Fixed Point Theorems For Pairs of Occasionally Weakly Semi Compatible Hybrid Mappings

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Abstract: *In this paper we obtain some common fixed point theorems for hybrid pairs of single and multivalued occasionally weakly semi-compatible mappings.*

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I. Introduction

In 1965, Zadeh[10] introduced the concept of fuzzy sets. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [6] and George and Veeramani [5] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. The concepts of weak commutativity, compatibility, and weak compatibility were frequently used to prove existence theorems in fixed and common fixed points for single and multivalued maps satisfying certain conditions in different spaces. Al-Thagafi and Shahzad [9] weakened the concept of weakly compatible maps by giving the concept of occasionally weakly compatible maps. recently, Abbas and Rhoades [4] extended the definition of o.w.c. maps to the setting of set-valued maps and they proved some common fixed point theorems satisfying generalized contractive condition of integral type. More recently, Aamri et.al. [3,2 ,1], Moutawaki [7], and Abbas and Rhoades [4], obtained fixed and common fixed point theorems in symmetric spaces for single and multi valued mappings. In this paper we obtain some common fixed point theorems for hybrid pairs of single and Multivalued occasionally weakly semi-compatible mappings defind on a fuzzy symmetric spaces satisfying a contractive condition under implicit function.

We start with some preliminaries:

II. Preliminaries

Definition 2.1 : A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norms if $*$ satisfying conditions:

i. $*$ is commutative and associative;
ii. $*$ is continuous;
iii. $a*1=a$ for all $a\in[0,1]$;
iv. $a*b\le c*d$ whenever $a\le c$ and $b\le d$, and $a,b,c,d\in[0,1]$.

Definition 2.3 : A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions, $\forall\ x,y,z\in X, s,t>0$,

$(f1)$ $M(x,y,t)>0$;

$(f2)$ $M(x,y,t)=1$ if and only if $x=y$.

(f3) $M(x,y,t) = M(y,x,t)$;

(f4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$

(f5) $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t. If only $f1, f2, f3$ hold, the 3-tuple $(X,M,*)$ is said to be symmetric fuzzy metric space.

Example 2.1 (Induced fuzzy metric) **:** Let (X,d) be a metric space. Denote a $*$ b =ab for all $a,b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric *d* as the standard fuzzy metric.

We extend the concept of owc and owsc in symmetric fuzzy metric space for a pair of hybrid maps.

Definition 2.2 : Let $(X,M,*)$ be a *symmetric fuzzy metric space* and $f : X \to X, F : X \to B(X)$. The hybrid pairs (f,F) is said to be occasionally weakly

compatible(o.w.c.) iff there exists some point $x \in X$ such that $fx \in Fx$ and $fFx \subseteq Ffx$.

Definition 2.3 : Let $(X,M,*)$ be a symmetric fuzzy metric space and $f : X \to X, F : X \to B(X)$. The hybrid pairs (f,F) is said to be *occasionally weakly semi- compatible*(o.w.s.c.) iff there exists some point $x \in X$ such that $fx \in Fx$ and $f^2 x \in Ffx$.

It is clear that owc hybrid pair is owsc pair, but not the converse in view of the following example.

Example: 2.2: Let $X = [0,1]$ and Define $fx = 1-x$ and $Fx = [0,\frac{1}{2}]$. Then $f(\frac{1}{2}) = \frac{1}{2} \in F(\frac{1}{2})$, $fF(\frac{1}{2}) = [\frac{1}{2},1] \not\subset Ff(\frac{1}{2}) = [0,\frac{1}{2}]$ and $f^2(\frac{1}{2}) = \frac{1}{2} \in Ff(\frac{1}{2})$.

Thus the hybrid pair (f,F) is owsc, but not owc.

Definition: 2.4 : Let $CB(X)$ be the set of all nonempty bounded and closed subsets of X, we define the functions;

$$M^{\nabla}(a,B,t) = \max\{M(a,b,t) : b \in B\}$$

If $a \in B$ then from above definition $M^{\nabla}(a,B,t)=1$

Definition 2.5 : Let $CB(X)$ be the set of all nonempty bounded and closed subsets of X ,we define the functions;

$$\delta(A,B,t)=\inf\left\{M(a,b,t): a\in A, b\in B\right\}$$

III. Main Results

Theorem 3. 1: Let $(M,X,*)$ be a symmetric fuzzy metric space and $f,g: X\to X, F,G: X\to CB(X)$ be mappings satisfying;

(i) the pairs (f,F) and (g,G) are occasionally weakly semi-compatible, (ii)
$\delta(Fx,Gy,qt)\ge \min\left(M(fx,gy,t),M^{\nabla}(fx,Fx,t),M^{\nabla}(gy,Gy,t),M^{\nabla}(fx,Gy,t),M^{\nabla}(gy,Fx,t)\right)$ for all $x,y\in X$, Where $q\in(0,1)$. Then f,g,F and G have a unique common fixed point in X .

