Some Results on Graceful Labeling for Step Grid Related Graphs

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Abstract

In this paper we investigated some Step Grid St_n related families with their graceful labeling. We obtained graceful labeling for $t-$super subdivision of the step grid graph, open star of step grid graph $S(t \cdot St_n)$ and one point union for path of step grid graph ${P_n}^t(tn \cdot St_m)$. We also obtained graceful labeling for path union of cycle C_m $(m \equiv 0 \pmod 4)$ with step grid graph.

Key words : Graceful labeling, $t-$super subdivision of graph, open star of graph, one point union of path graphs, path union, step grid graph.

AMS **subject classification number :** 05C78.

1 Introduction :

The graceful labeling was introduced by A. Rosa [1] during 1967. Golomb [2] named such labeling as graceful labeling, which was called earlier as $\beta-$valuation. Kaneria and makadia [3] introduced a Step Grid Graph St_n, $n \geq 3$ and proved that it is graceful. Kaneria et al. [4] introduced open star of graph, one point union of path graphs. They discussed gracefulness of open star of t copies of $K_{m,n}$, $P_n \times P_m$ and also discussed gracefulness of one point union for path of $K_{m,n}$. Kaneria and makadia [5] discussed gracefulness of $t-$super subdivision of $P_n \times P_m$ and path union of cycle $C_p(p \equiv 0 \pmod 4)$ with grid graph $P_n \times P_m$.

We begin with a simple, undirected finite graph $G = (V, E)$ with $|V| = p$ vertices and $|E| = q$ edges. For all terminology and notations we follows Harary [6]. Here are some of the definitions which are used in this paper.

Definition-1.1 **:** A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^{\star} : E \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^{\star}(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition-1.2 **:** A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the graphs G_1, G_2, $\ldots$,G_n is known as *open star of graphs*. We shall denoted such graph by $S(G_1, G_2, \ldots, G_n)$.

If we replace each vertices of $K_{1,n}$ except the apex vertex be a graph G.i.e. $G_1 = G$, $G_2 = G$, $\ldots$,$G_n = G$, such cycle of a graph G is denoted by $S(n \cdot G)$.

Definition-1.3 **:** A graph G is obtained be replacing each edge of $K_{1,t}$ by a path P_n of length n on $n+1$ vertices is called one point union for t copies of path P_n. We shall denoted such graph G by ${P_n}^t$.

Definition-1.4 **:** A graph G is obtained by replacing each vertices of ${P_n}^t$ except the central vertex by the graphs $G_1, G_2, \ldots, G_{tn}$ is known as *one point union for path of graphs*. We shall denoted such graph G by ${P_n}^t$ $(G_1, G_2, \ldots, G_{tn})$, where ${P_n}^t$ is the one point union of t copies of path P_n.

If we replace each vertices of ${P_n}^t$ except the central vertex by a graph H. i.e. $G_1 = G_2 = \ldots = G_{tn} = H$, such one point union of path graph, we shall denoted it by ${P_n}^t(tn \cdot H)$.

Above three definitions 1.2, 1.3 and 1.4 are introduced by Kaneria et al. [4].

Definition−1.5 **:** Let $G = (V, E)$ be a graph with p vertices and q edges. A graph H is said to be a $t-$ super subdivision of G if H is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,t}$ for some $t \in N$.

Definition−1.6 **:** Let G be a graph and $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} $(1 \leq i \leq n-1)$ is called *path union* of G.

Definition−1.7 **:** Take $P_n, P_n, P_{n-1}, \ldots, P_2$ paths on $n, n, n-1, n-2, \ldots, 3, 2$ vertices and arrange them vertically.

A graph obtained by joining horizontal vertices of given successive paths is known as a step grid graph of size n, where $n \geq 3$. We shall denote it by St_n.

Obviously $|V(St_n)| = \frac{1}{2}(n^2 + 3n - 2)$ and $|E(St_n)| = n^2 + n - 2$. Above definition 1.7 was introduced by Kaneria and Makadia [3].

In this paper we have discussed graceful labeling for $t-$super subdivision, open star and one point union for path of step grid graph St_n. By joining cycle C_m ($m \equiv 0$ (mod 4)) and a step grid graph St_n, with a path of arbitrary length t is graceful labeling graph as well. An extensive survey on graph labeling and bibliographic references are given in Gallian [7].

