Coupled Fixed Point Theorems in Vector b-metric Space

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Abstract: *This paper consists of some coupled and common coupled fixed point theorems in vector b-metric spaces. Vector b-metric space or E-b-metric space was introduced by Petre [6] merging the concepts of vector metric space as introduced by Cevik [4] and b-metric space as introduced by Czerwik [5]. We generalize the results of Shatnanawi and Hani [8] and Rao et al. [7].*

Keywords - *coupled and common coupled fixed point theorems, vector b-metric spaces*

I. Introduction

The notion of coupled fixed point for metric spaces was initiated by Bhaskar and Lakshmikantan [12]. Thereafter several authors investigated coupled fixed point theorems for various general metric spaces [12-14]. In 1989, Bakhtin [3] introduced the concept of b-metric space. Chao et al. [13] and Lakshikanthan [14] have probed coupled fixed point theorems on b-metric space. Cevik and Altun [4] introduced the concept of vector metric space and proved some fixed point theorems on this space. Petre [6] defined the vector b-metric space or E-b-metric space. We prove some results related to coupled fixed points on E-b-metric space.

II. Preliminaries

We present here various definitions and results that will be used in the sequel.. For definitions and results related to Riesz space, we refer Aliprantis and Border [2] and that for vector metric spaces, one can see Cevik and Altun [4].

Definition 2.1 A set Z with binary relation ($\leq$) which is reflexive, antisymmetric and transitive is called partial ordered set(poset).

A poset ($Z, \leq$) is said to be linearly ordered or totally ordered or chain if for each pair $u, v \in Z,$ we have either $u \leq v$ or $v \leq u.$

Definition 2.2 A poset in which every subset with two elements has a supremum or an infimum is called lattice. A lattice in Z is said to be complete if every subset has a supremum or infimum. A lattice is Dedekind complete if every non empty subset of a lattice which is bounded below (above) has a infimum (supremum).

Definition 2.3 A partially ordered vector space is poset $(E, \leq)$ where E is a real vector space such that for all $u, v, w \in E$ and $\lambda > 0$

(i) $u \leq v \quad \Rightarrow \quad u + w \leq v + w$

(ii) $u \le v \quad \Rightarrow \quad \lambda u \le \lambda v$

Definition 2.4 A partially ordered vector space which is also a lattice under its ordering is called a Riesz space.

Notation In a Riesz space, for a decreasing sequence $\{u_n\}$ whose inf $u_n = u$. We use the notation $u_n \downarrow u$.

Definition 2.5 An Archimedean Riesz space E is a Riesz space in which $\frac{1}{n}u \downarrow 0$ for every $u \in E_+$, where $E_+ = \{u \in E : u \ge 0\}$.

Definition 2.6 In a Riesz space E, A sequence $\{u_n\}$ is order convergent to u written as $u_n \xrightarrow{0} u$, if there exists a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $|u_n - u| \le a_n$ for all n, where $|u| = u \vee -u$.

The sequence $\{u_n\}$ in a Riesz space E is order-Cauchy if there exists a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $|u_n - u_{n+m}| \le a_n$ *forall n* and *forall m.*

Lemma 2.7 [15] In a Riesz space E, if $u \le ku$, where $u \in E_+$, $k \in [0,1)$ and $E_+ = \{u \in E : u \ge 0\}$, then $u = 0$.

Example 2.8 $\mathbb{R}^2$ is a Riesz space with coordinate wise ordering defined by $(u_1, u_2) \le (v_1, v_2)$ iff $u_1 \le v_1,\ u_2 \le v_2$ for all $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$.

Definition 2.9 A function $d : Z \times Z \to E$, where Z is nonempty set and E is Riesz space is called a E-metric(vector metric) on Z if it satisfies the following properties :

(i) $d(z_1, z_2) = 0$ iff $z_1 = z_2$

(ii) $d(z_1, z_2) \le d(z_1, z_3) + d(z_3, z_2) \quad \forall z_1, z_2, z_3 \in Z$

Then triplet (Z, d, E) is said to be vector metric space.

For any z_1, z_2, z_3, z_4 in a vector metric space, some inequalities listed below are trivial

(a) $0 \le d(z_1, z_2)$

(b) $d(z_1, z_2) = d(z_2, z_1)$

(c) $|d(z_1, z_3) - d(z_2, z_3)| \le d(z_1, z_2)$

(d) $|d(z_1, z_3) - d(z_2, z_4)| \le d(z_1, z_2) + d(z_3, z_4)$.

Example 2.10 Every Riesz space E is a vector metric space with $d(z_1, z_2) = |z_1 - z_2|$ for all $z_1, z_2 \in E$.

Example 2.11 Let $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$ be defined as $d(x,y)=(\alpha_1|x-y|,\alpha_2|x-y|)$, where $\alpha_1,\alpha_2\geq 0$ and $\alpha_1+\alpha_2>0$. Then d is a vector metric with coordinatewise or lexicographical ordering and $(\mathbb{R},d,\mathbb{R}^2)$ is a vector metric space.

Example 2.12 $\mathbb{R}^n$ is a Riesz space corresponds to partial order defined by

$$(u_1,u_2,...,u_n)\leq(v_1,v_2,...,v_n) \text{ if and only if } u_1\leq v_1,u_2\leq v_2,...,u_n\leq v_n.$$

Define $d:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by

$$d((u_1,u_2,...,u_n),(v_1,v_2,...,v_n))=(\alpha_1|u_1-v_1|,\alpha_2|u_2-v_2|,...,\alpha_n|u_n-v_n|)$$

where $\alpha_i, i\leq 1\leq n,$ are non-negative real numbers with $\alpha_1+\alpha_2+....+\alpha_n>0$ .Then $(\mathbb{R}^n,d,\mathbb{R}^n)$ is a vector metric space.

