A Non-uniform Bound on Poisson Approximation for Random Sums of Negative Binomial Random Variables

Kanint Teerapabolarn [#1]

[#] *Department of Mathematics, Faculty of Science, Burapha University, Chonburi, 20131, Thailand*

Abstract *This paper uses the Stein-Chen method to determine a non-uniform bound for the point metric between the distribution of random sums of independent negative binomial random variables and a Poisson distribution. Three examples are provided to illustrate applications of the result obtained.*

Keywords *Negative binomial distribution, Poisson approximation, point metric, random sums, Stein-Chen method.*

I. INTRODUCTION

Let $X_1, X_2,...$ be a sequence of independent non-negative integer-valued random variables. Let us consider the sum $S_M = \sum_{i=1}^{M} X_i$, where M is a positive integer-valued random variable, which is independent of the X_i's. The sum S_M is called *random sums*. For $X_1, X_2,...$ is a sequence of independent Bernoulli random variables, some authors have tried to determine uniform and non-uniform bounds in the Poisson approximation to the distribution of S_M, which can be found in [11], [3] and [6]. For $X_1, X_2,...$ is a sequence of independent geometric random variables, [7] and [8] gave uniform and non-uniform bounds for approximating the distribution of S_M by a Poisson distribution.

Let $X_1, X_2,...$ be a sequence of independent negative binomial random variables, each with probability function $p_{X_i}(x) = \frac{\Gamma(r_i + x)}{\Gamma(r_i)x!} q_i^x p_i^{r_i}$, $x \in \mathbb{N} \cup \{0\}$, where $q_i = 1 - p_i$, and $\wp_\lambda$ a Poisson random variable with mean λ. In the case, [9] gave a uniform bound for the total variation distance between the distributions of S_M and $\wp_\lambda$ as follows:

$$d_A(S_M, \wp_\lambda) \le \min\left\{ E\left(\sum_{i=1}^{M} \frac{r_i q_i^2}{p_i} \right), E\left(\frac{\sum_{i=1}^{M} \frac{r_i q_i^2}{p_i}}{\sqrt{2e\lambda_M}} \right) \right\} + \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_M - \lambda|, \tag{1}$$

where $d_A(S_M, \wp_\lambda) = \sup_{A \subseteq \mathbb{N} \cup \{0\}} |P(S_M \in A) - P(\wp_\lambda \in A)|$, $\lambda_M = \sum_{i=1}^{M} r_i q_i$ and $\lambda = E(\lambda_M)$.

Consider the inequality (1), if $A = \{x_0\}$, $x_0 \in \mathbb{N} \cup \{0\}$, then a uniform counterpart of the bound in (1) for the point metric between the distributions of S_M and $\wp_\lambda$ is as follows:

$$d_{x_0}(S_M, \wp_\lambda) \le \min\left\{ E\left(\sum_{i=1}^{M} \frac{r_i q_i^2}{p_i} \right), E\left(\frac{\sum_{i=1}^{M} \frac{r_i q_i^2}{p_i}}{\sqrt{2e\lambda_M}} \right) \right\} + \min\left\{1, \sqrt{\frac{2}{e\lambda}}\right\} E|\lambda_M - \lambda|, \tag{2}$$

where $d_{x_0}(S_M, \wp_\lambda) = |P(S_M = x_0) - P(\wp_\lambda = x_0)|$.

It can be seen that the bound in (2) is a uniform bound with respect to x_0, does not depend on x_0. Thus, it may not be good enough for measuring the accuracy of the approximation. In this case, a non-uniform bound with respect to x_0 is required. However, the probability $P(S_M=0)=E\left(\prod_{i=1}^{N} p_i^{r_i}\right)$ can be directly computed. In this paper, we focus on determining a non-uniform bound for the point metric $d_{x_0}(S_M,\wp_\lambda)$ when $x_0\in\mathbb{N}$.

II. METHOD

In 1972, [5] introduced a power full method for the normal approximation, which is Stein's method. Later, [2] developed and applied Stein's method to the Poisson approximation, which is referred to as the Stein-Chen method. Stein's equation for Poisson distribution with mean $\Lambda>0$, for given *h*, is of the form

$$h(x)-P_\Lambda(h)=\Lambda f(x+1)-xf(x), \tag{3}$$

where $P_\Lambda(h)=e^{-\Lambda}\sum_{k=0}^{\infty}h(k)\frac{\Lambda^k}{k!}$ and f and h are bounded real valued functions defined on $\mathbb{N}\cup\{0\}$. For $A\subseteq\mathbb{N}\cup\{0\}$, let $h_A:\mathbb{N}\cup\{0\}\to\mathbb{R}$ be defined by

$$h_A(x)=\begin{cases}1 \text{ , if } x\in A,\\ 0 \text{ , if } x\notin A.\end{cases} \tag{4}$$

