Existence of Positive Periodic Solutions of Two Species Discrete Lotka-Volterra System with Diffusion

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Abstract– We deal with two species discrete Lotka-Volterra system with diffusion. By applying the continuation theorem of coincidence degree theory, we establish a set of sufficient conditions on the existence of at least one positive periodic solution with period p. Some examples are provided to illustrate the results.

Keywords– Discrete Lotka-Volterra system, diffusion, continuation theorem, periodic solution

1 Introduction

One of the extensive interactions among species is the predator-prey interconnection, and it has been broadly investigated, because of its global existence. Predator-prey models are agreeably the most crucial building blocks of any bio and ecosystems as all biomasses are grown out of their resource masses. Alfred James Lotka [1] and Vito Volterra [2] imported the first predator-prey model in 1925 and 1926 respectively, the following form:

$$\begin{cases} x' = rx\left(1 - \frac{x}{K}\right) - g(x)y, \\ y' = y(-d + \mu g(x)). \end{cases}$$

In [19], the authors proposed the following Lotka-Volterra periodic cooperative systems with linear diffusion

$$\begin{cases} x_1'(t) = x_1(t)(r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)y_1(t)) + D_1(t)(x_2(t) - x_1(t)), \\ y_1'(t) = y_1(t)(r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)y_1(t)) + D_2(t)(y_2(t) - y_1(t)), \\ x_2'(t) = x_2(t)(s_1(t) - b_{11}(t)x_2(t) + b_{12}(t)y_2(t)) + D_1(t)(x_1(t) - x_2(t)), \\ y_2'(t) = y_2(t)(s_2(t) + b_{21}(t)x_2(t) - b_{22}(t)y_2(t)) + D_2(t)(y_1(t) - y_2(t)). \end{cases}$$

By means of making Lyapunov function, then the sufficient conditions on the existence of a unique almost periodic solution and its global asymptotic stability are established for the above system. Based on the above system in [15], the authors obtain that asymptotically periodic systems have a unique solution which is asymptotically stable. Furthermore, the authors [20] explored the following nonautonomous Lotka-Volterra periodic competitive systems with linear diffusion

$$\begin{cases} x_1'(t) = x_1(t)(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)y_1(t)) + D_1(t)(x_2(t) - x_1(t)), \\ y_1'(t) = y_1(t)(r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)y_1(t)) + D_2(t)(y_2(t) - y_1(t)), \\ x_2'(t) = x_2(t)(s_1(t) - b_{11}(t)x_2(t) - b_{12}(t)y_2(t)) + D_1(t)(x_1(t) - x_2(t)), \\ y_2'(t) = y_2(t)(s_2(t) - b_{21}(t)x_2(t) - b_{22}(t)y_2(t)) + D_2(t)(y_1(t) - y_2(t)). \end{cases}$$

By applying the Brouwer's fixed point theorem and forming a suitable Lyapunov function, under some appropriate conditions, the authors obtain that the system has a unique periodic solution which is globally asymptotically stable. Many authors [3–5] have explored that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations and also [6–10] discussed the periodic solution of discrete time predator-prey system.

Furthermore, the more discussions about periodic solution, global stability and persistence of predator-prey system with diffusion and time delays could be found in references [11–18, 21, 23–26]. In this paper, we investigate the following form of two species discrete Lotka-Volterra system with diffusion.