Definition 2.13 Suppose (Z,d,E) is a vector metric space. A sequence $\langle z_n\rangle$ in Z is said to be E-converges(or vectorial converges) to some $z\in Z$, written as $z_n\xrightarrow{d,E}z$, if there exists a sequence $\langle a_n\rangle$ in E such that $a_n\downarrow 0$ and $d(z_n,z)\leq a_n$ for all n.

Definition 2.14 A sequence $\langle z_n\rangle$ in a vector metric space (Z,d,E) is said to be E-Cauchy if if there exists a sequence $\langle a_n\rangle$ in E such that $a_n\downarrow 0$ and $d(z_n,z_{n+m})\leq a_n$ $\forall$ n and m.

Definition 2.15 Let Y be any subset of a vector metric space Z. Y is said to be E-closed if for every sequence $\{z_n\}\subseteq Y$ and $z_n\xrightarrow{d,E}z$, imply $z\in Y$.

It is easy to see that if $z_n\xrightarrow{d,E}z$, then the limit of the sequence z_n is unique and every subsequence of $\langle z_n\rangle$ E-converges to z. If $y_n\xrightarrow{d,E}y$, then $d(z_n,y_n)\xrightarrow{0}d(z,y)$.

The concepts of convergence in metric similar to vectorial convergence when $E=\mathbb{R}$. Also, If d is the absolute valued metric and $Z=E$ then concepts of convergence in order and vectorial convergence coincide.

Definition 2.16 An E-complete vector metric space Z is a vector metric space in which every E-Cauchy sequence in Z E-converges to a limit in Z.

Definition 2.17 [16] A mapping $f:(Z,d,E)\to(Y,d',F)$ is vectorially continuous at z if $z_n\xrightarrow{d,E}z$ in Z implies $f(z_n)\xrightarrow{d',F}f(z)$ in Y and the function f is vectorially continuous on Z if it is vectorically continuous at each element of Z.

Definition 2.18 [6]A function $d: Z\times Z\to E_+$,where E is Riesz space and Z is nonempty set, is said to be E-b-metric if, for any $z_1, z_2, z_3 \in Z$ and $s\geq 1$ any real number, the following conditions are satisfied:

(i) $d(z_1,z_2)\leq s\left[d(z_1,z_3)+d(z_2,z_3)\right]$

(i) $d(z_1,z_2)=0$ if and only if $z_1=z_2$.

The triple (Z,d,E) is said to be E-b-metric space.

Example 2.19 Let $Z=L^p[0,1]$ with $0<p<1$ and $E=\mathbb{R}^2$. Let $d: L^p[0,1]\times L^p[0,1]\to \mathbb{R}^2_+$ be defined by

$$d(f_1,f_2)=\left(\alpha\|f_1-f_2\|_p, \beta\|f_1-f_2\|_p\right)$$

where $\alpha,\beta\geq 0$ and $\alpha+\beta>0$. Then we can deduce that

$$d(f_1,f_2)\leq 2^{\frac{1}{p}}\left[d(f_1,f_2)+d(f_2,f_3)\right].$$

Hence $(Z,d,\mathbb{R}^2)$ is E-b-metric space with parameter $s=2^{\frac{1}{p}}>1$.

Example 2.20 Suppose $0<p<1, Z=l_p,$ and $d: l_p\times l_p\to\mathbb{R}^2_+$ is defined as

$$d(u,v)=\left(\alpha\|u-v\|_p, \beta\|u-v\|_p\right),$$

where $\alpha,\beta\geq 0$ and $\alpha+\beta>0$, then $(Z,d,\mathbb{R}^2)$ is E-b-metric space with parameter $s=2^{\frac{1}{p}}>1$.

Definition 2.21 Let $Z=C[-1,1]=E$ and $d: Z\times Z\to E_+$ be defined as

$$d(f_1,f_2)=(f_1-f_2)^p, p>1.$$

Then (Z,d,E) is E-b-metric space with parameter $s=2^{\frac{1}{p}}>1$. Since the function $x^p\,(p>1)$ is convex, we have

$$\left(\frac{1}{2}x+\frac{1}{2}y\right)^p\leq\frac{1}{2}x^p+\frac{1}{2}y^p$$

so that $(x+y)^p\leq 2^{p-1}(x^p+y^p)$

Therefore

$$\begin{aligned} d(f_1,f_3)&=(f_1-f_3)^p=(f_1-f_2+f_2-f_3)^p\\ &\leq 2^{p-1}\left[(f_1-f_2)^p+(f_2-f_3)^p\right]\\ &=2^{p-1}\left[d(f_1,f_2)+d(f_2,f_3)\right]\end{aligned}$$

Thus the relaxed triangular inequality holds with $s=2^{\frac{1}{p}}>1$

Example 2.22 Let $Z=\mathbb{R}^2, E=\mathbb{R}^2$ and $d:Z\times Z\to\mathbb{R}_+^2$ be defined as

$$d\left(\left(u_1,v_1\right),\left(u_2,v_2\right)\right)=\left(\alpha\left|u_1-u_2\right|^2,\beta\left|v_1-v_2\right|^2\right).$$

where $\alpha,\beta\geq 0$ and $\alpha+\beta>0$, then $\left(Z,d,\mathbb{R}^2\right)$ is E-b-metric space with parameter $s=2>1$.