Following [1], putting $h_{\{x\}}$ by h_x and for $x_0\in\mathbb{N}\cup\{0\}$, the solution $f_{x_0}=f_{\{x_0\}}$ of (3) can be expressed as

$$f_{x_0}(x)=\begin{cases}\frac{(x-1)!}{x_0!}\Lambda^{x_0-x}P_\Lambda(1-h_{C_{x-1}}) \text{ , if } x_0<x,\\ -\frac{(x-1)!}{x_0!}\Lambda^{x_0-x}P_\Lambda(h_{C_{x-1}}) \text{ , if } x_0\ge x>0,\\ 0 \text{ , if } x=0,\end{cases} \tag{5}$$

where $x\in\mathbb{N}$ and $C_{x-1}=\{0,1,...,x-1\}$.

For giving the desired result, we also need the following lemmas.

Lemma 1. *For* $x_0,x\in\mathbb{N}$, *then the following inequality holds*:

$$\sup_{x\ge1}\left|f_{x_0}(x)\right|\le\min\left\{\frac{1}{x_0},\frac{1-e^{-\Lambda}}{\Lambda}\right\} \quad \text{([4]).} \tag{6}$$

Lemma 2. *If* $g:\mathbb{N}\cup\{0\}\to R$ *is any bounded function and* Z *is the Poisson random variable with mean* Λ *then the following identity holds:*

$$E\left[\Lambda g(Z+1)\right]=E\left[Zg(Z)\right] \quad \text{([1]).} \tag{7}$$

Lemma 3. *Let* $x_0,n\in\mathbb{N}$, $S_n=\sum_{i=1}^{n}X_i$ *and* $\Lambda=\lambda_n=\sum_{i=1}^{n}r_iq_i$, *then we have the following.*

$$d_{x_0}\left(S_n,\wp_{\lambda_n}\right)\le\frac{1}{x_0}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i} \tag{8}$$

and

$$d_{x_0}\left(S_n,\wp_{\lambda_n}\right)\le\frac{1-e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i}. \tag{9}$$

Proof. [10] showed that

$$\begin{aligned}d_{x_0}\left(S_n,\wp_{\lambda_n}\right)&\le\sum_{i=1}^{n}\sum_{x=1}^{\infty}xq_ip_{X_i}(x)\sup_x\left|f_{x_0}(x+1)\right|\\ &\le\sum_{i=1}^{n}\sum_{x=1}^{\infty}xq_ip_{X_i}(x)\min\left\{\frac{1}{x_0},\frac{1-e^{-\lambda_n}}{\lambda_n}\right\} \quad \text{(by Lemma 2.1)}\\ &=\min\left\{\frac{1}{x_0}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i},\frac{1-e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n}\frac{r_iq_i^2}{p_i}\right\},\end{aligned}$$

which gives the inequalities (8) and (9). □

III. RESULT

The main point of this study is to determine a non-uniform bound for the point metric $d_{x_0}(S_M,\wp_\lambda)$. The following theorem gives the desired result.

Theorem 1. *For* $x_0 \in \mathbb{N}$, *then we have the following inequality.*

$$d_{x_0}(S_M, \wp_\lambda) \le \min\left\{\frac{1}{x_0} E\left(\sum_{i=1}^{M} \frac{r_i q_i^2}{p_i}\right), E\left(\frac{1-e^{-\lambda_M}}{\lambda_M}\sum_{i=1}^{M}\frac{r_i q_i^2}{p_i}\right)\right\} + \min\left\{\frac{1}{x_0}, \frac{1-e^{-\lambda}}{\lambda}\right\} E|\lambda_M - \lambda|. \quad (10)$$

Proof. It follows the fact that

$$\begin{aligned}
d_{x_0}(S_M, \wp_\lambda) &\le d_{x_0}(S_M, \wp_{\lambda_M}) + d_{x_0}(\wp_{\lambda_M}, \wp_\lambda)\\
&= \sum_{n=1}^{\infty} P(M=n) d_{x_0}(S_n, \wp_{\lambda_n}) + d_{x_0}(\wp_{\lambda_M}, \wp_\lambda)\\
&\le \min\left\{\frac{1}{x_0}\sum_{n=1}^{\infty} P(M=n)\sum_{i=1}^{n}\frac{r_i q_i^2}{p_i}, \sum_{n=1}^{\infty} P(M=n)\frac{1-e^{-\lambda_n}}{\lambda_n}\sum_{i=1}^{n}\frac{r_i q_i^2}{p_i}\right\} + d_{x_0}(\wp_{\lambda_M}, \wp_\lambda) \quad \text{(by Lemma 3)}\\
&= \min\left\{\frac{1}{x_0} E\left(\sum_{i=1}^{M}\frac{r_i q_i^2}{p_i}\right), E\left(\frac{1-e^{-\lambda_M}}{\lambda_M}\sum_{i=1}^{M}\frac{r_i q_i^2}{p_i}\right)\right\} + d_{x_0}(\wp_{\lambda_M}, \wp_\lambda) \quad (11)
\end{aligned}$$