$$\begin{cases} x_1(l+1) = x_1(l)e^{\left[r_1(l)-a_{11}(l)x_1(l)-a_{12}(l)y_1(l)+D_1(l)\left(\frac{x_2(l)}{x_1(l)}-1\right)\right]}, \\ y_1(l+1) = y_1(l)e^{\left[r_2(l)-a_{21}(l)x_1(l)-a_{22}(l)y_1(l)+D_2(l)\left(\frac{y_2(l)}{y_1(l)}-1\right)\right]}, \\ x_2(l+1) = x_2(l)e^{\left[s_1(l)-b_{11}(l)x_2(l)-b_{12}(l)y_2(l)+D_1(l)\left(\frac{x_1(l)}{x_2(l)}-1\right)\right]}, \\ y_2(l+1) = y_2(l)e^{\left[s_2(l)-b_{21}(l)x_2(l)-b_{22}(l)y_2(l)+D_2(l)\left(\frac{y_1(l)}{y_2(l)}-1\right)\right]}. \end{cases} \tag{1.1}$$

where $x_1(l), x_2(l)$ denote the densities of prey species and $y_1(l), y_2(l)$ denote the densities of predator species while the prey and predator species can disperse between two patches; $r_i(l), s_i(l)$ are intrinsic growth rate of the prey and predator; $a_{ii}(l), b_{ii}(l)$ are intraspecies prey and predator; $a_{ij}(l), b_{ij}(l)$ are interspecies prey and predator; $D_i(l)$ is the dispersion rate of prey and predator species. By using the technical idea of continuation theorem of coincidence degree theory by Gaines and Mawhin [27], we will to derived set of sufficient conditions for the new result of existence of positive periodic solution of system (1.1), finally some examples are provided to illustrate the main results.

2 Main Results

Before exploring the existence of periodic solutions of system, we will make some arrangements.

Let Y and Z be real Banach spaces, $L : \text{Dom}L \subset Y \to Z$ be a linear mapping and $N : Y \to Z$ be a continuous mapping. The mapping L is said to be a Fredholm mapping of index zero, if $\text{dimKer}L = \text{codimIm}L < +\infty$ and $\text{Im}L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exists continuous projectors $P : Y \to Y$ and $Q : Z \to Z$ such that $\text{Im}P = \text{Ker}L$, $\text{Ker}Q = \text{Im}L = \text{Im}(I - Q)$. It follows that the restriction L_P of L to $\text{Dom}L \cap \text{Ker}P : (I - P)Y \to \text{Im}L$ is invertible. Denote the inverse of L_P by K_P.

Lemma 2.1. *(See [27]) Let $\Omega \subset Y$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume*

(I) for all $\lambda \in (0, 1)$, $z \in \partial\Omega \cap DomL$, $Lz \neq \lambda Nz$,

(II) for all $z \in \partial\Omega \cap KerL$, $QNz \neq 0$,

(III) $deg(JQN, \Omega \cap KerL, 0) \neq 0$.

Then $Lz = Nz$ has at least one solution in $\overline{\Omega} \cap DomL$.

Lemma 2.2. *(See [6]) Let $g : \mathbb{Z} \to \mathbb{R}$ be p-periodic, i.e., $g(l + p) = g(l)$. Then for any fixed $l_1, l_2 \in I_p$, and any $l \in \mathbb{Z}$, one has*

$$g(l) \leq g(l_1) + \sum_{s=0}^{p-1} |g(s + 1) - g(s)|$$

and

$$g(l) \geq g(l_2) - \sum_{s=0}^{p-1} |g(s + 1) - g(s)|.$$

For convenience, we shall use the notation:

$$I_p = \{0, 1, \ldots, p-1\},\ \overline{g} = \frac{1}{p}\sum_{l=0}^{p-1} g(l),\ g^M = \max_{l\in I_p} g(l),\ g^m = \min_{l\in I_p} g(l).$$

Theorem 2.1. *For system (1.1), we assume that:*

(i) $r_i(l),\ s_i(l),\ D_i(l),\ a_{ij}(l)$ *and* $b_{ij}(l),\ i,j = 1,2$ *are continuous positive periodic functions with period* $p > 0$,

(ii) $a_{11}^m > 0,\ a_{22}^m > 0,\ b_{11}^m > 0$ *and* $b_{22}^m > 0$,

(iii) $\overline{a_{11}}b_{22}^m > 0,\ \overline{a_{22}}b_{11}^m > 0,\ \overline{b_{11}}a_{22}^m > 0$ *and* $\overline{b_{22}}a_{11}^m > 0$.