Example 2.23 Let $Z=\{0,1,2\}$, $E=\mathbb{R}^2$ and $d:Z\times Z\to\mathbb{R}^2$ be defined as

$$d(0,1)=d(1,0)=(1,1)$$

$$d(1,2)=d(2,1)=(1,1)$$

$$d(0,2)=d(2,0)=(4,4)$$

Since $d(0,2)=(4,4)\nleq d(0,1)+d(1,2),(Z,d,E)$ is E-b-metric space with $s=2$ but not a metric space.

Definition 2.24 Suppose $(Z,\leq)$ is a poset and $T:Z\times Z\to Z$ be a map. If $T(u,v)$ is monotone nondecreasing in first argument i.e. u and is monotone nonincreasing in second argument i.e. v, that is, for all $p,q\in Z$, $p\leq q$ implies $T(p,v)\leq T(q,v)$ for any $v\in Z$ and for all $x,y\in Z$, $y\leq x$ implies $T(u,x)\leq T(u,y)$ for any $u\in Z$, then one can say that T has mixed monotone property .

Definition 2.25 Suppose $(Z,\leq)$ is a poset and $T:Z\times Z\to Z$ and $g:Z\to Z$ be two mappings. T has the mixed g-monotone property if for any $p,q\in Z$, $gp\leq gq$ implies $T(p,y)\leq T(q,y)$ for any $y\in Z$ and for any $u,v\in Z$, $gu\leq gv$ implies $T(z,v)\leq T(z,u)$ for any $z\in Z$.

Definition 2.26 Suppose (Z,d,E) be E-b-metric space. An element $(z_1,z_2)\in Z\times Z$ is said to be coupled fixed point of a function $T:Z\times Z\to Z$ if $T(z_1,z_2)=z_1$ and $T(z_2,z_1)=z_2$.

Definition 2.27 An element of $(z_1,z_2)\in Z\times Z$ is said to be coupled coincidence point of the mapping $T:Z\times Z\to Z$ and $g:Z\to Z$ if $T(z_1,z_2)=g(z_1)=z_1$ and $T(z_2,z_1)=g(z_2)=z_2$.

Definition 2.28 Suppose Z is a non empty set. The mapping $T:Z\times Z\to Z$ and $g:Z\to Z$ are said to be commutative if $g\left(T(z_1,z_2)\right)=T\left(g(z_1),g(z_2)\right)\ \forall x,y\in Z$.

Definition 2.29 Suppose $Z\neq\phi$ and $T:Z\times Z\to Z$ and $g:Z\to Z$. The pair (T,g) is said to be weakly compatible if $g\left(T(z_1,z_2)\right)=T\left(g(z_1),g(z_2)\right)$ whenever $g(z_1)=T(z_1,z_2)$ and $g(z_2)=T(z_2,z_1)\ \forall\ z_1,z_2\in Z\times Z$.

III. Main Results

Theorem 3.1 Suppose $(Z,\leq_Z)$ is a poset and $d: Z\times Z \to E_+$ be E-b-metric defined on Z with coefficient $s\geq 1$ and E-Archimedean. Let $T: Z\times Z \to Z$ and $g: Z \to Z$ be two mappings such that

$$d\left(T(u,v),T(p,q)\right)+d\left(T(v,u),T(q,p)\right)\leq_E k\left[d(gu,gp)+d(gv,gq)\right]$$

for some $k\in\left(0,\frac{1}{s}\right)$ and for all $u,v,p,q\in Z$ with $gp\leq_Z gu$ and $gv\leq_Z gq$. We further assume the following hypothesis

(1) $T(Z\times Z)\subseteq g(Z)$

(2) $g(Z)$ is E-complete.

(3) g is vectorially continuous and commute with T.

(4) T has the mixed g - monotone property on Z.

(5) Either T is vectorially continuous or

   A. for every non decreasing sequence, if $\{u_n\}\to u$ then $u_n\leq_Z u$.

   B. for every increasing sequence if $\{v_n\}\to v$ then $v\leq_Z v_n$.

Then if there exists two elements $u_0,v_0\in Z$ with $g(u_0)\leq_Z T(u_0,v_0)$ and $T(u_0,v_0)\leq_Z g(v_0)$, T and g have coupled coincident fixed point.

Proof: - Let $u_0,v_0\in Z$ be such that $gu_0\leq_Z T(u_0,v_0)$ and $T(v_0,u_0)\leq_Z gv_0$.

Since $T(Z\times Z)\subseteq g(Z)$, it is to find $u_1,v_1\in Z$ such that

$$g(u_1)=T(u_0,v_0) \text{ and } g(v_1)=T(v_0,u_0).$$

Again since $T(Z\times Z)\subseteq g(Z)$, we can select $u_2,v_2\in Z$ such that

$$g(u_2)=T(u_1,v_1) \text{ and } g(v_2)=T(v_1,u_1).$$

Continuing this process, we can construct two sequences $\{u_n\}$ and $\{v_n\}$ in Z such that,

$$g(u_{n+1})=T(u_n,v_n) \tag{1}$$

and $$g(v_{n+1})=T(v_n,u_n),\quad \forall n \tag{2}$$

Now we will prove that $\forall n \geq 0,$

$$g(u_n) \leq_Z g(u_{n+1}) \qquad (3)$$

$$g(v_{n+1}) \leq_Z g(v_n) \qquad (4)$$

We will prove (3) and (4) by the use of principle mathematical induction.

