On Relative Fix Points of Functions of ClassII

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Abstract : *We introduce the idea of relative iteration of four functions and using this, extend a theorem of Lahiri and Banerjee for a special class of functions involving exact factor order.*

Keywords **:** *Relative fix points, exact factor order, class II functions.*

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I. INTRODUCTION

A single valued function $f(z)$ of the complex variable z is said to belong to (i) class I if $f(z)$ is entire transcendental, (ii) class II if it is regular in the complex plane punctured at $a, b(a \neq b)$ and has an essential singularitiy at b, and a singularity at a and if $f(z)$ does not assume the values a and b anywhere in the complex plane except possible at the point a.

We may take $a = 0$ and $b = \infty$.

The iterated functions $f_n(z)$ of $f(z)$ are defined inductively by

$$f_0(z) = z \text{ and } f_{n+1}(z) = f\big(f_n(z)\big) \text{ for } n = 0,1,2,\ldots .$$

A point α is called a fix point of $f(z)$ of order n if α is a solution of $f_n(z) = z$ and called a fix point of exact order n if α is a solution of $f_n(z) = z$ but not a solution of $f_k(z) = z, k = 1,2,\ldots,n-1$.

In [1] Baker proved the following theorem.

Theorem 1.1 *If $f(z)$ belongs to class I, then $f(z)$ has fix points of exact order n, except for at most one value of n.*

Bhattacharyya [4] extended it to the functions belonging to class II as follows.

Theorem 1.2 *If $f(z)$ belongs to class II, then $f(z)$ has infinitely many fix points of exact order n, for every positive integer n.*

In [7], Lahiri and Banerjee introduced the concept of relative iteration defined as follows .

Let $f(z)$ and $g(z)$ be functions of the complex variable z.

Let

$$f_1(z) = f(z)$$

$$f_2(z) = f(g(z)) = f(g_1(z))$$

$$f_3(z) = f(g(f(z))) = f(g(f_1(z)))$$

$$\vdots$$

$$f_n(z) = f(g(f(g\ldots(f(z) \text{ or } g(z) \text{ according as } n \text{ is odd or even})\ldots)))$$

$$= f(g_{n-1}(z)) = f(g(f_{n-2}(z)))$$

and so

$$g_1(z) = g(z)$$

$$g_2(z) = g(f(z)) = g(f_1(z))$$
$$g_3(z) = g(f_2(z)) = g(f(g_1(z)))$$
$$\vdots$$
$$g_n(z) = g(f_{n-1}(z)) = g(f(g_{n-2}(z))).$$

Clearly all $f_n(z)$ and $g_n(z)$ are functions in class II, if $f(z)$ and $g(z)$ are so.

A point α is called a fix point of $f(z)$ of order n with respect to $g(z)$, if $f_n(\alpha) = \alpha$ and a fix point of exact order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha, k = 1,2,3,\ldots, n-1$. Such points α are also called relative fix points.

Theorem 1.3 ([7]) *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has infinitely many relative fix points of exact order n for every positive integer n, provided $\frac{T(r,g_n)}{(T(r,f_n)}$ is bounded.*

Recently Banerjee and Mandal [2] introducing the idea of relative fix point of exact factor order n, proved the result of Lahiri and Banerjee [7] for exact factor order.

A point α is called a relative fix point of $f(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$ and $g_k(\alpha) \neq \alpha$ for all divisors k $(< n)$ of n.

Banerjee and Mandal [2] proved the following theorem.

Theorem 1.4 ([2]) *If $f(z)$ and $g(z)$ belong to class II, then $f(z)$ has infinitely many relative fix points of exact factor order n for every positive integer n, provided $\frac{T(r,f_{n-1})}{T(r,f_n)}$ is bounded.*

In 2015, Banerjee and Mandal [3] introduced the idea of relative iteration of three functions as follows.

Let $f(z), g(z)$ and $h(z)$ be three functions of complex variable z and $m \geq 2$ be any fixed positive integer.

Let

$$f_1(z) = f(z)$$
$$f_2(z) = f(g(z)) = f(g_1(z))$$
$$f_3(z) = f(g(h(z))) = f(g(h_1(z))) = f(g_2(z))$$
$$f_4(z) = f(g(h(f(z)))) = f(g(h_2(z))) = f(g_3(z))$$
$$\vdots$$
$$f_n(z) = f(g(h(f..(f(z) \text{ or } g(z) \text{ or } h(z) \text{ according as } n = 3m-2 \text{ or } 3m-1$$
$$\text{or } 3m)\ldots)))$$
$$= f(g_{n-1}(z)) = f(g(h_{n-2}(z))).$$

Similarly,

$$g_1(z) = g(z)$$
$$g_2(z) = g(h(z)) = g(h_1(z))$$
$$g_3(z) = g(h(f(z))) = g(h(f_1(z))) = g(h_2(z))$$
$$g_4(z) = g(h(f(g(z)))) = g(h(f_2(z))) = g(h_3(z))$$
$$\vdots$$
$$g_n(z) = g(h(f(g\ldots(g(z) \text{ or } h(z) \text{ or } f(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \text{ or } 3m)\ldots)))$$
$$= g(h_{n-1}(z)) = g(h(f_{n-2}(z)))$$

and so are

$$h_1(z) = h(z)$$
$$h_2(z) = h(f(z)) = h(f_1(z))$$
$$h_3(z) = h(f(g(z))) = h(f(g_1(z))) = h(f_2(z))$$
$$h_4(z) = h(f(g(h(z)))) = h(f(g_2(z))) = h(f_3(z))$$
$$\vdots$$
$$h_n(z) = h(f(g(h\ldots(h(z) \text{ or } f(z) \text{ or } g(z) \text{ according as } n = 3m-2 \text{ or } 3m-1$$
$$\text{or } 3m)\ldots)))$$
$$= h(f_{n-1}(z)) = h(f(g_{n-2}(z))).$$

Clearly all $f_n(z), g_n(z)$ and $h_n(z)$ are functions in class II, if $f(z), g(z)$ and $h(z)$ are so.