Then system (1.1) has at least one positive periodic solution.

Proof. By the biological meaning, we only focus on the positive periodic solutions to system (1.1).
Let $x_i(l) = e^{u_i(l)}$ and $y_i(l) = e^{v_i(l)},\ i = 1,2$.
From system (1.1) becomes

$$\begin{cases} u_1(l+1) - u_1(l) = r_1(l) - a_{11}(l)e^{u_1(l)} - a_{12}(l)e^{v_1(l)} + D_1(l)e^{u_2(l)-u_1(l)} - D_1(l), \\ v_1(l+1) - v_1(l) = r_2(l) - a_{21}(l)e^{u_1(l)} - a_{22}(l)e^{v_1(l)} + D_2(l)e^{v_2(l)-v_1(l)} - D_2(l), \\ u_2(l+1) - u_2(l) = s_1(l) - b_{11}(l)e^{u_2(l)} - b_{12}(l)e^{v_2(l)} + D_1(l)e^{u_1(l)-u_2(l)} - D_1(l), \\ v_2(l+1) - v_2(l) = s_2(l) - b_{21}(l)e^{u_2(l)} - b_{22}(l)e^{v_2(l)} + D_2(l)e^{v_1(l)-v_2(l)} - D_2(l). \end{cases} \tag{2.1}$$

It is easy to see that if (2.1) has an p-periodic solution $(u_1^*(l), v_1^*(l), u_2^*(l), v_2^*(l))^T$, then $(x_1^*(l), y_1^*(l), x_2^*(l), y_2^*(l))^T = (e^{u_1^*(l)}, e^{v_1^*(l)}, e^{u_2^*(l)}, e^{v_2^*(l)})^T$ is a positive p-periodic solution of system of (1.1). So to complete the proof, it suffices to show that (2.1) has a p-periodic solution.
Define

$$j_4 = \{z = (z(l)) : z(l) \in \mathbb{R}^4,\ l \in \mathbb{Z}\}.$$

For $c = (c_1, c_2, c_3, c_4)^T \in \mathbb{R}^4$, define $|c| = \max\{|c_1|, |c_2|, |c_3|, |c_4|\}$.
Let $j^p \subset j_4$ denote the subspace of all p-periodic sequences equipped with the usual supremum norm $\|.\|$, *i.e.*, $\|z\| = \max_{l\in I_p}|z(l)|$, for any $z = \{z(l) : l \in \mathbb{Z}\} \in j^p$. It is easy to see that j^p is a finite dimensional Banach space.
Let

$$j_0^p = \{z = z(l) \in j^p : \sum_{l=0}^{p-1} z(l) = 0\},\ j_c^p = \{z = z(l) \in j^p : z(l) = h \in \mathbb{R}^4, l \in \mathbb{Z}\}.$$

Then it follows that j_0^p and j_c^p are both closed linear subspaces of j^p and $j^p = j_c^p \oplus j_0^p,\ \dim j_c^p = 4$.
We take

$$j^p = Z = \{z(l) = (u_1(l), v_1(l), u_2(l), v_2(l))^T \in \mathbb{R}^4,\ u_i(l+p) = u_i(l),\ v_i(l+p) = v_i(l),\ i = 1,2\},$$

$$\|z\| = \|(u_1(l), v_1(l), u_2(l), v_2(l))^T\| = \max_{l\in I_p}|u_1(l)| + \max_{l\in I_p}|v_1(l)| + \max_{l\in I_p}|u_2(l)| + \max_{l\in I_p}|v_2(l)|.$$