Suppose $n = 0$.

Since $g(u_0) \leq_Z T(u_0, v_0)$ and $T(v_0, u_0) \leq_Z g(v_0)$.

Thus we have $g(u_0) \leq_Z g(u_1)$ and $g(v_1) \leq_Z g(v_0)$.

So (3) and (4) hold for $n = 0$.

Now we suppose that (3) and (4) hold for some $n > 0$.

Since the mapping T is mixed g-monotone and $g(u_n) \leq_Z g(u_{n+1})$ and $g(v_{n+1}) \leq_Z g(v_n)$.
We get

$$g(u_{n+1}) = T(u_n, v_n) \leq_Z T(u_{n+1}, v_n)$$

and $$T(v_{n+1}, u_n) \leq_Z T(v_n, u_n) = g(v_{n+1})$$

Also , we have

$$T(u_{n+1}, v_n) \leq_Z T(u_{n+1}, v_{n+1}) = g(u_{n+2})$$

and $$g(v_{n+2}) = T(v_{n+1}, u_{n+1}) \leq_Y T(v_{n+1}, u_n)$$

Then from (1) and (2),

$$g(u_{n+1}) \leq_Z g(u_{n+2}) \text{ and } g(v_{n+2}) \leq_Z g(v_{n+1})$$

Thus by the use of principle of Mathematical Induction, (3) and (4) holds for all $n \geq 0$. After repeating the above process, one can deduce the following

$$g(u_0) \leq_Z g(u_1) \leq_Z g(u_2) \leq_Z g(u_3) \leq_Z g(u_4) \leq_Z \ldots\ldots\ldots \leq_Z g(u_{n+1}) \leq_Z \ldots\ldots \qquad (5)$$

and $$\ldots\ldots\ldots\ldots \leq_Z g(v_{n+1}) \leq_Z \ldots\ldots \leq_Z g(v_2) \leq_Z g(v_1) \leq_Z g(v_0) \qquad (6)$$

Now, if we assume $(u_{n+1}, v_{n+1}) = (u_n, v_n)$, then the result is trivial.

So, we assume $(u_{n+1}, v_{n+1}) \neq (u_n, v_n) \quad \forall n \geq 0$

Thus we assume that either

$$g(u_{n+1}) = T(u_n, v_n) \neq g(u_n) \text{ or}$$

$$g(v_{n+1}) = T(v_n, u_n) \neq g(v_n)$$

Again

$$d(gu_n, gu_{n+1}) + d(gv_n, gv_{n+1}) = d(T(u_{n-1}, v_{n-1}), T(u_n, v_n)) + d(T(v_{n-1}, u_{n-1}), T(v_n, u_n))$$
$$\leq_E k[d(gu_{n-1}, gu_n) + d(gv_{n-1}, gv_n)]$$

Let $d(gu_n, gu_{n+1}) + d(gv_n, gv_{n+1}) = d_n$ (7)

where d_n is some element of E.

Then $d_n \leq_E kd_{n-1}$

$\Rightarrow d_n \leq_E kd_{n-1} \leq_E k^2 d_{n-2} \leq_E \text{..........} \leq_E k^n d_0$

Again let m, n be two positive integers, such that $m > n$. Then we can write

$$d(gu_n, gu_m) \leq_E s\,d(gu_n, gu_{n+1}) + s^2\, d(gu_{n+1}, gu_{n+2}) + s^3\, d(gu_{n+2}, gu_{n+3}) + \text{........}$$
$$\text{...........} + s^{m-n-1} d(gu_{m-2}, gu_{m-1}) + s^{m-n-1} d(gu_{m-1}, gu_m)$$

(By repeated use of triangular inequality)

$$d(gv_n, gv_m) \leq_E s\,d(gv_n, gv_{n+1}) + s^2\, d(gv_{n+1}, gv_{n+2}) + s^3\, d(gv_{n+2}, gv_{n+3}) + \text{........}$$
$$\text{.....} + s^{m-n-1}\, d(gv_{m-2}, gv_{m-1}) + s^{m-n-1} d(gv_{m-1}, gv_m)$$

Therefore,

$$d(gu_n, gu_m) + d(gv_n, gv_m) \leq_E sd_n + s^2 d_{n+1} + s^3 d_{n+2} + \text{.......} + s^{m-n-1} d_{m-2} + s^{m-n-1} d_{m-1}$$

[using (7)]

$$\leq_E sk^n d_0 + s^2 k^{n+1} d_0 + s^3 k^{n+2} d_0 + \text{.......} + s^{m-n-1} k^{m-2} d_0 + s^{m-n-1} k^{m-1} d_0$$
$$\leq_E sk^n d_0 + s^2 k^{n+1} d_0 + s^3 k^{n+2} d_0 + \text{.......} + s^{m-n-1} k^{m-2} d_0 + s^{m-n} k^{m-1} d_0$$
$$\leq_E sk^n d_0 [1 + sk + s^2 k^2 + \text{.......} + s^{m-n-1} k^{m-n-1}]$$
$$\leq_E sk^n d_0 [1 + sk + s^2 k^2 + \text{..........}]$$
$$\leq_E sk^n d_0 \frac{1}{1 - sk}$$

Thus $d(gu_n, gu_m) + d(gv_n, gv_m)_E \leq 0$

$\Rightarrow d(gu_n, gu_m) \leq_E 0, d(gv_n, gv_m) \leq_E 0$

Hence $\{gu_n\}$ and $\{gv_n\}$ are two E-Cauchy sequence in gZ and we supposed the hypothesis that gZ is E-complete.