A point α is called a fix point of $f(z)$ of order n with respect to $g(z)$ and $h(z)$, if $f_n(\alpha) = \alpha$ and a fix point $f(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha$, $g_k(\alpha) \neq \alpha$ and $h_k(\alpha) \neq \alpha$ for all divisors k $(< n)$ of n.

Theorem 1.5 ([3]) *If $f(z)$, $g(z)$ and $h(z)$ belong to class II, then $f(z)$ has infinitely many fix points of exact factor order n for every positive integer n (≥ 3) provided $\frac{T(r,g_n)}{T(r,f_n)}$ and $\frac{T(r,h_n)}{T(r,f_n)}$ are bounded.*

In this paper we extend the definition of relative iterations for four functions and proved the result of Banerjee and Mandal[3] in that direction.

Let $f(z)$, $g(z)$, $h(z)$ and $w(z)$ be four functions of complex variable z and $m \geq 3$ be any fixed positive integer. We set

$$f_1(z) = f(z)$$
$$f_2(z) = f(g(z)) = f(g_1(z))$$
$$f_3(z) = f(g(h(z))) = f(g(h_1(z))) = f(g_2(z))$$
$$f_4(z) = f(g(h(w(z)))) = f(g(h(w_1(z)))) = f(g(h_2(z))) = f(g_3(z))$$
$$f_5(z) = f(g(h(w(f(z))))) = f(g(h(w_2(z)))) = f(g(h_3(z))) = f(g_4(z))$$
$$\vdots$$
$$f_n(z) = f(g(h(w(f\ldots(f(z) \text{ or } g(z) \text{ or } h(z) \text{ or } w(z) \text{ according as } n = 4m-3$$
$$\text{or } 4m-2 \text{ or } 4m-1 \text{ or } 4m)\ldots))))$$
$$= f(g_{n-1}(z)) = f(g(h_{n-2}(z))) = f(g(h(w_{n-3}(z))))$$

and

$$g_1(z) = g(z)$$
$$g_2(z) = g(h(z)) = g(h_1(z))$$
$$g_3(z) = g(h(w(z))) = g(h(w_1(z))) = g(h_2(z))$$
$$g_4(z) = g(h(w(f(z)))) = g(h(w(f_1(z)))) = g(h(w_2(z))) = g(h_3(z))$$
$$g_5(z) = g(h(w(f(g(z))))) = g(h(w(f_2(z)))) = g(h(w_3(z))) = g(h_4(z))$$
$$\vdots$$

$$g_n(z) = g(h(w(f(g\ldots(g(z) \text{ or } h(z) \text{ or } w(z) \text{ or } f(z) \text{ according as } n = 4m-3 \text{ or } 4m-2 \text{ or } 4m-1 \text{ or } 4m)\ldots))))$$
$$= g(h_{n-1}(z)) = g\left(h(w_{n-2}(z))\right) = g\left(h\left(w(f_{n-3}(z))\right)\right).$$

Similarly,

$$h_1(z) = h(z)$$
$$h_2(z) = h(w(z)) = h(w_1(z))$$
$$h_3(z) = h(w(f(z))) = h(w(f_1(z))) = h(w_2(z))$$
$$h_4(z) = h(w(f(g(z)))) = h(w(f(g_1(z)))) = h(w(f_2(z))) = h(w_3(z))$$
$$h_5(z) = h(w(f(g(h(z))))) = h(w(f(g_2(z)))) = h(w(f_3(z))) = h(w_4(z))$$
$$\vdots$$
$$h_n(z) = h(w(f(g(h\ldots(h(z) \text{ or } w(z) \text{ or } f(z) \text{ or } g(z) \text{ according as } n = 4m-3 \text{ or } 4m-2 \text{ or } 4m-1 \text{ or } 4m)\ldots)))$$
$$= h(w_{n-1}(z)) = h(w(f_{n-2}(z))) = h(w(f(g_{n-3}(z))))$$

and so are

$$w_1(z) = w(z)$$
$$w_2(z) = w(f(z)) = w(f_1(z))$$
$$w_3(z) = w(f(g(z))) = w(f(g_1(z))) = w(f_2(z))$$
$$w_4(z) = w(f(g(h(z)))) = w(f(g(h_1(z)))) = w(f(g_2(z))) = w(f_3(z))$$
$$w_5(z) = w(f(g(h(w(z))))) = w(f(g(h_2(z)))) = w(f(g_3(z))) = w(f_4(z))$$
$$\vdots$$
$$w_n(z) = w(f(g(h(w\ldots(w(z) \text{ or } f(z) \text{ or } g(z) \text{ or } h(z) \text{ according as } n = 4m-3 \text{ or } 4m-2 \text{ or } 4m-1 \text{ or } 4m)\ldots))))$$
$$= w(f_{n-1}(z)) = w(f(g_{n-2}(z))) = w(f(g(h_{n-3}(z)))) .$$

Clearly $f_n(z), g_n(z), h_n(z)$ and $w_n(z)$ are functions in class II, if $f(z), g(z), h(z)$ and $w(z)$ are so.

The following definition is now introduced.

Definition 1.6 *A point α is called a fix point of $f(z)$ of order n with respect to $g(z), h(z)$ and $w(z)$,* if *$f_n(\alpha) = \alpha$ and a fix point of $f(z)$ of exact factor order n if $f_n(\alpha) = \alpha$ but $f_k(\alpha) \neq \alpha, g_k(\alpha) \neq \alpha, h_k(\alpha) \neq \alpha$ and $w_k(\alpha) \neq \alpha$, for all divisors $k(< n)$ of n.*

Example 1.7 *Let $f(z) = z + 1, g(z) = \frac{1}{z-1}, h(z) = 2z, w(z) = \frac{1}{z}$.*

$$\textit{Clearly } f_4(z) = \frac{2}{2-z}. \textit{ Here } z = 1 \pm i \textit{ are fix points of } f(z) \textit{ of exact factor order } 4.$$

Let $f(z)$ be meromorphic in $r_0 \leq |z| < \infty, r_0 > 0$. Here we use the following notations [5]:

$n(t, a, f)$ = the number of roots of $f(z) = a$ in $r_0 < |z| \leq t$, counted according to multiplicity,

$N(r,a,f) = \frac{\int_{r_0}^{r} n(t,a,f)}{t}\, dt,$

$n(t,\infty,f) = n(t,f)$ =number of poles of $f(z)$ in $r_0 < |z| \le t$, counted

with due to multiplicity

$N(t,\infty,f) = N(t,f),$

$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} log^+|f(re^{i\theta})|d\theta$ and

$m(r,a,f) = \frac{1}{2\pi}\int_0^{2\pi} log^+|\frac{1}{f(re^{i\theta}-a)}|d\theta$

With this notations Jensen's formula can be written as ,

$$m(r,f) + N(r,f) = m(r,\frac{1}{f}) + N(r,\frac{1}{f}) + O(log\, r).$$

Since $m(r,f) + N(r,f) = T(r,f)$, so we have

$T(r,f) = T(r,\frac{1}{f}) + O(log\, r).$

Also first fundamental theorem takes the form

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(log\, r), \quad (1.1)$$

where $r_0 \le |z| < \infty, r_0 > 0$.

Suppose that $f(z)$ is non-constant. Let $a_1, a_2, \ldots, a_q;\ q \ge 2$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_v| \ge \delta\ for\ 1 \le \mu \le v \le q$. Then

$$m(r,f) + \sum_{v=1}^{q} m(r,a_v,f) \le 2T(r,f) - N_1(r) + S(r), \quad (1.2)$$

where

$$N_1(r) = N(r,\frac{1}{f'}) + 2N(r,f) - N(r,f'),$$

and

$$S(r) = m(r,\frac{f'}{f}) + \sum_{v=1}^{q} m(r,\frac{f'}{f-a_v}) + O(log\, r).$$

Adding $N(r,f) + \sum_v^q N(r,a_v,f)$ to both sides of (1.2) and from (1.1), we get

$$(q-1)T(r,f) \le \overline{N}(r,f) + \sum_{v=1}^{q} \overline{N}(r,a_v,f) + S_1(r), \quad (1.3)$$

where $S_1(r) = O(logT(r,f))$ and $\overline{N}$ corresponds to distinct roots.

Again since f_n has an essential singularity at ∞, we have $\frac{logr}{T(r,f_n)} \to 0$ as $r \to \infty$ [5].

II. LEMMAS

The following lemmas will be needed to prove our main result.

Lemma 2.1 *If n is any positive integer and f,g,h,w are functions in class II, then for any $r_0 > 0$ and a positive constant M_1, we have*

$$\frac{T(r,f_{n+p})}{T(r,f_n)} > M_1\ or\ \frac{T(r,g_{n+p})}{T(r,f_n)} > M_1\ or\ \frac{T(r,h_{n+p})}{T(r,f_n)} > M_1\ or\ \frac{T(r,w_{n+p})}{T(r,f_n)} > M_1$$

according as $p = 4m$ or $4m-1$ or $4m-2$ or $4m-3;\ m \in \mathbb{N}$ for all large r, except a set of r intervals of total finite length.

Proof. CASE I. When $p = 4m, m \in \mathbb{N}$.

In this case we consider the equation $f_{n+p}(z) = a$, where $a \neq 0, \infty$, i.e, $f_p(f_n(z)) = a$. This is equivalent to $f_p(u_i) = a$ and $f_n(z) = u_i, (i = 1,2,\dots)$. f_p being transcendental, $f_p(u_i) = a$ has infinitely many roots for every complex number a with two exceptions $a = 0, \infty$.

Now from (1.1)

$$\begin{aligned} T(r, f_{n+p}) &= m(r, a, f_{n+p}) + N(r, a, f_{n+p}) + O(\log r) \\ &\geq \bar{N}(r, a, f_{n+p}) + O(\log r) \\ &\geq \sum_{i=1}^{M} \bar{N}(r, u_i, f_n) \end{aligned}$$

for a fixed $M(> 4)$.

Taking $a_v = u_i, f = f_n$ and $q = M$ in (1.3), we have

$$\sum_{i=1}^{M} \bar{N}(r, u_i, f_n) \geq (M-1)T(r, f_n) - \bar{N}(r, f_n) - S_1(r).$$

Since for large r

$$S_1(r) \leq T(r, f_n),$$

so

$$\begin{aligned} \sum_{i=1}^{M} \bar{N}(r, u_i, f_n) &\geq (M-3)T(r, f_n) \\ &> (M-4)T(r, f_n). \end{aligned} \quad (2.1)$$

Therefore

$$T(r, f_{n+p}) > M_1 T(r, f_n), \text{ where } M_1 = M - 4$$

outside a set of r intervals of total finite length.

CASE II. When $p = 4m - 1, m \in \mathbb{N}$.

In this case we consider the equation $g_{n+p}(z) = a$, where $a \neq 0, \infty$, i.e, $g_p(f_n(z)) = a$. This is equivalent to $g_p(u_i') = a$ and $f_n(z) = u_i', (i = 1,2,\dots)$.

As in case I, we have

$$\begin{aligned} T(r, g_{n+p}) &= m(r, a, g_{n+p}) + N(r, a, g_{n+p}) + O(\log r) \\ &\geq \bar{N}(r, a, g_{n+p}) + O(\log r) \\ &\geq \sum_{i=1}^{M} \bar{N}(r, u_i', f_n) \end{aligned}$$

for a fixed $M(> 4)$.

Therefore

$$T(r, g_{n+p}) > M_1 T(r, f_n), \text{ where } M_1 = M - 4$$

outside a set of r intervals of total finite length.

CASE III. When $p = 4m - 2, m \in \mathbb{N}$.

In this case we consider the equation $h_{n+p}(z) = a$, where $a \neq 0, \infty$, i.e, $h_p\left(f_n(z)\right) = a$. This is equivalent to $h_p(u_i^{''}) = a$ and $f_n(z) = u_i^{''}, (i = 1,2,\ldots)$

and finally we get

$$T(r, h_{n+p}) > M_1 T(r, f_n), \text{ where } M_1 = M - 4$$

outside a set of r intervals of total finite length.

