii-Cl($\{x\}$) $\subset$ U. Therefore, $(X, \Im)$ is ii-R_0.

Corollary 6.23. A topological space $(X, \Im)$ is ii-R_1 if and only if for $x, y \in X$, ii-ker($\{x\}$) $\neq$ ii-ker($\{y\}$), there exist disjoint ii-open sets U and V such that ii-Cl($\{x\}$) $\subset$ U and ii-Cl($\{y\}$) $\subset$ V.

Proof. Follows from **Theorem 6.11**.

Theorem 6.24. A topological space $(X, \Im)$ is ii-R_1 if and only if $x \in X -$ ii-Cl($\{y\}$) implies that x and y have disjoint ii-open neighbourhoods.

Proof. **Necessity**. Let $x \in X -$ ii-Cl($\{y\}$). Then ii-Cl($\{x\}$) $\neq$ ii-Cl($\{y\}$), so, x and y have disjoint ii-open neighbourhoods.

Sufficiency. First, we show that $(X, \Im)$ is ii-R_0. Let U be an ii-open set and $x \in U$. Suppose that $y \notin U$. Then, ii-Cl($\{y\}$) $\cap U = \phi$ and $x \notin$ ii-Cl($\{y\}$). There exist ii-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, ii-Cl($\{x\}$) $\subset$ ii-Cl(U_x) and ii-Cl($\{x\}$) $\cap U_y \subset$ ii-Cl(U_x) $\cap U_y = \phi$. Therefore, $y \notin$ ii-Cl($\{x\}$). Consequently, ii-Cl($\{x\}$) $\subset$ U and $(X, \Im)$ is ii-R_0. Next, we show that $(X, \Im)$ is ii-R_1. Suppose that ii-Cl($\{x\}$) $\neq$ ii-Cl($\{y\}$). Then, we can assume that there exists $z \in$ ii-Cl($\{x\}$) such that $z \notin$ ii-Cl($\{y\}$). There exist ii-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \phi$. Since $z \in$ ii-Cl($\{x\}$), $x \in V_z$. Since $(X, \Im)$ is ii-R_0, we obtain ii-Cl($\{x\}$) $\subset V_z$, ii-Cl($\{y\}$) $\subset V_y$ and $V_z \cap V_y = \phi$. This shows that $(X, \Im)$ is ii-R_1.

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