So there exists two points, say u and v in Z, such that the two E-Cauchy sequences $gu_n \to gu = \xi$ and $gv_n \to gv = \eta$ as $n \to \infty$.

Now let (5) holds, T is vectorially continuous and so

$g(g(u_{n+1})) = g(T(u_n, v_n)) = T(gu_n, gv_n)$ since T and g are commutative.

$\Rightarrow g(\xi)=T(\xi,\eta)$, since T and g are vectorially continuous.

Similarly, we can show that $g(\eta)=T(\eta,\xi)$.

Hence (ξ,η) is a point of coincidence for T and g.

Again, let (5A) hold, by (5) we get that $\{gu_n\}$ is a non-decreasing sequence and $gu_n \to \xi$, therefore $gu_n \leq_Z \xi$, for all n.

Similarly by (5B) and (6), we get that $\{gv_n\}$ is a non-increasing sequence and $gv_n \to \eta$, therefore $\eta \leq_Z gv_n$, for all n.

Then

$$\begin{aligned} d(g\xi,T(\xi,\eta)) &\leq_E s\,d(g\xi,ggu_{n+1})+s\,d(ggu_{n+1},T(\xi,\eta)) \\ &= s\,d(g\xi,ggu_{n+1})+s\,d(g(T(u_n,v_n)),T(\xi,\eta)) \\ &= s\,d(g\xi,ggu_{n+1})+s\,d(T(gu_n,gv_n),T(\xi,\eta)) \end{aligned}$$

Add both side $s\,d(T(gv_n,gu_n),T(\eta,\xi))$, thus we have

$$\begin{aligned} d(g\xi,T(\xi,\eta))+s\,d(T(gv_n,gu_n),T(\eta,\xi)) &\leq_E s\,d(g\xi,ggu_{n+1}) \\ &\quad + s\,d(T(gu_n,gv_n),T(\xi,\eta))+s\,d(T(gv_n,gu_n),T(\eta,\xi)) \\ d(g\xi,T(\xi,\eta))+s\,d(T(gv_n,gu_n),T(\eta,\xi)) &\leq_E s\,d(g\xi,ggu_{n+1})+s\,k\left[d(ggu_n,g\xi)+d(ggv_n,g\eta)\right] \end{aligned}$$

(8)

Since, g is vectorially E-continuous, $ggu_n \to g\xi$ and $ggv_n \to g\eta$ as $n\to\infty$ and hence (8) gives $g\xi=T(\xi,\eta)$.

Similarly, we can show that, $g\eta=T(\eta,\xi)$

Again, $d(g\xi,g\eta)+d(g\eta,g\xi)$

$$=d(T(\xi,\eta),T(\eta,\xi))+d(T(\eta,\xi),T(\xi,\eta))$$

$$\leq_E k\left[d(g\xi,g\eta)+d(g\eta,g\xi)\right]$$

$$\Rightarrow 2d(g\xi,g\eta)\leq_E 2k\,d(g\xi,g\eta)$$

$$\Rightarrow d(g\xi,g\eta)\leq_E k\,d(g\xi,g\eta)$$

Since $k<\frac{1}{s}$, $d(g\xi,g\eta)=0$, thus $g\xi=g\eta$.

Hence $T(\xi,\eta)=g\xi=g\eta=T(\eta,\xi)$

Finally,

$$d(\xi,g\xi)\preceq_E s\,d(\xi,gu_{n+1})+s\,d(gu_{n+1},g\xi)$$
$$=s\,d(\xi,gu_{n+1})+s\,d(T(u_n,v_n),T(\xi,\eta))$$

and in the same manner

$$d(\eta,g\eta)\preceq_E s\,d(\eta,gv_{n+1})+s\,d(gv_{n+1},g\eta)$$
$$=s\,d(\eta,gv_{n+1})+s\,d(T(v_n,u_n),T(\eta,\xi))$$

Therefore,

$$d(\xi,g\xi)+d(\eta,g\eta)\preceq_E s\left[d(\xi,gu_{n+1})\right]+d(\eta,gv_{n+1})$$
$$+s\left[d(T(u_n,v_n),T(\xi,\eta))+d(T(v_n,u_n),T(\eta,\xi))\right]$$
$$\preceq_E s\left[d(\xi,gu_{n+1})\right]+d(\eta,gv_{n+1})+sk\left[d(gu_n,g\xi)+d(gv_n,g\eta)\right]$$
$$\preceq_E s\left[d(\xi,gu_{n+1})\right]+d(\eta,gv_{n+1})+s^2k\left[d(gu_n,\xi)+d(\xi,g\xi)+d(gv_n,\eta)+d(\eta,g\eta)\right]$$

Thus $(1-ks^2)\left[d(\xi,g\xi)+d(\eta,g\eta)\right]$

$\preceq_E s\left[d(\xi,gu_{n+1})+d(\eta,gv_{n+1})\right]+s^2k\left[d(gu_n,\xi)+d(gv_n,\eta)\right]\to 0$ as $n\to\infty$

Thus $d(\xi,g\xi)=0=d(\eta,g\eta)$

$\Rightarrow \xi=g\xi$ and $\eta=g\eta$

$\Rightarrow g(\xi)=\xi=T(\xi,\eta)$, $g(\eta)=\eta=T(\eta,\xi)$.