CASE IV. When $p = 4m - 3, m \in \mathbb{N}$.

In this case we consider the equation $w_{n+p}(z) = a$, where $a \neq 0, \infty$, i.e, $w_p\left(f_n(z)\right) = a$. This is equivalent to $w_p(u_i^{'''}) = a$ and $f_n(z) = u_i^{'''}, (i = 1,2,\ldots)$.

In this case we have

$$T(r, w_{n+p}) > M_1 T(r, f_n), \text{ where } M_1 = M - 4$$

outside a set of r intervals of total finite length.

This proves the lemma.

If we interchange f, g, h, w in cyclic order, then we obtain the following three lemmas.

Lemma 2.2 *If n is any positive integer and f, g, h, w are functions in class II, then for any $r_0 > 0$ and a positive constant M_1 we have*

$$\frac{T(r,g_{n+p})}{T(r,g_n)} > M_1 \text{ or } \frac{T(r,h_{n+p})}{T(r,g_n)} > M_1 \text{ or } \frac{T(r,w_{n+p})}{T(r,g_n)} > M_1 \text{ or } \frac{T(r,f_{n+p})}{T(r,g_n)} > M_1$$

according as $p = 4m$ or $4m - 1$ or $4m - 2$ or $4m - 3$; $m \in \mathbb{N}$ for all large r, except a set of r intervals of total finite length.

Lemma 2.3 *If n is any positive integer and f, g, h, w are functions in class II, then for any $r_0 > 0$ and a positive constant M_1 we have*

$$\frac{T(r,h_{n+p})}{T(r,h_n)} > M_1 \text{ or } \frac{T(r,w_{n+p})}{T(r,h_n)} > M_1 \text{ or } \frac{T(r,f_{n+p})}{T(r,h_n)} > M_1 \text{ or } \frac{T(r,g_{n+p})}{T(r,h_n)} > M_1$$

according as $p = 4m$ or $4m - 1$ or $4m - 2$ or $4m - 3$; $m \in \mathbb{N}$ for all large r, except a set of r intervals of total finite length.

Lemma 2.4 *If n is any positive integer and f, g, h, w are functions in class II, then for any $r_0 > 0$ and a positive constant M_1 we have*

$$\frac{T(r,w_{n+p})}{T(r,w_n)} > M_1 \text{ or } \frac{T(r,f_{n+p})}{T(r,w_n)} > M_1 \text{ or } \frac{T(r,g_{n+p})}{T(r,w_n)} > M_1 \text{ or } \frac{T(r,h_{n+p})}{T(r,w_n)} > M_1$$

according as $p = 4m$ or $4m - 1$ or $4m - 2$ or $4m - 3$; $m \in \mathbb{N}$ for all large r, except a set of r intervals of total finite length.

III. MAIN RESULT

Our main result is the following theorem.

Theorem 3.1 *If $f(z), g(z), h(z)$ and $w(z)$ belong to class II, then $f(z)$ has infinitely many fix* points *of exact factor order n for every positive integer $n (\geq 4)$ provided*

$$\frac{T(r,g_n)}{T(r,f_n)}, \frac{T(r,h_n)}{T(r,f_n)} \text{ and } \frac{T(r,w_n)}{T(r,f_n)} \text{ are bounded.}$$

Proof. We assume that $f(z)$ has only a finite number of fix points of exact factor order n.

We consider the function

$$\varphi(z) = \frac{f_n(z)}{z} \quad , r_0 < |z| < \infty.$$

Then

$$T(r,\varphi) = T(r,f_n) + O(\log r). \tag{3.1}$$

Taking $q = 2, a_1 = 0, a_2 = 1$ in (1.3) we have for φ

$$T(r,\varphi) \leq \bar{N}(r,\infty,\varphi) + \bar{N}(r,0,\varphi) + \bar{N}(r,1,\varphi) + S_1(r,\varphi),$$

where $S_1(r,\varphi) = O(\log T(r,\varphi))$ outside a set of r intervals of finite length [6].

We have

$$\bar{N}(r,0,\varphi) = \int_{r_0}^{r} \frac{\bar{n}(t,0,\varphi)}{t} dt$$

where $\bar{n}(t,0,\varphi)$ is the number of roots of $\varphi(z) = 0$ in $r_0 < |z| \leq t$, each multiple root taken once at a time. The distinct roots of $\varphi(z) = 0$ in $r_0 < |z| \leq t$ are the roots of $f_n(z) = 0$ in $r_0 < |z| \leq t$. Now $\bar{n}(t,0,\varphi) = 0$, since $f_n(z)$ has a singularity at $z = 0$, an essential singularity at $z = \infty$, and $f_n(z) \neq 0, \infty$. So $\bar{N}(r,0,\varphi) = 0$. By similar argument $\bar{N}(r,\infty,\varphi) = 0$. So

$$T(r,\varphi) \leq \bar{N}(r,1,\varphi) + S_1(r,\varphi). \tag{3.2}$$

We now calculate $\bar{N}(r,1,\varphi)$. If $\varphi(z) = 1$, then $f_n(z) = z$. Now we consider the following four cases :