In the next step, we have to bound $d_{x_0}(\wp_{\lambda_M}, \wp_\lambda)$. Appling Stein's equation in (3), it follows that

$$\begin{aligned}
d_{x_0}(\wp_{\lambda_M}, \wp_\lambda) &= \left|E\left[\lambda f_{x_0}(\wp_{\lambda_M}+1) - \wp_{\lambda_M} f_{x_0}(\wp_{\lambda_M})\right]\right|\\
&= \left|E\left[\lambda f_{x_0}(\wp_{\lambda_M}+1)\right] - E\left[\wp_{\lambda_M} f_{x_0}(\wp_{\lambda_M})\right]\right|\\
&= \left|E\left[\lambda f_{x_0}(\wp_{\lambda_M}+1)\right] - E\left\{E\left[\left(\wp_{\lambda_M} f_{x_0}(\wp_{\lambda_M})\right)|\lambda_M\right]\right\}\right|.\\
&= \left|E\left[\lambda f_{x_0}(\wp_{\lambda_M}+1)\right] - E\left\{E\left[\left(\lambda_M f_{x_0}(\wp_{\lambda_M}+1)\right)|\lambda_M\right]\right\}\right| \quad \text{(by Lemma 2)}\\
&= \left|E\left[\lambda f_{x_0}(\wp_{\lambda_M}+1)\right] - E\left[\lambda_M f_{x_0}(\wp_{\lambda_M}+1)\right]\right|\\
&= \left|E\left[(\lambda-\lambda_M) f_{x_0}(\wp_{\lambda_M}+1)\right]\right|\\
&\le E\left|(\lambda-\lambda_M) f_{x_0}(\wp_{\lambda_M}+1)\right|\\
&\le \sup_{x\ge 1}\left|f_{x_0}(x)\right| E|\lambda-\lambda_M|\\
&\le \min\left\{\frac{1}{x_0}, \frac{1-e^{-\lambda}}{\lambda}\right\} E|\lambda_M-\lambda| \quad \text{(by Lemma 1).} \quad (12)
\end{aligned}$$

Combining (11) and (12), the inequality (10) is obtained. □

If $r_1 = r_2 = \dots = 1$, then the result in (10) is a Poisson approximation to the distribution of random sums of independent geometric random variables with respect to $\lambda_M = \sum_{i=1}^{M} q_i$. The following corollary presents the result mentioned.

Corollary 1. *For* $x_0 \in \mathbb{N}$, *if* $r_1 = r_2 = \dots = 1$ *and* $\lambda_M = \sum_{i=1}^{M} q_i$, *then the following inequality holds:*

$$d_{x_0}(S_M, \wp_\lambda) \le \min\left\{\frac{1}{x_0} E\left(\sum_{i=1}^{M}\frac{q_i^2}{p_i}\right), E\left(\frac{1-e^{-\lambda_M}}{\lambda_M}\sum_{i=1}^{M}\frac{q_i^2}{p_i}\right)\right\} + \min\left\{\frac{1}{x_0}, \frac{1-e^{-\lambda}}{\lambda}\right\} E|\lambda_M-\lambda|. \quad (13)$$

If X_i's are identically distributed, then the following corollary is a consequence of Theorem 1.

Corollary 2. *For* $x_0 \in \mathbb{N}$, *if* $r_1 = r_2 = \dots = r$ *and* $p_1 = p_2 = \dots = p$, *then we have the following.*

$$d_{x_0}(S_M, \wp_\lambda) \le \frac{q}{p}\min\left\{\frac{rqE(M)}{x_0}, 1-e^{-rqE(M)}\right\} + \min\left\{\frac{rq}{x_0}, \frac{1-e^{-rqE(M)}}{E(M)}\right\} E|M-E(M)|. \quad (14)$$

IV. EXAMPLES

This section gives three examples to illustrate applications of the result when all X_i's are identically distributed.