2 Main Results :

Theorem−2.1 **:** $t-$super subdivision of St_n, where $n \geq 3$ is graceful.

Proof : Let G be a step grid St_n with $t-$super subdivision. Let $V(St_n)$ be vertex set of step grid graph St_n, where $V(St_n) = \{u_{i,j}/i = n, n-1, \ldots, 1, j = 1, 2, \ldots, l$, where $l = \min(n, n+2-i)\}$.

We know that $|V(St_n)| = p = \frac{1}{2}(n^2 + 3n - 2)$, $|E(St_n)| = q = n^2 + n - 2$ and $|V(G)| = P = p + tq$, $|E(G)| = Q = 2tq$.

Let $v_{i,j,k}$ $(i = n, n-1, \ldots, n, j = 1, 2, \ldots, i, k = 1, 2, \ldots, t)$ be vertices for the vertical edges in G and $w_{i,j,k}$ $(i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, i+1, k = 1, 2, \ldots, t)$ be vertices for the horizontal edges in G.

Thus $V(G) = V(St_n) \bigcup \{v_{i,j,k} \;/\; i = n, n-1, \ldots, n, j = 1, 2, \ldots, i, k = 1, 2, \ldots, t\} \bigcup \{w_{i,j,k} \;/\; i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, i+1, k = 1, 2, \ldots, t\}$.

We define the labeling function $f : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows.

$f(u_{i,j}) = Q - \frac{t}{2}(n-i)(n+3-i) - t(n+2-(i+j))$,

$\forall\; i = n, n-1, \ldots, 2, \forall\; j = 1, 2, \ldots, l$,where $l = \min(n, n+2-i)$;

$f(u_{1,j}) = Q - \frac{t}{2}(n-1)(n+2) - t(n-j), \quad \forall\; j = 1, 2 \ldots, n$;

$f(v_{i,j,k}) = [\frac{(i-1)}{2}(3i+2) + (i-j)]t + (k-1)$,

$\forall\; i = 1, 2, \ldots, n-1, \forall\; j = 1, 2, \ldots, i, \forall\; k = 1, 2, \ldots, t$;

$f(v_{n,j,k}) = [\frac{(n-1)}{2}(3n+2) + (n-j-1)]t + (k-1), \quad \forall\; j = 1, 2 \ldots, n-1, \forall\; k = 1, 2, \ldots, t$;

$f(w_{i,j,k}) = [\frac{i}{2}(3i+1) + 2j - 3]t + (k-1)$,

$\forall\; i = 1, 2, \ldots, n-1, \forall\; j = 1, 2, \ldots, i+1, \forall\; k = 1, 2, \ldots, t$.

Above labeling patten give rise a graceful labeling to the graph G. So G is a graceful graph.

Illustration-2.2 **:** St_4 with $3-$super subdivision and its graceful labeling shown in *figure*-1.

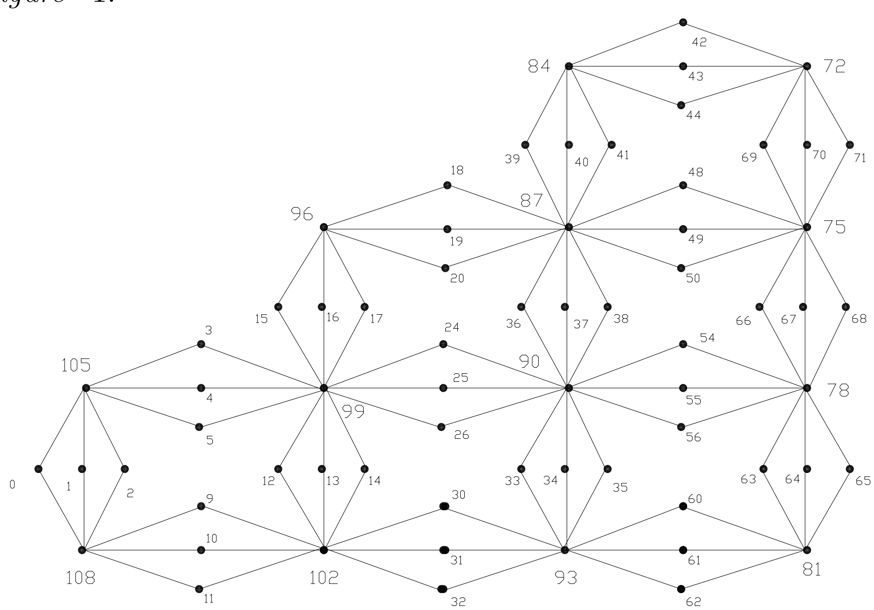


Figure-1 St_4 with $3-$super subdivision and its graceful labeling.