Proof: Since the pairs (f,F) and (g,G) are occasionally weakly semi-compatible, there exist $u,v\in X$ such that $fu\in Fu, f^2u\in Ffu, gv\in Gv$ and $g^2v\in Ggv$.

Suppose $fu\ne gv$ .From (ii), we have

$$M(fu,gv,qt)\ge\delta(Fu,Gv,qt)\ge\min\left(M(fu,gv,t),M^{\nabla}(fu,Fu,t),M^{\nabla}(gv,Gv,t),M^{\nabla}(fu,Gv,t),M^{\nabla}(gv,Fu,t)\right)$$
$$M(fu,gv,qt)\ge\min\left(M(fu,gv,t),1,1,M(fu,gv,t),M(fu,gv,t)\right)$$
$$M(fu,gv,qt)\ge M(fu,gv,t)$$

Which is a contradiction. Hence $fu=gv$.

Suppose $f^2u\ne fu$ .i.e. $f^2u\ne gv$ .Since $fu\in Fu$ and $f^2u\in Ffu$,from (ii),we have

$$M(f^2u,gv,qt)\ge\delta(Ffu,Gv,qt)\ge\min\begin{pmatrix}M(f^2u,gv,t),M^{\nabla}(f^2u,Ffu,t),M^{\nabla}(gv,Gv,t),\\ M^{\nabla}(f^2u,Gv,t),M^{\nabla}(gv,Ffu,t)\end{pmatrix}$$

$$M(f^2u,gv,qt)\ge\delta(Ffu,Gv,qt)\ge\min\left(M(f^2u,gv,t),1,1,M^{\nabla}(f^2u,Gv,t),M^{\nabla}(gv,Ffu,t)\right)$$

$$M(f^2u,fu,qt)\ge\min\left(M(f^2u,fu,t),1,1,M(f^2u,fu,t),M(fu,f^2u,t)\right)$$

$$M(f^2u,fu,qt)\ge M(f^2u,fu,t).$$

Which is a contradiction. Hence $f^2u=fu$ .Thus $fu=f^2u\in Ffu$ .Hence fu is a common fixed point of f and F .Similarly we can show that gv is common fixed point of g and G since $fu=gv$,it follows that fu is common fixed point of f,g,F and G .Uniqueness of common fixed point follows easily from (ii).

We give an example to furnish our theorem.

Ex. Let $X=[0,1]$ endowed with the fuzzy metric $M(x,y,t)=\frac{t}{t+|x-y|}$. Define $f,g:X\to X$, $F,G:X\to CB(X)$ as follows:

$$Fx=\left\{\frac{1}{2}\right\} \qquad Gx=\begin{cases}\left\{\frac{1}{2}\right\}, x\in[0,\frac{1}{2}]\\ [\frac{3}{8},\frac{1}{2}], x\in(\frac{1}{2},1]\end{cases}$$

$$fx=\begin{cases}\frac{1}{2}, x\in[0,\frac{1}{2}]\\ \frac{x+1}{4}, x\in(\frac{1}{2},1]\end{cases} \qquad gx=\begin{cases}1-x, x\in[0,\frac{1}{2}]\\ 0, x\in(\frac{1}{2},1]\end{cases}$$

It is clear that pairs (f,F) and (g,G) are occasionally weakly semi-compatible and satisfy condition (ii) of theorem 1 and so have a common fixed point $x=\frac{1}{2}$.

Theorem 3.2: Let $(M,X,*)$ be a symmetric fuzzy metric space and $f,g:X\to X$, $F,G:X\to CB(X)$ be mappings satisfying:

(i)the pairs (f,F) and (g,G) are occasionally weakly semi-compatible,

(ii) $M^{\nabla}(fx,Gy,qt)\le\phi\left(M(fx,gy,t),M^{\nabla}(fx,Fx,t),M^{\nabla}(gy,Gy,t)\right)$

for all $x,y\in X$, Where $q>1$ and $\phi:[0,1]^3\to[0,1]$,is a continuous function and increasing function such that $\phi(t,1,1)<t$.Then f,g,F and G have a unique common fixed point in X.