Then Z is a Banach space with norm $\|.\|$.
$L : \mathrm{Dom}L \subset Z \to Z$ and $N : Z \to Z$ are defined by

$$L\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1(l+1) - u_1(l) \\ v_1(l+1) - v_1(l) \\ u_2(l+1) - u_2(l) \\ v_2(l+1) - v_2(l) \end{pmatrix},$$

$$N\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} r_1(l) - D_1(l) - a_{11}(l)e^{u_1(l)} - a_{12}(l)e^{v_1(l)} + D_1(l)e^{u_2(l)-u_1(l)} \\ r_2(l) - D_2(l) - a_{21}(l)e^{u_1(l)} - a_{22}(l)e^{v_1(l)} + D_2(l)e^{v_2(l)-v_1(l)} \\ s_1(l) - D_1(l) - b_{11}(l)e^{u_2(l)} - b_{12}(l)e^{v_2(l)} + D_1(l)e^{u_1(l)-u_2(l)} \\ s_2(l) - D_2(l) - b_{21}(l)e^{u_2(l)} - b_{22}(l)e^{v_2(l)} + D_2(l)e^{v_1(l)-v_2(l)} \end{pmatrix}.$$

for any $(u_1, v_1, u_2, v_2)^T \in Z$ and $l \in \mathbb{Z}$, it is trivial to see that L is a bounded linear operator and

$$\mathrm{Ker}L = j_c^p,\ \mathrm{Im}L = j_0^p$$

and $\mathrm{Im}L$ is closed in Z. Therefore,

$$\mathrm{dimKer}L = 4 = \mathrm{codimIm}L.$$

So it follows that L is a Fredholm mapping of index zero.
Now we set the continuous projectors $P : Z \to Z,\ Q : Z \to Z$ are defined by

$$P\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix} = Q\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix} = \begin{pmatrix}\frac{1}{p}\left[\sum\limits_{l=0}^{p-1} u_1(l)\right]\\ \frac{1}{p}\left[\sum\limits_{l=0}^{p-1} v_1(l)\right]\\ \frac{1}{p}\left[\sum\limits_{l=0}^{p-1} u_2(l)\right]\\ \frac{1}{p}\left[\sum\limits_{l=0}^{p-1} v_2(l)\right]\end{pmatrix},$$

such that

$$\mathrm{Im}P = \mathrm{Ker}L,\ \mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q).$$

Furthermore, K_p denote the inverse of $L\mid_{\mathrm{Dom}L\cap\mathrm{Ker}L}$

$$K_p\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix} = \begin{pmatrix}\sum\limits_{s=0}^{p-1} u_1(s) - \frac{1}{p}\left[\sum\limits_{s=0}^{p-1}(p-s)u_1(s)\right]\\ \sum\limits_{s=0}^{p-1} v_1(s) - \frac{1}{p}\left[\sum\limits_{s=0}^{p-1}(p-s)v_1(s)\right]\\ \sum\limits_{s=0}^{p-1} u_2(s) - \frac{1}{p}\left[\sum\limits_{s=0}^{p-1}(p-s)u_2(s)\right]\\ \sum\limits_{s=0}^{p-1} v_2(s) - \frac{1}{p}\left[\sum\limits_{s=0}^{p-1}(p-s)v_2(s)\right]\end{pmatrix},$$

$QN : Z \to Z$ and $K_p(I-Q)N : Z \to Z$ are defined by

$$QN\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix} = \begin{pmatrix}\frac{1}{p}\sum\limits_{l=0}^{p-1}\left[r_1(l) - D_1(l) - a_{11}(l)e^{u_1(l)} - a_{12}(l)e^{v_1(l)} + D_1(l)e^{u_2(l)-u_1(l)}\right]\\ \frac{1}{p}\sum\limits_{l=0}^{p-1}\left[r_2(l) - D_2(l) - a_{21}(l)e^{u_1(l)} - a_{22}(l)e^{v_1(l)} + D_2(l)e^{v_2(l)-v_1(l)}\right]\\ \frac{1}{p}\sum\limits_{l=0}^{p-1}\left[s_1(l) - D_1(l) - b_{11}(l)e^{u_2(l)} - b_{12}(l)e^{v_2(l)} + D_1(l)e^{u_1(l)-u_2(l)}\right]\\ \frac{1}{p}\sum\limits_{l=0}^{p-1}\left[s_2(l) - D_2(l) - b_{21}(l)e^{u_2(l)} - b_{22}(l)e^{v_2(l)} + D_2(l)e^{v_1(l)-v_2(l)}\right]\end{pmatrix},$$