Theorem 3.2 Suppose (Z,d,E) is a E-b-metric space with $s>1,$ E-Archimedean and $T:Z\times Z\to Z$ and $g:Z\to Z$ be two mappings on Z. Suppose that there exists non-negative constant $t_i,\ i=1,2,...,10$ such that

$$d(T(u,v),T(p,q))\preceq_E t_1d(gu,gp)+t_2d(gv,gq)+t_3d(T(u,v),gu)+t_4d(T(v,u),gv)$$
$$+t_5d(T(p,q),gp)+t_6d(T(q,p),gq)+t_7d(T(u,v),gp)+t_8d(T(v,u),gq)$$
$$+t_9d(T(p,q),gu)+t_{10}d(T(q,p),gv)$$

holds for all $u,v,p,q\in Z.$

Suppose

(i) $g(Z)$ is E-complete subspace of Z.

(ii) $T(Z\times Z)\subseteq g(Z)$

(iii) $s(t_1+t_2+t_3+t_4)+t_5+t_6+t_7+t_8+(s^2+s)(t_9+t_{10})<1$

Then T and g have coupled coincidence point. Further, if T and g are weakly compatible then T and g have unique coupled fixed point.

Proof: - Take $v_0 \in Z$.

Since $T(Z\times Z)\subseteq g(Z)$, then can find $v_1 \in Z$ such that $T(v_0,v_0)=g(v_1.)$

Again since $T(Z\times Z)\subseteq g(Z)$, then $\exists\, v_2 \in Z$ such that $T(v_1,v_1)=g(v_2)$.

Repeating the above process , we will get a sequence $\{u_n\}$ in $g(Z)$ such that

$$u_n = g(v_{n+1}) = T(v_n,v_n) \tag{9}$$

$$\begin{aligned}
d(gv_{n+1},gv_{n+2}) &= d(T(v_n,v_n),T(v_{n+1},v_{n+1})) \le_E t_1 d(gv_n,gv_{n+1})+t_2 d(gv_n,gv_{n+1})\\
&+t_3 d(T(v_n,v_n),gv_n)+t_4 d(T(v_n,v_n),gv_n)+t_5 d(T(v_{n+1},v_{n+1}),gv_n)\\
&+t_6 d(T(v_{n+1},v_{n+1}),gv_{n+1})+t_7 d(T(v_n,v_n),gv_{n+1})+t_8 d(T(v_n,v_n),gv_{n+1})\\
&+t_9 d(T(v_{n+1},v_{n+1}),gv_n)+t_{10} d(T(v_{n+1},v_{n+1}),gv_n)
\end{aligned}$$

by using (9), we have

$$\begin{aligned}
\Rightarrow d(gv_{n+1},gv_{n+2}) \le_E\ & t_1 d(gv_n,gv_{n+1})+t_2 d(gv_n,gv_{n+1})+t_3 d(gv_{n+1},gv_n)+t_4 d(gv_{n+1},gv_n)\\
&+t_5 d(gv_{n+2},gv_{n+1})+t_6 d(gv_{n+2},gv_{n+1})+t_7 d(gv_{n+1},gv_{n+1})\\
&+t_8 d(gv_{n+1},gv_{n+1})+t_9 d(gv_{n+2},gv_n)+t_{10} d(gv_{n+2},gv_n)
\end{aligned}$$

$$(1-t_5-t_6)d(gv_{n+1},gv_{n+2}) \le_E (t_1+t_2+t_3+t_4)d(gv_n,gv_{n+1})+(t_9+t_{10})d(gv_{n+2},gv_n)$$

Since, $d(gv_n,gv_{n+2}) \le_E sd(gv_n,gv_{n+1})+sd(gv_{n+1},gv_{n+2})$

$$\begin{aligned}
(1-t_5-t_6)d(gv_{n+1},gv_{n+2}) \le_E\ & (t_1+t_2+t_3+t_4)d(gv_n,gv_{n+1})\\
&+s(t_9+t_{10})\left[d(gv_n,gv_{n+1})+d(gv_{n+1},gv_{n+2})\right]
\end{aligned}$$

$$\Rightarrow (1-t_5-t_6-s(t_9+t_{10}))d(gv_{n+1},gv_{n+2}) \le_E (t_1+t_2+t_3+t_4+s(t_9+t_{10}))d(gv_n,gv_{n+1})$$

$$\Rightarrow d(gv_{n+1},gv_{n+2}) \le_E \frac{t_1+t_2+t_3+t_4+s(t_9+t_{10})}{1-t_5-t_6-s(t_9+t_{10})}d(gv_n,gv_{n+1})$$

$$\Rightarrow d(gv_{n+1},gv_{n+2}) \le_E r\ d(gv_n,gv_{n+1})$$

where $r = \dfrac{t_1+t_2+t_3+t_4+s(t_9+t_{10})}{1-t_5-t_6-s(t_9+t_{10})}$

$$sr = \frac{s(t_1+t_2+t_3+t_4)+s^2(t_9+t_{10})}{1-t_5-t_6-s(t_9+t_{10})}$$

Since $s(t_1+t_2+t_3+t_4)+t_5+t_6+(s^2+s)(t_9+t_{10})<1$

By using the condition (i),

$s(t_1+t_2+t_3+t_4)+t_5+t_6+t_7+t_8+(s^2+s)(t_9+t_{10})<1$ and $r \le sr$, we get $r<1$,

$$\Rightarrow d(gv_{n+1}, gv_{n+2}) \le_E r\ d(gv_n, gv_{n+1}) \qquad (10)$$

Repeating n-times,

$$d(gv_{n+1}, gv_{n+2}) \le_E r^{n+1} d(gv_0, gv_1) \qquad (11)$$

Let $n,m \in N,\ m>n.$

Therefore

$$d(gv_n, gv_m) \le_E sd(gv_n, gv_{n+1})+s^2 d(gv_{n+1}, gv_{n+2})+..........+s^{m-n} d(gv_{m-1}, gv_m)$$