Theorem-2.3 **:** Open star of step grid graph $S(t \cdot St_n)$ is graceful, Where $t \in N-1$.

Proof : Let $G = S(t \cdot St_n)$ be a graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex of $K_{1,t}$ by the graph St_n. Let u_0 is the apex vertex of $K_{1,t}$. i.e. It is central vertex of the graph G.

Let $u_{k,i,j}$ $(i = n, n-1, \ldots, 1, j = 1, 2, \ldots, m,$ where $m = \min(n, n+2-i))$ be the vertices of k^{th} copy of St_n in G, $\forall\ k = 1, 2, \ldots, t$. Where the vertices of k^{th} copy of St_n is $p = \frac{1}{2}(n^2 + 3n - 2)$ and edges of k^{th} copy of St_n is $q = n^2 + n - 2$.

We know that the step grid graph St_n is graceful graph, where f be the function for graceful labeling of St_n as we proved in [3]. In graph G, we see that the vertices $|V(G)| = P = tp + 1$ and edges $|E(G)| = Q = t(q + 1)$. Join the vertex $u_{k,n,2}$ with the vertex u_0 by and edge to from the open star of graph G, $\forall\ k = 1, 2, \ldots, t$.

We define labeling function $g : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows.

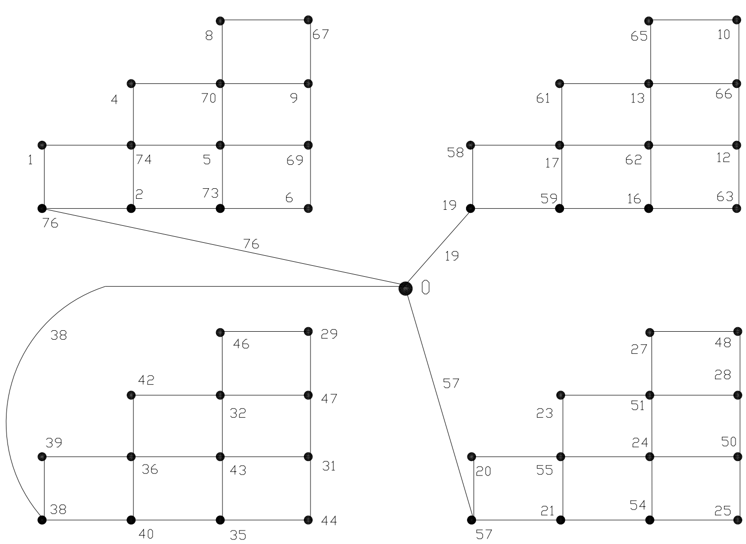
$g(u_0) = 0;$

$$
\begin{aligned}
g(u_{1,i,j}) &= f(u_{i,j}) + 1, && \text{when } f(u_{i,j}) < \tfrac{q}{2}, \\
&= f(u_{i,j}) + (Q - q) && \text{when } f(u_{i,j}) > \tfrac{q}{2}; \\
g(u_{2,i,j}) &= g(u_{1,i,j}) + (Q - q - 1), && \text{when } g(u_{1,i,j}) < \tfrac{Q}{2}, \\
&= g(u_{1,i,j}) - (Q - q - 1) && \text{when } g(u_{1,i,j}) > \tfrac{Q}{2}; \\
g(u_{k,i,j}) &= g(u_{k-2,i,j}) + (q + 1), && \text{when } g(u_{k-2,i,j}) < \tfrac{Q}{2}, \\
&= g(u_{k-2,i,j}) - (q + 1) && \text{when } g(u_{k-2,i,j}) > \tfrac{Q}{2}; \quad \forall\ k = 3, 4, \ldots, t;
\end{aligned}
$$

$\forall\ i = n, n-1, \ldots, 1,\ \forall\ j = 1, 2, \ldots, m,$ where $m = \min(n, n+2-i)$.