$$\begin{aligned} K_p(I-Q)N\begin{pmatrix}u_1\\v_1\\u_2\\v_2\end{pmatrix} &= \begin{pmatrix}\sum\limits_{s=0}^{p-1}\left[r_1(s) - D_1(s) - a_{11}(s)e^{u_1(s)} - a_{12}(s)e^{v_1(s)} + D_1(s)e^{u_2(s)-u_1(s)}\right]\\ \sum\limits_{s=0}^{p-1}\left[r_2(s) - D_2(s) - a_{21}(s)e^{u_1(s)} - a_{22}(s)e^{v_1(s)} + D_2(s)e^{v_2(s)-v_1(s)}\right]\\ \sum\limits_{s=0}^{p-1}\left[s_1(s) - D_1(s) - b_{11}(s)e^{u_2(s)} - b_{12}(s)e^{v_2(s)} + D_1(s)e^{u_1(s)-u_2(s)}\right]\\ \sum\limits_{s=0}^{p-1}\left[s_2(s) - D_2(s) - b_{21}(s)e^{u_2(s)} - b_{22}(s)e^{v_2(s)} + D_2(s)e^{v_1(s)-v_2(s)}\right]\end{pmatrix} \\ &- \begin{pmatrix}\frac{1}{p}\sum\limits_{s=0}^{p-1}(p-s)\left[r_1(s) - D_1(s) - a_{11}(s)e^{u_1(s)} - a_{12}(s)e^{v_1(s)} + D_1(s)e^{u_2(s)-u_1(s)}\right]\\ \frac{1}{p}\sum\limits_{s=0}^{p-1}(p-s)\left[r_2(s) - D_2(s) - a_{21}(s)e^{u_1(s)} - a_{22}(s)e^{v_1(s)} + D_2(s)e^{v_2(s)-v_1(s)}\right]\\ \frac{1}{p}\sum\limits_{s=0}^{p-1}(p-s)\left[s_1(s) - D_1(s) - b_{11}(s)e^{u_2(s)} - b_{12}(s)e^{v_2(s)} + D_1(s)e^{u_1(s)-u_2(s)}\right]\\ \frac{1}{p}\sum\limits_{s=0}^{p-1}(p-s)\left[s_2(s) - D_2(s) - b_{21}(s)e^{u_2(s)} - b_{22}(s)e^{v_2(s)} + D_2(s)e^{v_1(s)-v_2(s)}\right]\end{pmatrix}\end{aligned}$$

$$-\begin{pmatrix} \left(\frac{s}{p}-\frac{p+1}{2p}\right)\sum_{l=0}^{p-1}\left[r_1(l)-D_1(l)-a_{11}(l)e^{u_1(l)}-a_{12}(l)e^{v_1(l)}+D_1(l)e^{u_2(l)-u_1(l)}\right] \\ \left(\frac{s}{p}-\frac{p+1}{2p}\right)\sum_{l=0}^{p-1}\left[r_2(l)-D_2(l)-a_{21}(l)e^{u_1(l)}-a_{22}(l)e^{v_1(l)}+D_2(l)e^{v_2(l)-v_1(l)}\right] \\ \left(\frac{s}{p}-\frac{p+1}{2p}\right)\sum_{l=0}^{p-1}\left[s_1(l)-D_1(l)-b_{11}(l)e^{u_2(l)}-b_{12}(l)e^{v_2(l)}+D_1(l)e^{u_1(l)-u_2(l)}\right] \\ \left(\frac{s}{p}-\frac{p+1}{2p}\right)\sum_{l=0}^{p-1}\left[s_2(l)-D_2(l)-b_{21}(l)e^{u_2(l)}-b_{22}(l)e^{v_2(l)}+D_2(l)e^{v_1(l)-v_2(l)}\right] \end{pmatrix}.$$