$$\Rightarrow d(gv_n, gv_m) \le_E s r^n d(gv_0, gv_1)+s^2 r^{n+1} d(gv_0, gv_1)+..........+s^{m-n} r^{m-1} d$$

$$= s r^n d(gv_0, gv_1)\left[1+sr+sr^2+.........+s^{m-n-1} r^{m-n-1}\right]$$

$$\le_E s r^n d(gv_0, gv_1)\left[\frac{1}{1-sr}\right]$$

Since

$r<1 \quad \Rightarrow r^n \to 0$ as $n \to \infty,$

$sr<1 \quad \Rightarrow (1-sr)>0$

$\Rightarrow \{gv_n\}$ is an E-Cauchy sequence in $(g(Z), d)$.

Since $g(Z)$ is complete, $\exists\, t \in g(Z)$ such that

$$\lim_{n\to\infty} g(v_n) = g(v) = t$$

We will now prove that (y, y) is the coupled coincident point of T and g.

$$d\left(gv_{n+1},T\left(v,v\right)\right)=d\left(T\left(v_n,v_n\right),T\left(v,v\right)\right)$$
$$\le_E t_1d\left(gv_n,gv\right)+t_2d\left(gv_n,gv\right)+t_3d\left(T\left(v_n,v_n\right),gv_n\right)+t_4d\left(T\left(v_n,v_n\right),gv_n\right)$$
$$+t_5d\left(T\left(v,v\right),gv\right)+t_6d\left(T\left(v,v\right),gv\right)+t_7d\left(T\left(v_n,v_n\right),gv\right)+t_8d\left(T\left(v_n,v_n\right),gv\right)$$
$$+t_9d\left(T\left(v,v\right),gv_n\right)+t_{10}d\left(T\left(v,v\right),gv_n\right)$$

$$d\left(gv_{n+1},T\left(v,v\right)\right)=d\left(T\left(v_n,v_n\right),T\left(v,v\right)\right)$$
$$\le_E t_1d\left(gv_n,gv\right)+t_2d\left(gv_n,gv\right)+t_3d\left(gv_{n+1},gv_n\right)+t_4d\left(gv_{n+1},gv_n\right)$$
$$+t_5d\left(T\left(v,v\right),gv\right)+t_6d\left(T\left(v,v\right),gv\right)+t_7d\left(gv_{n+1},gv\right)+t_8d\left(gv_{n+1},gv\right)$$
$$+t_9d\left(T\left(v,v\right),gv_n\right)+t_{10}d\left(T\left(v,v\right),gv_n\right) \qquad (12)$$

Since $d\left(gv,T\left(v,v\right)\right)\le_E s\left[d\left(gv,gv_{n+1}\right)+d\left(gv_{n+1},T\left(v,v\right)\right)\right]$

$$\Rightarrow \frac{1}{s}d\left(gv,T\left(v,v\right)\right)\le_E \lim_{n\to\infty} d\left(gv_{n+1},T\left(v,v\right)\right) \qquad (13)$$

Also, $d\left(gv_{n+1},gv_n\right)\le_E s\left[d\left(gv_{n+1},gv\right)+d\left(gv,gv_n\right)\right]$

Thus $\lim d\left(gv_{n+1},gv_n\right)=0$ (14)

Further,

$$d\left(T\left(v,v\right),gv_n\right)\le_E s\left[d\left(T\left(v,v\right),gv\right)+d\left(gv,gv_n\right)\right]$$

Letting $n\to\infty$ in above inequality,

$$\lim_{n\to\infty} d\left(T\left(v,v\right),gv_n\right)\le_E sd\left(T\left(v,v\right),gv\right) \qquad (15)$$

Taking $\lim n\to\infty$ in (12) and using (15)

$$\frac{1}{s}d\left(gv,T\left(v,v\right)\right)\le_E t_5d\left(T\left(v,v\right),gv\right)+t_6d\left(T\left(v,v\right),gv\right)+s\,t_9\,d\left(T\left(v,v\right),gv\right)+s\,t_{10}\,d\left(T\left(v,v\right),gv\right)$$

$$\Rightarrow d\left(gv,T\left(v,v\right)\right)\le_E \left(st_5+st_6+s^2t_9+s^2t_{10}\right)d\left(T\left(v,v\right),gv\right)$$

Since $st_5+st_6+s^2t_9+s^2t_{10}<1$ Using condition (i)

and E is Archimedean

$$\Rightarrow g\left(v\right)=T\left(v,v\right)$$

$\Rightarrow \left(v,v\right)$ is a coupled coincidence point of T and g.