Above labeling patten give rise a graceful labeling to given graph G.

Illustration−2.4 : Open star of 4 copies of St_4 and its graceful labeling shown in *figure*−2.



Figure−2 An open star of St_4 and its graceful labeling.

Theorem-2.5 **:** one point union for path of step grid graph ${P_n}^t(tn \cdot St_m)$.

Proof : Let $G = {P_n}^t(tn \cdot St_m)$ be a graph obtained by replacing each vertices of ${P_n}^t$ except the central vertex by the graph St_m. i.e. G is the graph obtained by replacing each vertices of $K_{1,t}$ except the apex vertex by the union of n copies of the graph St_m. Let u_0 be the central vertex for the graph G with t branches.

We know that the step grid graph St_m is graceful graph, where f be the function for graceful labeling of St_m as we proved in [3]. Where the vertices of St_m is $p_1 = \frac{1}{2}(m^2 + 3m - 2)$ and edges of St_m is $q_1 = m^2 + m - 2$.

Now first we shall defined the path union of n copies of St_m is called graph H. Let $u_{k,i,j}$ ($i = m, m-1, \ldots, 1, j = 1, 2, \ldots, r$, where $r = \min(m, m+2-i)$ and $k = 1, 2, \ldots, n$) be the vertices of k^{th} copy of St_m. Join the vertices $u_{k,1,1}$ to $u_{k+1,m,1}$ for $k = 1, 2, \ldots, n-1$ be an edge to from the path union of n copies of step grid graph. In graph H, the vertices $|V(H)| = \frac{n}{2}(m^2 + m - 1) = p_2$ and the edges $|E(H)| = n(m^2 + m - 1) - 1 = q_2$.

We define labeling function $g : V(H) \longrightarrow \{0, 1, \ldots, q_2\}$ as follows

$$\begin{aligned}
g(u_{1,i,j}) &= f(u_{i,j}) && \text{if } f(u_{i,j}) < \tfrac{q_1}{2}, \\
&= f(u_{i,j}) + (q_2 - q_1) && \text{if } f(u_{i,j}) > \tfrac{q_1}{2}; \\
g(u_{k,i,j}) &= g(u_{k-1,i,j}) + \tfrac{q_1}{2} && \text{if } g(u_{k-1,i,j}) < \tfrac{q_2}{2}, \\
&= g(u_{k-1,i,j}) - (\tfrac{q_1}{2} + 1) && \text{if } g(u_{k-1,i,j}) > \tfrac{q_2}{2}; \\
& \forall\ k = 2, 3, \ldots, n; \\
& \forall\ i = m, m-1, \ldots, 1,\ \forall\ j = 1, 2, \ldots, r, \text{ where } r = \min(m, m+2-i).
\end{aligned}$$

Above labeling pattern give rise graceful labeling to the path union of n copies of St_m. Which lies in s^{th} branch of the graph G, where $s = 1, 2, \ldots, t$.

Now we shall define the graph G. Let $u_{s,k,i,j}$ ($i = m, m-1, \ldots, 1, j = 1, 2, \ldots, r$, where $r = \min(m, m+2-i)$ and $k = 1, 2, \ldots, n$.) be the vertices of k^{th} copy of path union of n copies of St_m lies in s^{th} branch of the graph G, $\forall\ s = 1, 2, \ldots, t$. Join the vertices of $u_{s,1,m,2}$ with u_0 by an edge to from the one point union for path of step grid graph G. The vertices of graph G is $|V(G)| = tn(p_1 + 1) = P$ and the edges of graph G is $|E(G)| = tn(q_1 + 1) = t(q_2 + 1) = Q$.