CASE I. When $n = 4m, m \in \mathbb{N}$. We have for all large r

$$\begin{aligned}\bar{N}(r,1,\varphi) &= \bar{N}(r,0,f_n - z)\\ &\leq \sum_{j/n, j=1}^{n-2} [\bar{N}(r,0,f_j - z) + \bar{N}(r,0,g_j - z) + \bar{N}(r,0,h_j - z)\\ &\qquad + \bar{N}(r,0,w_j - z)] + O(\log r),\end{aligned}$$

the term $O(\log r)$ arises due to the assumption that $f(z)$ has only a finite number of relative fix points of exact factor order n

$$\leq \sum_{j/n, j=1}^{n-2} [T(r,f_j - z) + O(\log r) + T(r,g_j - z) + O(\log r) + T(r,h_j - z) + O(\log r) + T(r,w_j - z) + O(\log r)] + O(\log r)$$

$$= \sum_{j/n, j=1}^{n-2} [T(r,f_j) + T(r,g_j) + T(r,h_j) + T(r,w_j)] + O(\log r)$$

$$\begin{aligned}= \{&T(r,f_{j_1}) + T(r,f_{j_5}) + \ldots + T(r,f_{j_{4p-3}}) + T(r,f_{j_2}) + T(r,f_{j_6}) + \ldots + T(r,f_{j_{4q-2}}) + T(r,f_{j_3})\\ &+ T(r,f_{j_7}) + \ldots + T(r,f_{j_{4r-1}}) + T(r,f_{j_4}) + T(r,f_{j_8}) +\\ &\ldots + T(r,f_{j_{4s}})\} + \{T(r,g_{j_1}) + T(r,g_{j_5}) + \ldots + T(r,g_{j_{4p-3}}) + T(r,g_{j_2})\\ &+ T(r,g_{j_6}) + \ldots + T(r,g_{j_{4q-2}}) + T(r,g_{j_3}) + T(r,g_{j_7}) + \ldots + T(r,g_{j_{4r-1}}) + T(r,g_{j_4})\\ &+ T(r,g_{j_8}) + \ldots + T(r,g_{j_{4s}})\} + \{T(r,h_{j_1}) + T(r,h_{j_5}) + \ldots + T(r,h_{j_{4p-3}}) + T(r,h_{j_2})\\ &+ T(r,h_{j_6}) + \cdots + T(r,h_{j_{4q-2}}) + T(r,h_{j_3}) + T(r,h_{j_7}) + \ldots + T(r,h_{j_{4r-1}}) + T(r,h_{j_4})\\ &+ T(r,h_{j_8}) + \cdots + T(r,h_{j_{4s}})\} + \{T(r,w_{j_1}) + T(r,w_{j_5}) + \cdots + T(r,w_{j_{4p-3}}) + T(r,w_{j_2})\\ &+ T(r,w_{j_6}) + \cdots + T(r,w_{j_{4q-2}}) + T(r,w_{j_3}) + T(r,w_{j_7}) + \ldots + T(r,w_{j_{4r-1}}) +\\ &T(r,w_{j_4}) + T(r,w_{j_8}) + \ldots + T(r,w_{j_{4s}})\} + O(\log r),\end{aligned}$$

where $j_1, j_5, \ldots, j_{4p-3}; j_2, j_6, \ldots, j_{4q-2}; j_3, j_7, \ldots, j_{4r-1}; j_4, j_8, \ldots, j_{4s}$ are divisors of $n = 4m$ and are strictly less than n and are of the form $4p-3, 4q-2, 4r-1$ and $4s (p, q, r, s \in \mathbb{N})$.

$$=T(r,f_n)[\frac{T(r,f_{j_4})}{T(r,f_n)}+\frac{T(r,f_{j_8})}{T(r,f_n)}+\ldots+\frac{T(r,f_{j_{4s}})}{T(r,f_n)}+\frac{T(r,g_{j_1})}{T(r,f_n)}+\frac{T(r,g_{j_5})}{T(r,f_n)}+\ldots$$

$$+\frac{T(r,g_{j_{4p-3}})}{T(r,f_n)}+\frac{T(r,h_{j_2})}{T(r,f_n)}+\frac{T(r,h_{j_6})}{T(r,f_n)}+\ldots+\frac{T(r,h_{j_{4q-2}})}{T(r,f_n)}+\frac{T(r,w_{j_3})}{T(r,f_n)}$$

$$+\frac{T(r,w_{j_7})}{T(r,f_n)}+\ldots+\frac{T(r,w_{j_{4r-1}})}{T(r,f_n)}]+T(r,g_n)[\frac{T(r,f_{j_3})}{T(r,g_n)}+\frac{T(r,f_{j_7})}{T(r,g_n)}+\ldots+\frac{T(r,f_{j_{4r-1}})}{T(r,g_n)}+\frac{T(r,g_{j_4})}{T(r,g_n)}+\frac{T(r,g_{j_8})}{T(r,g_n)}+\ldots+\frac{T(r,g_{j_{4s}})}{T(r,g_n)}+\frac{T(r,h_{j_1})}{T(r,g_n)}+\frac{T(r,h_{j_5})}{T(r,g_n)}+\ldots+\frac{T(r,h_{j_{4p-3}})}{T(r,g_n)}+\frac{T(r,w_{j_2})}{T(r,g_n)}$$

$$+\frac{T(r,w_{j_6})}{T(r,g_n)}+\ldots+\frac{T(r,w_{j_{4q-2}})}{T(r,g_n)}]+T(r,h_n)[\frac{T(r,f_{j_2})}{T(r,h_n)}+\frac{T(r,f_{j_6})}{T(r,h_n)}+\ldots+\frac{T(r,f_{j_{4q-2}})}{T(r,h_n)}+\frac{T(r,g_{j_3})}{T(r,h_n)}+\frac{T(r,g_{j_7})}{T(r,h_n)}+\ldots+\frac{T(r,g_{j_{4r-1}})}{T(r,h_n)}+\frac{T(r,h_{j_4})}{T(r,h_n)}+\frac{T(r,h_{j_8})}{T(r,h_n)}+\ldots+\frac{T(r,h_{j_{4s}})}{T(r,h_n)}+\frac{T(r,w_{j_1})}{T(r,h_n)}$$

$$+\frac{T(r,w_{j_5})}{T(r,h_n)}+\ldots+\frac{T(r,w_{j_{4p-3}})}{T(r,h_n)}]+T(r,w_n)[\frac{T(r,f_{j_1})}{T(r,w_n)}+\frac{T(r,f_{j_5})}{T(r,w_n)}+\ldots+\frac{T(r,f_{j_{4p-3}})}{T(r,w_n)}+\frac{T(r,g_{j_2})}{T(r,w_n)}+\frac{T(r,g_{j_6})}{T(r,w_n)}+\ldots+\frac{T(r,g_{j_{4q-2}})}{T(r,w_n)}+\frac{T(r,h_{j_3})}{T(r,w_n)}+\frac{T(r,h_{j_7})}{T(r,w_n)}+\ldots+\frac{T(r,h_{j_{4r-1}})}{T(r,w_n)}+\frac{T(r,w_{j_4})}{T(r,w_n)}$$

$$+\frac{T(r,w_{j_8})}{T(r,w_n)}+\ldots+\frac{T(r,w_{j_{4s}})}{T(r,w_n)}]+O(logr).$$

$$<\frac{n-1}{8n}T(r,f_n)+\frac{n-1}{8n}T(r,g_n)+\frac{n-1}{8n}T(r,h_n)+\frac{n-1}{8n}T(r,w_n)+O(log\, r),$$

by Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4.