For uniqueness of couple coincidence point $\left(v,v\right)$, suppose $\left(v',v'\right)$ be another coupled coincidence point of T and g. Then

$$d\left(gv,gv^{'}\right)=d\left(T\left(v,v\right),T\left(v^{'},v^{'}\right)\right)$$
$$\leq_E t_1 d\left(gv,gv^{'}\right)+t_2 d\left(gv,gv^{'}\right)+t_3 d\left(T\left(v,v\right),gv\right)+t_4 d\left(T\left(v,v\right),gv\right)$$
$$+t_5 d\left(T\left(v^{'},v^{'}\right),gv\right)+t_6 d\left(T\left(v^{'},v^{'}\right),gv^{'}\right)+t_7 d\left(T\left(v,v\right),gv^{'}\right)$$
$$+t_8 d\left(T\left(v,v\right),gv^{'}\right)+t_9 d\left(T\left(v^{'},v^{'}\right),gv\right)+t_{10} d\left(T\left(v^{'},y^{'}\right),gv\right)$$

$$d\left(gv,gv^{'}\right)\leq\left(t_1+t_2+t_7+t_8+t_9+t_{10}\right)d\left(gv,gv^{'}\right)$$

Since $t_1+t_2+t_7+t_8+t_9+t_{10}<1$

$$\Rightarrow g\left(v\right)=g\left(v^{'}\right)$$

Hence $\left(v,v\right)$ is the unique coupled coincidence point of T and g.

Since T and g are weakly compatible, then

$$\Rightarrow g\left(T\left(v,v\right)\right)=T\left(gv,gv\right)$$

Put $w=g\left(v\right)$

$$g\left(w\right)=g\left(gv\right)=g\left(T\left(v,v\right)\right)=T\left(gv,gv\right)$$

$\Rightarrow\left(w,w\right)$ is coupled coincidence point of T and g.

By uniqueness of coupled coincidence point of T and g , $w=v$ i.e. $g\left(y\right)=v$

But $T\left(v,v\right)=gv=v$

$\Rightarrow\left(v,v\right)$ is coupled fixed point of T and g.

Corollary 3.3 Let $\left(Z,d,E\right)$ be Archimedean E-b-metric space with constant $s\geq1$ and E-Complete. Let $T:Z\times Z\to Z$ be a mapping. Suppose there exist non-negative constants $t_i, 1\leq i\leq10, i\in\mathbb{N}$ such that

$$d\left(T\left(u,v\right),T\left(p,q\right)\right)\leq_E t_1 d\left(u,p\right)+t_2 d\left(v,q\right)+t_3 d\left(T\left(u,y\right),u\right)$$
$$+t_4 d\left(T\left(v,u\right),v\right)+t_5 d\left(T\left(p,q\right),p\right)$$
$$+t_6 d\left(T\left(q,p\right),q\right)+t_7 d\left(T\left(u,v\right),p\right)$$
$$+t_8 d\left(T\left(v,u\right),q\right)+t_9 d\left(T\left(p,q\right),u\right)+t_{10} d\left(T\left(q,p\right),v\right)$$

holds for all $u,v,p,q\in Z.$

If $s\left(\sum_{i=1}^{6} t_i\right)+t_7+t_8+\left(s^2+s\right)t_9+\left(s^2+s\right)t_{10}<1$, then T has a unique coupled fixed point.

Proof: Simply take g=I (identity) in theorem 3.2.and repeat the above proof.

Corollary 3.4 Let (Z,d,E) be Archimedean E-b-metric space with constant $s \geq 1$ and E-Complete. Let $T : Z \times Z \to Z$ be a mapping. Suppose $\exists t_i, i = 1,2,\dots,10$ where $t_i \geq 0$ such that

$$\begin{aligned} d\left(T(u,v),T(p,q)\right) \leq_E & \ t_1 d(u,p) + t_2 d(v,q) + t_3 d\left(T(u,v),u\right) \\ & + t_4 d\left(T(v,u),v\right) + t_5 d\left(T(p,q),p\right) \\ & + t_6 d\left(T(q,p),q\right) + t_7 d\left(T(u,v),p\right) \\ & + t_8 d\left(T(v,u),q\right) + t_9 d\left(T(p,q),u\right) + t_{10} d\left(T(q,p),v\right) \end{aligned}$$

holds $\forall u,v,p,q \in Z.$

If $\sum_{i=1}^{8} t_i + 2t_9 + 2t_{10} < 1$, then T has a unique coupled fixed point.

Proof: Put s=1 in the proof of above corollary 3.3.

Example 3.5 If $Z = \Box$, $E = \Box^2$, let $d(u,v) = \left(\alpha|u-v|^2, \beta|u-v|^2\right)$ where $\alpha,\beta \geq 0, \alpha+\beta > 0$.

Delineate $T : Z \times Z \to Z$ and $g : Z \to Z$ by

$$T(u,v) = \frac{u-v}{12}, \ g(u) = 1 - \frac{u}{2}$$

Then $T(Z \times Z) \subseteq g(Z) = Z$

Then

$$\begin{aligned} d\left(T(u,v),T(p,q)\right) &= d\left(\frac{u-v}{12}, \frac{p-q}{12}\right) \\ &= \left(\alpha\frac{|u-v-p+q|^2}{(12)^2}, \beta\frac{|u-v-p+q|^2}{(12)^2}\right) \\ &\leq \left(\alpha.\frac{\left(|u-p|^2+|v-q|^2\right)}{(12)^2}, \beta.\frac{\left(|u-p|^2+|v-q|^2\right)}{(12)^2}\right) \\ &= \frac{2}{72}\left(\alpha\frac{|u-p|^2}{2^2} + \alpha\frac{|v-q|^2}{2^2}, \beta\frac{|u-p|^2}{2^2} + \beta\frac{|v-q|^2}{2^2}\right) \\ &= \frac{1}{36}\left[d(gu,gp) + d(gv,gq)\right] \end{aligned}$$

$$\Rightarrow t_1 = t_2 = \frac{1}{36}, \quad t_i = 0 \;\; for \;\; i = 3, 4, \ldots, 10$$

$$t_1 + t_2 = \frac{1}{18} < 1$$

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