We define labeling function $h : V(G) \longrightarrow \{0, 1, \ldots, Q\}$ as follows.

$$
\begin{aligned}
h(u_0) &= 0; \\
h(u_{1,k,i,j}) &= g(u_{k,i,j}) + 1 && \text{when } g(u_{k,i,j}) < \tfrac{q_2}{2}, \\
&= g(u_{k,i,j}) + (Q - q_2) && \text{when } g(u_{k,i,j}) > \tfrac{q_2}{2}; \\
h(u_{2,k,i,j}) &= h(u_{1,k,i,j}) + [Q - (q_2 + 1)] && \text{when } h(u_{1,k,i,j}) < \tfrac{Q}{2}, \\
&= h(u_{1,k,i,j}) - [Q - (q_2 + 1)] && \text{when } h(u_{1,k,i,j}) > \tfrac{Q}{2}; \\
h(u_{s,k,i,j}) &= h(u_{s-2,k,i,j}) + (q_2 + 1) && \text{when } h(u_{s-2,k,i,j}) < \tfrac{Q}{2}, \\
&= h(u_{s-2,k,i,j}) - (q_2 + 1) && \text{when } h(u_{s-2,k,i,j}) > \tfrac{Q}{2}; \\
&\qquad \forall\ s = 3, 4, \ldots, t;
\end{aligned}
$$

$\forall\ i = m, m-1, \ldots, 1,\ \forall\ j = 1, 2, \ldots, r$, where $r = \min(m, m+2-i)$ and $\forall\ k = 1, 2, \ldots, n$.

Above labeling patten give rise a graceful labeling to the graph G and so it is a graceful graph.

Illustration-2.6 **:** One point union for path of St_3 is ${P_4}^3(3 \cdot 4 \cdot St_3)$ and its graceful labeling shown in *figure*-3.

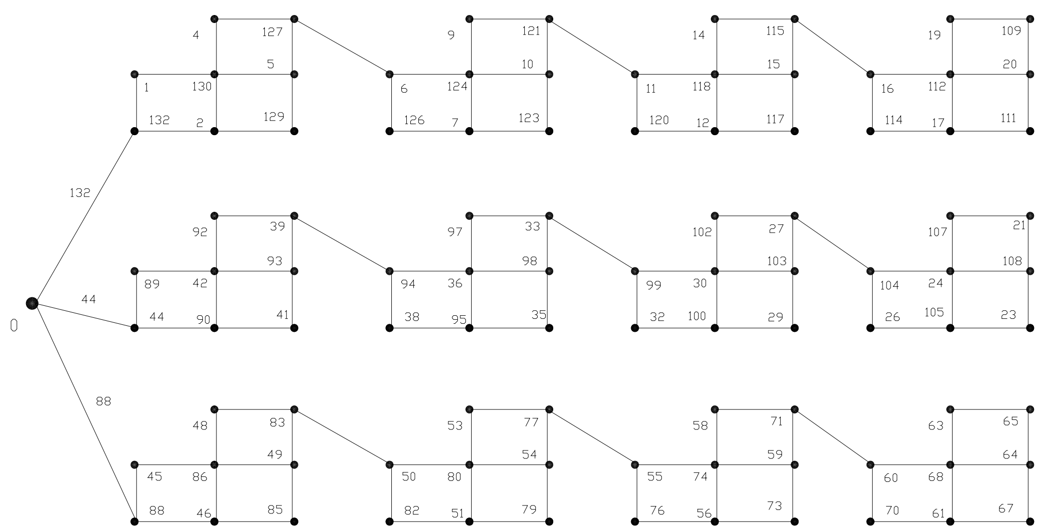


Figure-3 One point union for path of St_3 with 3 copies and its graceful labeling.

Theorem-2.7 **:** A graph obtain by joining $C_m (m \equiv 0 \bmod 4)$ and a step grid graph St_n with a path of arbitrary length t is graceful.

Proof : Let G be the graph obtained by joining a cycle $C_m (m \equiv 0 \bmod 4)$ and a step grid graph St_n with P_t, a path of length t on $t+1$ vertices.

Let $u_1, u_2, \dots, u_m$ be the vertices of cycle C_m and $v_1 = u_m, v_2, \dots, v_{t+1}$ be the vertices of path P_t of t length and $w_{i,j}$ $(i = n, n-1, \dots, 1, j = 1, 2, \dots, r$, where $r = \min(n, n+2-i))$ be the vertices of the step grid graph St_n, where $v_{t+1} = w_{n,2}$ if t is odd and $v_{t+1} = w_{n,1}$ is t is even.