Example 1. Let k be a fixed positive integer and M a discrete random variable with the probability function

$$p_M(n) = \begin{cases} \frac{1}{3}, & n = k, k+1, k+2,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, we have $E(M)=k+1$ and $E|M-E(M)|=\frac{2}{3}$. In the case of $r_1=r_2=\cdots=r$ and $p_1=p_2=\cdots=p$, we have $\lambda=(k+1)rq$ and for $x_0\in\mathbb{N}$,

$$d_{x_0}(S_M,\wp_\lambda)\le\frac{q}{p}\min\left\{\frac{(k+1)rq}{x_0},1-e^{-(k+1)rq}\right\}+\frac{2}{3}\min\left\{\frac{rq}{x_0},\frac{1-e^{-(k+1)rq}}{k+1}\right\}.$$

Example 2. Let M be a discrete random variable with the probability function

$$p_M(n)=\begin{cases}\frac{1}{2}\left(\frac{2}{3}\right)^n, & n=1,2,...,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, $E(M)=3$ and $E|M-E(M)|=\frac{37}{27}$. If $r_1=r_2=\cdots=r$ and $p_1=p_2=\cdots=p$, then $\lambda=3rq$, and for $x_0\in\mathbb{N}$, we have

$$d_{x_0}(S_M,\wp_\lambda)\le\frac{q}{p}\min\left\{\frac{3rq}{x_0},1-e^{-3rq}\right\}+\frac{37}{27}\min\left\{\frac{rq}{x_0},\frac{1-e^{-3rq}}{3}\right\}.$$

Example 3. Let M be a discrete random variable with the probability function

$$p_M(n)=\begin{cases}\frac{n-1}{2^n}, & n=2,3,...,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, $E(M)=4$ and $E|M-E(M)|=\frac{3}{2}$. If $r_1=r_2=\cdots=r$ and $p_1=p_2=\cdots=p$, then $\lambda=4rq$, and for $x_0\in\mathbb{N}$, we have

$$d_{x_0}(S_M,\wp_\lambda)\le\frac{q}{p}\min\left\{\frac{4rq}{x_0},1-e^{-4rq}\right\}+\frac{3}{2}\min\left\{\frac{rq}{x_0},\frac{1-e^{-4rq}}{4}\right\}.$$

V. CONCLUSION

In this study, the Stein-Chen method was used to determine a non-uniform bound for the point metric between the distribution of random sums of independent negative binomial random variables and a Poisson distribution with mean $\lambda=E\left(\sum_{i=1}^{M}r_iq_i\right)$, where r_i and $p_i=1-q_i$ are parameters of each negative binomial distribution and M is a positive integer-valued random variable, which is independent of all negative binomial random variables. In view of this bound, the Poisson distribution with this mean can be used as an approximation of the distribution of the random sums mentioned above if all q_i are small or λ is small.

REFERENCES

[1] A. D. Barbour, L. Holst and S. Janson, S. "Poisson Approximation" (Oxford Studies in Probability 2), Clarendon Press, Oxford, 1992.

[2] L. H. Y. Chen, "Poisson approximation for dependent trials". Annals of Probability, vol. 3, pp. 534-545, 1975.

[3] S. Kongudomthrap and N. Chaidee, N. 2012. "Bounds in Poisson approximation of random sums of Bernoulli random variables", Journal of Mathematics Research, vol. 4, pp. 29-35. 2012.

[4] R. Kun and K. Teerapabolarn, "A pointwise Poisson approximation by w-functions", Applied Mathematical Sciences, vol. 6, pp. 5029-5037, 2012.

[5] C. M. Stein, "A bound for the error in normal approximation to the distribution of a sum of dependent random variables", Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, California, Vol. 19-22, pp. 583-602, 1972.

[6] K. Teerapabolarn, "A new bound on Poisson approximation for random sums of Bernoulli random variables", International Journal of Pure and Applied Mathematics, vol. 89, pp. 141-146, 2013.

[7] K. Teerapabolarn, "A pointwise Poisson approximation for random sums of geometric random variables", International Journal of Pure and Applied Mathematics, vol. 89, pp. 353-356, 2013.

[8] K. Teerapabolarn, "A non-uniform bound on Poisson approximation for random sums of geometric random variables", International Journal of Pure and Applied Mathematics, vol. 90, pp. 5-9, 2014.

[9] K. Teerapabolarn, "Poisson approximation for random sums of independent negative binomial random variables", International Journal of Pure and Applied Mathematics, vol. 93, pp. 783-787, 2014.

[10] K. Teerapabolarn, "A non-uniform bound on Poisson approximation for a sum of negative binomial random variables", Songklanakarin Journal of Science and Technology, vol. 3, pp. 355-358, 2017.

[11] N. Yannaros, "Poisson approximation for random sums of Bernoulli random variables", Statistics & Probability Letters, vol. 11, pp. 161-165, 1991.