CASE II. When $n = 4m+1, m \in \mathbb{N}$. In this case we have

$$\bar{N}(r,1,\varphi) = \bar{N}(r,0,f_n-z)$$

$$\le \sum_{j/n,j=1}^{n-4}[\bar{N}(r,0,f_j-z)+\bar{N}(r,0,g_j-z)+\bar{N}(r,0,h_j-z)$$

$$+\bar{N}(r,0,w_j-z)]+O(log\, r),$$

$$\le \sum_{j/n,j=1}^{n-4}[T(r,f_j-z)+O(log\, r)+T(r,g_j-z)+O(log\, r)+T(r,h_j-z)$$

$$+O(log\, r)+T(r,w_j-z)+O(log\, r)]+O(log\, r)$$

$$=\sum_{j/n,j=1}^{n-4}[T(r,f_j)+T(r,g_j)+T(r,h_j)+T(r,w_j)]+O(log\, r)$$

$$=\{T(r,f_{j_1})+T(r,f_{j_5})+\ldots+T(r,f_{j_{4p-3}})+T(r,f_{j_3})+T(r,f_{j_7})+\ldots+T(r,f_{j_{4r-1}})\}$$

$$+\{T(r,g_{j_1})+T(r,g_{j_5})+\ldots+T(r,g_{j_{4p-3}})+T(r,g_{j_3})+T(r,g_{j_7})+\ldots+T(r,g_{j_{4r-1}})\}$$

$$+\{T(r,h_{j_1})+T(r,h_{j_5})+\ldots+T(r,h_{j_{4p-3}})+T(r,h_{j_3})+T(r,h_{j_7})+\ldots+T(r,h_{j_{4r-1}})+$$

$$\{T(r,w_{j_1})+T(r,w_{j_5})+\cdots+T(r,w_{j_{4p-3}})+T(r,w_{j_3})+T(r,w_{j_7})+\ldots+T(r,w_{j_{4r-1}})\}+O(log\, r),$$

where $j_1, j_5, \ldots, j_{4p-3}; j_3, j_7, \ldots, j_{r-1}$ are divisors of $n = 4m+1$ and are strictly less than n and are of the form $4p-3, 4r-1$ $(p, r \in \mathbb{N})$.

$$=T(r,f_n)[\frac{T(r,f_{j_1})}{T(r,f_n)}+\frac{T(r,f_{j_5})}{T(r,f_n)}+\ldots+\frac{T(r,f_{j_{4p-3}})}{T(r,f_n)}+\frac{T(r,g_{j_3})}{T(r,f_n)}+\frac{T(r,g_{j_7})}{T(r,f_n)}+\ldots+\frac{T(r,g_{j_{4r-1}})}{T(r,f_n)}]$$

$$+T(r,g_n)\left[\frac{T(r,g_{j_1})}{T(r,g_n)}+\frac{T(r,g_{j_5})}{T(r,g_n)}+\ldots+\frac{T(r,g_{j_{4p-3}})}{T(r,g_n)}+\frac{T(r,h_{j_3})}{T(r,g_n)}+\frac{T(r,h_{j_7})}{T(r,g_n)}+\ldots+\frac{T(r,h_{j_{4r-1}})}{T(r,g_n)}\right]+T(r,h_n)\left[\frac{T(r,h_{j_1})}{T(r,h_n)}+\right.$$
$$\left.\frac{T(r,h_{j_5})}{T(r,h_n)}+\ldots+\frac{T(r,h_{j_{4p-3}})}{T(r,h_n)}+\frac{T(r,w_{j_3})}{T(r,h_n)}+\frac{T(r,w_{j_7})}{T(r,h_n)}+\ldots+\frac{T(r,w_{j_{4r-1}})}{T(r,h_n)}\right]+T(r,w_n)[\frac{T(r,w_{j_1})}{T(r,w_n)}+$$
$$\frac{T(r,w_{j_5})}{T(r,w_n)}+\ldots+\frac{T(r,w_{j_{4p-3}})}{T(r,w_n)}+\frac{T(r,f_{j_3})}{T(r,w_n)}+\frac{T(r,f_{j_7})}{T(r,w_n)}+\ldots+\frac{T(r,f_{j_{4r-1}})}{T(r,w_n)}]+O(logr).$$

$$<\frac{n-1}{8n}T(r,f_n)+\frac{n-1}{8n}T(r,g_n)+\frac{n-1}{8n}T(r,h_n)+\frac{n-1}{8n}T(r,w_n)+O(log\,r),$$

by Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4.