We know that the step grid graph St_n is graceful graph, where f be the function for graceful labeling of St_n as we proved in [3]. Where the vertices of St_n is $p = \frac{1}{2}(n^2+3n-2)$ and edges of St_n is $q = n^2+n-1$. In the graph G, the vertices $|V(G)| = P = m+(t+1)+p$ and edges $|E(G)| = Q = m + t + q$.

We define labeling function $g : V(G) \longrightarrow \{0, 1, \dots, Q\}$, we shall take following two cases.

Case$-$**I :** t is odd

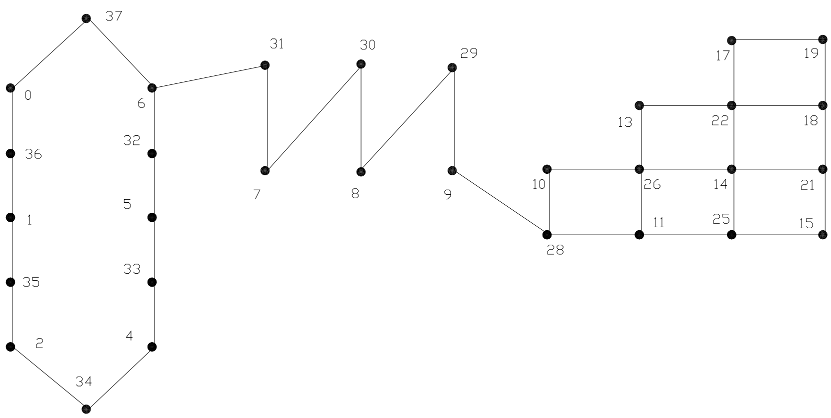
$$\begin{aligned}
g(u_i) &= Q - (\tfrac{i-1}{2}), && \forall\ i = 1, 3, \dots, m-1, \\
&= \tfrac{i-2}{2}, && \forall\ i = 2, 4 \dots, \tfrac{m}{2}, \\
&= \tfrac{i}{2}, && \forall\ i = \tfrac{m}{2}+2, \tfrac{m}{2}+4, \dots, m; \\
g(v_i) &= Q + 1 - (\tfrac{m+i}{2}), && \forall\ i = 2, 4, \dots, t+1, \\
&= \tfrac{m+i-1}{2}, && \forall\ i = 1, 3 \dots, t; \\
g(w_{i,j}) &= f(w_{i,j}) + (\tfrac{m+t+1}{2}), && \forall\ w_{i,j} \in V(St_n).
\end{aligned}$$

Case$-$**II :** t is even

$$\begin{aligned}
g(u_i) &= Q - (\tfrac{i-1}{2}), && \forall\ i = 1, 3, \dots, m-1, \\
&= \tfrac{i-2}{2}, && \forall\ i = 2, 4 \dots, \tfrac{m}{2}, \\
&= \tfrac{i}{2}, && \forall\ i = \tfrac{m}{2}+2, \tfrac{m}{2}+4, \dots, m; \\
g(v_i) &= Q + 1 - (\tfrac{m+i}{2}), && \forall\ i = 2, 4, \dots, t, \\
&= \tfrac{m+i-1}{2}, && \forall\ i = 1, 3 \dots, t+1; \\
g(w_{i,j}) &= f(w_{i,j}) + (\tfrac{m+t}{2}), && \forall\ w_{i,j} \in V(St_n).
\end{aligned}$$

Above labeling patten give rise graceful labeling to the given graph G.

Illustration−2.8 **:** A cycle C_{12}, a step grid graph St_4 joining by a path P_7 and its graceful labeling shown in *figure*−4.



Figure−4 A cycle C_{12}, a step grid graph St_4 joining by a path P_7 with its graceful labeling.

3 Concluding Remarks :

Here we have discussed the gracefulness of $t-$super subdivision of step grid graph, open star of step grid graph, one point union for path of step grid graph. We also discussed gracefulness of a cycle $C_m(m \equiv 0 \bmod 4)$ joining with a step grid graph by a path P_t of arbitrary length t. This work contributes some new results to the families of graceful labeling. The labeling pattern is demonstrated by means of illustrations.

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