Proof: Since the pairs (f,F) and (g,G) are occasionally weakly semi-compatible, there exist $u,v\in X$ such that $fu\in Fu, f^2u\in Ffu, gv\in Gv$ and $g^2v\in Ggv$.

Suppose $fu\ne gv$.From (ii),we have

$$M(fu,gv,qt)\le M^{\nabla}(fu,Gv,qt)\le\phi\left(M(fu,gv,t),M^{\nabla}(fu,Fu,t),M^{\nabla}(gv,Gv,t)\right)$$

$$=\phi\left(M(fu,gv,t),1,1\right)$$

$$<M(fu,gv,t)$$

Which is a contradiction.

Hence $fu=gv$.

Suppose $f^2u \neq fu$ .i.e. $f^2u \neq gv$ .Since $fu \in Fu$ and $f^2u \in Ffu$,from (ii),we have

$$M(f^2u, gv, qt) \leq M^{\nabla}(f^2u, Gv, qt) \leq \phi\left(M(f^2u, gv, t), M^{\nabla}(f^2u, Ffu, t), M^{\nabla}(gv, Gv, t)\right)$$

$$M(f^2u, fu, qt) \leq \phi\left(M(f^2u, fu, t), M^{\nabla}(f^2u, Ffu, t), M^{\nabla}(gv, Gv, t)\right)$$

$$= \phi\left(M(f^2u, fu, t), 1, 1\right)$$

$$< M(f^2u, fu, t)$$

Which is a contradiction. Hence $f^2u = fu$ .Thus $fu = f^2u \in Ffu$ .Hence fu is a common fixed point of f and F .Similarly we can show that gv is common fixed point of g and G since $fu = gv$,it follows that fu is common fixed point of f, g, F and G .Uniqueness of common fixed point follows easily from (ii).as follows: put $fu = w$ and suppose that w_1 be another fixed point s.t. $w \neq w_1$, then we have.

$$M(fw, gw_1, qt) \leq M^{\nabla}(fw, Gw_1, qt) \leq \phi\left(M(fw, gw_1, t), M^{\nabla}(fw, Fw, t), M^{\nabla}(gw_1, Gw_1, t)\right)$$

$$M(w, w_1, qt) \leq \phi\left(M(w, w_1, t), M^{\nabla}(w, Fw, t), M^{\nabla}(w_1, Gw_1, t)\right)$$

$$M(w, w_1, qt) \leq \phi\left(M(w, w_1, t), 1, 1\right)$$

$$M(w, w_1, qt) \leq M(w, w_1, t)$$

Which is a contradiction. Hence $w = w_1$.

References

[1] Aamri M., Bassou A.H., Moutawkil D.El. "Common fixed points for weakly compatible maps in symmetric spaces with application to probabilistic spaces," Applied Mathematics E- Notes, vol.5, pp. 171-175, 2005.

[2] Aamri M., Moutawakil D.El. "Common fixed points under contractive conditions in symmetric spaces," Applied Mathematics E-Notes, vol. 3, pp.156-162, 2003.

[3] Aamri, M. and Moutawakil, D. El, "Some new common fixed point theorems under strict contractive conditions", J. Math. Anal. Appl., vol. 270, no. 1, pp. 181-188 ,2002.

[4] Abbas M., Rhoades B.E.,: Common Fixed Point Theorems for Hybrid pairs of Occasionally Weakly Compatible Mappings Satisfying Generalized Contractive Condition of Intregal type . Fixed Point Theory and Applications , Article ID 54101,9 pages, 2007

[5] George A. and . Veeramani P., On some results in fuzzy metric spaces, Fuzzy sets and Systems vol. 64, pp. 395-399, 1994.

[6] Kramosil O. and Michalek J., Fuzzy metrics and statistical metric spaces. Kybernetica vol.11, pp.326-334, 1975.

[7] Moutawakil D.El. "Afixed point theorem for multivalued maps in symmetric spaces," Applied Mathematics E-Notes, 4, 26-32, 2004.

[8] Sharma, Sushil, Kutuku Servet, and Rathore R.S.,commun." Common fixed point for multivalued mappings in intuitionistic fuzzy metric spaces" Korean Math.Soc. vol.22, No.3,pp. 391-399, 2007.

[9] Thagafi A.Al - and Shahzad, Naseer, "Generalized I-Nonexpansive Selfmaps and Invariant Approximation", Acta Mathematica Sinica, English Series, Vol.24, No.5, pp.867-876, 2008.

[10] Zadeh L. A., "Fuzzy sets", Inform. and Control, vol. 8, pp. 338-353, 1965.