Clearly, QN and $K_p(I-Q)N$ are continuous. Since Z is a finite dimensional Banach space using Arzela Ascoli theorem [22], it is not difficult to see that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset Z$. Moreover, $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset Z$. The isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an identity mapping such that $\mathrm{Ker}L = \mathrm{Im}Q$. In the following, we consider the operator equation $Lz = \lambda Nz$, $\lambda \in (0, 1)$, that is,

$$\begin{cases} u_1(l+1)-u_1(l)=\lambda\Big[r_1(l)-a_{11}(l)e^{u_1(l)}-a_{12}(l)e^{v_1(l)}+D_1(l)e^{u_2(l)-u_1(l)}-D_1(l)\Big], \\ v_1(l+1)-v_1(l)=\lambda\Big[r_2(l)-a_{21}(l)e^{u_1(l)}-a_{22}(l)e^{v_1(l)}+D_2(l)e^{v_2(l)-v_1(l)}-D_2(l)\Big], \\ u_2(l+1)-u_2(l)=\lambda\Big[s_1(l)-b_{11}(l)e^{u_2(l)}-b_{12}(l)e^{v_2(l)}+D_1(l)e^{u_1(l)-u_2(l)}-D_1(l)\Big], \\ v_2(l+1)-v_2(l)=\lambda\Big[s_2(l)-b_{21}(l)e^{u_2(l)}-b_{22}(l)e^{v_2(l)}+D_2(l)e^{v_1(l)-v_2(l)}-D_2(l)\Big]. \end{cases} \tag{2.2}$$

Suppose that $(u_1(l), v_1(l), u_2(l), v_2(l))^T \in Z$ is a solution of (2.2) for a certain $\lambda \in (0, 1)$ and summing from 0 to $p-1$ on both sides, we get

$$\begin{cases} \sum_{l=0}^{p-1}\Big[r_1(l)-D_1(l)\Big]+\sum_{l=0}^{p-1}D_1(l)e^{u_2(l)-u_1(l)}=\sum_{l=0}^{p-1}\Big[a_{11}(l)e^{u_1(l)}+a_{12}(l)e^{v_1(l)}\Big] \\ \sum_{l=0}^{p-1}\Big[r_2(l)-D_2(l)\Big]+\sum_{l=0}^{p-1}D_2(l)e^{v_2(l)-v_1(l)}=\sum_{l=0}^{p-1}\Big[a_{21}(l)e^{u_1(l)}+a_{22}(l)e^{v_1(l)}\Big] \\ \sum_{l=0}^{p-1}\Big[s_1(l)-D_1(l)\Big]+\sum_{l=0}^{p-1}D_1(l)e^{u_1(l)-u_2(l)}=\sum_{l=0}^{p-1}\Big[b_{11}(l)e^{u_2(l)}+b_{12}(l)e^{v_2(l)}\Big] \\ \sum_{l=0}^{p-1}\Big[s_2(l)-D_2(l)\Big]+\sum_{l=0}^{p-1}D_2(l)e^{v_1(l)-v_2(l)}=\sum_{l=0}^{p-1}\Big[b_{21}(l)e^{u_2(l)}+b_{22}(l)e^{v_2(l)}\Big] \end{cases} \tag{2.3}$$

From the first equation of system (2.2) and the first equation of system (2.3) becomes

$$\sum_{l=0}^{p-1}\Big|u_1(l+1)-u_1(l)\Big| < 2\left[\sum_{l=0}^{p-1}a_{11}(l)e^{u_1(l)}+\sum_{l=0}^{p-1}a_{12}(l)e^{v_1(l)}\right]. \tag{2.4}$$