CASE III. When $n=4m+2, m\in\mathbb{N}$. In this case we have

$$\bar{N}(r,1,\varphi)=\bar{N}(r,0,f_n-z)$$
$$\leq\sum_{j/n,j=1}^{n-3}[\bar{N}(r,0,f_j-z)+\bar{N}(r,0,g_j-z)+\bar{N}(r,0,h_j-z)$$
$$+\bar{N}(r,0,w_j-z)]+O(log\,r),$$

the term $O(log\,r)$ arises due to the assumption that $f(z)$ has only a finite number of relative fix points of exact factor order n.

$$\leq\sum_{j/n,j=1}^{n-3}[T(r,f_j-z)+O(log\,r)+T(r,g_j-z)+O(log\,r)+T(r,h_j-z)$$
$$+O(log\,r)+T(r,w_j-z)+O(log\,r)]+O(log\,r)$$
$$=\sum_{j/n,n=1}^{n-3}[T(r,f_j)+T(r,g_j)+T(r,h_j)+T(r,w_j)]+O(log\,r)$$
$$=\{T(r,f_{j_1})+T(r,f_{j_5})+\ldots+T(r,f_{j_{4p-3}})+T(r,f_{j_2})+T(r,f_{j_6})+\ldots+T(r,f_{j_{4q-2}})$$
$$+T(r,f_{j_3})+T(r,f_{j_7})+\ldots+T(r,f_{j_{4r-1}})\}$$
$$+\{T(r,g_{j_1})+T(r,g_{j_5})+\ldots+T\left(r,g_{j_{4p-3}}\right)+T(r,g_{j_2})+T\left(r,g_{j_6}\right)+\ldots+T\left(r,g_{j_{4q-2}}\right)$$
$$+T(r,g_{j_3})+T(r,g_{j_7})+\ldots+T(r,g_{j_{4r-1}})\}+\{T(r,h_{j_1})+T(r,h_{j_5})+\ldots+T(r,h_{j_{4p-3}})$$
$$+T(r,h_{j_2})+T(r,h_{j_6})+\ldots$$
$$+T(r,h_{j_{4q-2}})+T(r,h_{j_3})+T(r,h_{j_7})+\ldots+T(r,h_{j_{4r-1}})\}+\{T(r,w_{j_1})+T\left(r,w_{j_5}\right)+\cdots$$
$$+T\left(r,w_{j_{4p-3}}\right)+T(r,w_{j_2})+T(r,w_{j_6})+\cdots+T\left(r,w_{j_{4q-2}}\right)+T(r,w_{j_3})$$
$$+T(r,w_{j_7})+\ldots+T(r,w_{j_{4r-1}})\}+O(log\,r),$$

where $j_1,j_5,\ldots,j_{4p-3};j_2,j_6,\ldots,j_{4q-2};j_3,j_7,\ldots j_{4r-1}$ are divisors of $n=4m+2$ and are strictly less than n and are of the form $4p-3, 4q-2, 4r-1$ $(p,q,r\in\mathbb{N})$.

$$=T(r,f_n)\left[\frac{T(r,f_{j_1})}{T(r,f_n)}+\frac{T(r,f_{j_5})}{T(r,f_n)}+\ldots+\frac{T(r,f_{j_{4p-3}})}{T(r,f_n)}+\frac{T(r,g_{j_2})}{T(r,f_n)}+\frac{T(r,g_{j_6})}{T(r,f_n)}+..+\frac{T(r,g_{j_{4q-2}})}{T(r,f_n)}+\frac{T(r,h_{j_3})}{T(r,f_n)}+\right.$$
$$\left.\frac{T(r,h_{j_7})}{T(r,f_n)}+\ldots+\frac{T(r,h_{j_{4r-1}})}{T(r,f_n)}\right]+T(r,g_n)[\frac{T(r,g_{j_1})}{T(r,g_n)}+\frac{T(r,g_{j_5})}{T(r,g_n)}+..+\frac{T(r,g_{j_{4p-3}})}{T(r,g_n)}+\frac{T(r,h_{j_2})}{T(r,g_n)}+\frac{T(r,h_{j_6})}{T(r,g_n)}+\ldots+\frac{T(r,h_{j_{4q-2}})}{T(r,g_n)}+$$
$$\frac{T(r,w_{j_3})}{T(r,g_n)}+\frac{T(r,w_{j_7})}{T(r,g_n)}+\ldots+\frac{T(r,w_{j_{4r-1}})}{T(r,g_n)}]+T(r,h_n)[\frac{T(r,h_{j_1})}{T(r,h_n)}+\frac{T(r,h_{j_5})}{T(r,h_n)}+\ldots+\frac{T(r,h_{j_{4p-3}})}{T(r,h_n)}+\frac{T(r,w_{j_2})}{T(r,h_n)}+$$

$\frac{T(r,w_{j_6})}{T(r,h_n)}+\ldots+\frac{T(r,w_{j_{4q-2}})}{T(r,h_n)}+\frac{T(r,f_{j_3})}{T(r,h_n)}+\frac{T(r,f_{j_7})}{T(r,h_n)}+\ldots+\frac{T(r,f_{j_{4r-1}})}{T(r,h_n)}]+T(r,w_n)[\frac{T(r,w_{j_1})}{T(r,w_n)}+\frac{T(r,w_{j_5})}{T(r,w_n)}+\ldots+\frac{T(r,w_{j_{4p-3}})}{T(r,w_n)}+$
$\frac{T(r,f_{j_2})}{T(r,w_n)}+\frac{T(r,f_{j_6})}{T(r,w_n)}+\ldots+\frac{T(r,f_{j_{4q-2}})}{T(r,w_n)}+\frac{T(r,g_{j_3})}{T(r,w_n)}+\frac{T(r,g_{j_7})}{T(r,w_n)}+\cdots+\frac{T(r,g_{j_{4r-1}})}{T(r,w_n)}]+O(logr).$

$<\frac{n-1}{8n}T(r,f_n)+\frac{n-1}{8n}T(r,g_n)+\frac{n-1}{8n}T(r,h_n)+\frac{n-1}{8n}T(r,w_n)+O(log\, r),$

by Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4.

CASE IV. When $n=4m+3, m\in\mathbb{N}$. In this case We have

$$\bar{N}(r,1,\varphi)=\bar{N}(r,0,f_n-z)$$
$$\leq\sum_{j/n,j=1}^{n-6}[\bar{N}(r,0,f_j-z)+\bar{N}(r,0,g_j-z)+\bar{N}(r,0,h_j-z)$$
$$+\bar{N}(r,0,w_j-z)]+O(log\, r),$$
$$\leq\sum_{j/n,j=1}^{n-6}[T(r,f_j-z)+O(log\, r)+T(r,g_j-z)+O(log\, r)+T(r,h_j-z)$$
$$+O(log\, r)+T(r,w_j-z)+O(log\, r)]+O(log\, r)$$
$$=\sum_{j/n,j=1}^{n-4}[T(r,f_j)+T(r,g_j)+T(r,h_j)+T(r,w_j)]+O(logr)$$
$$=\{T(r,f_{j_1})+T(r,f_{j_5})+\ldots+T(r,f_{j_{4p-3}})+T(r,f_{j_3})+T(r,f_{j_7})+\ldots+T(r,f_{j_{4r-1}})\}$$
$$+\{T(r,g_{j_1})+T(r,g_{j_5})+\ldots+T(r,g_{j_{4p-3}})+T(r,g_{j_3})+T(r,g_{j_7})+\ldots+T(r,g_{j_{4r-1}})\}$$
$$+\{T(r,h_{j_1})+T(r,h_{j_5})+\ldots+T(r,h_{j_{4p-3}})+T(r,h_{j_3})+T(r,h_{j_7})+\ldots+T(r,h_{j_{4r-1}})$$
$$+\{T(r,w_{j_1})+T\left(r,w_{j_5}\right)+\cdots+T\left(r,w_{j_{4p-3}}\right)+T(r,w_{j_3})+T(r,w_{j_7})+\ldots+T(r,w_{j_{4r-1}})\}+O(log\, r),$$

where $j_1,j_5,\ldots,j_{4p-3};j_3,j_7,\ldots,j_{4r-1}$are divisors of $n=4m+3$ and are strictly less than n and are of the form $4p-3,4r-1\ (p,r\in\mathbb{N})$.

$=T(r,f_n)[\frac{T(r,f_{j_1})}{T(r,f_n)}+\frac{T(r,f_{j_5})}{T(r,f_n)}+\ldots+\frac{T(r,f_{j_{4p-3}})}{T(r,f_n)}+\frac{T(r,g_{j_3})}{T(r,f_n)}+\frac{T(r,g_{j_7})}{T(r,f_n)}+\ldots+\frac{T(r,g_{j_{4r-1}})}{T(r,f_n)}]$

$+T(r,g_n)[\frac{T(r,g_{j_1})}{T(r,g_n)}+\frac{T(r,g_{j_5})}{T(r,g_n)}+..+\frac{T(r,g_{j_{4p-3}})}{T(r,g_n)}+\frac{T(r,h_{j_3})}{T(r,g_n)}+\frac{T(r,h_{j_7})}{T(r,g_n)}+\ldots+\frac{T(r,h_{j_{4r-1}})}{T(r,g_n)}]+T(r,h_n)[\frac{T(r,h_{j_1})}{T(r,h_n)}+$
$\frac{T(r,h_{j_5})}{T(r,h_n)}+\ldots+\frac{T(r,h_{j_{4p-3}})}{T(r,h_n)}+\frac{T(r,w_{j_3})}{T(r,h_n)}+\frac{T(r,w_{j_7})}{T(r,h_n)}+\ldots+\frac{T(r,w_{j_{4r-1}})}{T(r,h_n)}]$

$+T(r,w_n)[\frac{T(r,w_{j_1})}{T(r,w_n)}+\frac{T(r,w_{j_5})}{T(r,w_n)}+\ldots+\frac{T(r,w_{j_{4p-3}})}{T(r,w_n)}+\frac{T(r,f_{j_3})}{T(r,w_n)}+\frac{T(r,f_{j_7})}{T(r,w_n)}+\ldots+\frac{T(r,f_{j_{4r-1}})}{T(r,w_n)}]+O(logr).$

$$<\frac{n-1}{8n}T(r,f_n)+\frac{n-1}{8n}T(r,g_n)+\frac{n-1}{8n}T(r,h_n)+\frac{n-1}{8n}T(r,w_n)+O(log\, r),$$

by Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4.

So from (3.2) and since $\frac{T(r,g_n)}{T(r,f_n)},\frac{T(r,h_n)}{T(r,f_n)}$ and $\frac{T(r,w_n)}{T(r,f_n)}$ are bounded, we have

$$T(r,\varphi)<\frac{n-1}{8n}T(r,f_n)+\frac{n-1}{8n}T(r,g_n)+\frac{n-1}{8n}T(r,h_n)+\frac{n-1}{8n}T(r,w_n)$$

$$+O(\log r)+S_1(r,\varphi)$$

$$=\frac{n-1}{8n}T(r,f_n)+\frac{n-1}{8n}T(r,g_n)+\frac{n-1}{8n}T(r,h_n)+\frac{n-1}{8n}T(r,w_n)$$

$$+O(\log r)+O(\log T(r,\varphi))$$

$$\leq T(r,f_n)[\frac{n-1}{8n}+\frac{n-1}{8n}\frac{T(r,g_n)}{T(r,f_n)}+\frac{n-1}{8n}\frac{T(r,h_n)}{T(r,f_n)}+\frac{n-1}{8n}\frac{T(r,w_n)}{T(r,f_n)}+\frac{O\left(\log(T(r,f_n)+O(\log r))\right)}{T(r,f_n)}+\frac{O(\log r)}{T(r,f_n)}]$$

$$\leq\frac{T(r,f_n)}{T(r,f_n)}[\frac{n-1}{8n}+\frac{n-1}{8n}+\frac{n-1}{8n}+\frac{O\left(\log\left(T(r,f_n)\left(1+\frac{O(\log r)}{T(r,f_n)}\right)\right)\right)}{T(r,f_n)}+\frac{O(\log r)}{T(r,f_n)}]$$

$$< T(r,f_n)[\frac{1}{2}+\frac{O\left(\log\left(T(r,f_n)(1+\frac{O(\log r)}{T(r,f_n)})\right)\right)}{T(r,f_n)}+\frac{O(\log r)}{T(r,f_n)}]$$

$=\frac{1}{2}T(r,f_n)$, for all large r.

Therefore , $T(r,\varphi)<\frac{1}{2}T(r,f_n)$ for all large r. This contradicts (3.1).

Hence $f(z)$ has infinitely many relative fix points of exact factor order n.

This proves the theorem.

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