From the second equation of system (2.2) and the second equation of system (2.3) becomes

$$\sum_{l=0}^{p-1}\Big|v_1(l+1)-v_1(l)\Big| < 2\left[\sum_{l=0}^{p-1}a_{21}(l)e^{u_1(l)}+\sum_{l=0}^{p-1}a_{22}(l)e^{v_1(l)}\right]. \tag{2.5}$$

From the third equation of system (2.2) and the third equation of system (2.3) becomes

$$\sum_{l=0}^{p-1}\Big|u_2(l+1)-u_2(l)\Big| < 2\Big[\sum_{l=0}^{p-1}b_{11}(l)e^{u_2(l)}+\sum_{l=0}^{p-1}b_{12}(l)e^{v_2(l)}\Big]. \tag{2.6}$$

From the fourth equation of system (2.2) and the fourth equation of system (2.3) becomes

$$\sum_{l=0}^{p-1}\Big|v_2(l+1)-v_2(l)\Big| < 2\Big[\sum_{l=0}^{p-1}b_{21}(l)e^{u_2(l)}+\sum_{l=0}^{p-1}b_{22}(l)e^{v_2(l)}\Big]. \tag{2.7}$$

By using the inequalities

$$\left(\sum_{l=0}^{p-1}e^{u_i(l)}\right)^2 \le p\sum_{l=0}^{p-1}e^{2u_i(l)},\ i=1,2$$

and

$$\left(\sum_{l=0}^{p-1} e^{v_i(l)}\right)^2 \le p\sum_{l=0}^{p-1} e^{2v_i(l)},\ i=1,2.$$

Multiplying by the first equation of system (2.3) by $e^{u_1(l)}$ we get

$$\sum_{l=0}^{p-1} a_{11}(l)e^{2u_1(l)} < (r_1-D_1)^M \sum_{l=0}^{p-1} e^{u_1(l)} + D_1^M \sum_{l=0}^{p-1} e^{u_2(l)}$$

$$a_{11}^m\left(\sum_{l=0}^{p-1} e^{u_1(l)}\right)^2 < p(r_1-D_1)^M \sum_{l=0}^{p-1} e^{u_1(l)} + pD_1^M \sum_{l=0}^{p-1} e^{u_2(l)}. \tag{2.8}$$

If $\sum\limits_{l=0}^{p-1} e^{u_2(l)} \le \sum\limits_{l=0}^{p-1} e^{u_1(l)}$, then it follows from (2.8) we obtain

$$a_{11}^m\left(\sum_{l=0}^{p-1} e^{u_1(l)}\right)^2 < p(r_1-D_1)^M \sum_{l=0}^{p-1} e^{u_1(l)} + pD_1^M \sum_{l=0}^{p-1} e^{u_1(l)}$$

$$\sum_{l=0}^{p-1} e^{u_2(l)} \le \sum_{l=0}^{p-1} e^{u_1(l)} < \frac{p(r_1-D_1)^M + pD_1^M}{a_{11}^m}. \tag{2.9}$$

Multiplying by the second equation of system (2.3) by $e^{v_1(l)}$ we get

$$\sum_{l=0}^{p-1} a_{22}(l)e^{2v_1(l)} < (r_2-D_2)^M \sum_{l=0}^{p-1} e^{v_1(l)} + D_2^M \sum_{l=0}^{p-1} e^{v_2(l)}$$

$$a_{22}^m\left(\sum_{l=0}^{p-1} e^{v_1(l)}\right)^2 < p(r_2-D_2)^M \sum_{l=0}^{p-1} e^{v_1(l)} + pD_2^M \sum_{l=0}^{p-1} e^{v_2(l)}. \tag{2.10}$$

If $\sum\limits_{l=0}^{p-1} e^{v_2(l)} \le \sum\limits_{l=0}^{p-1} e^{v_1(l)}$, then it follows from (2.10) we obtain

$$a_{22}^m\left(\sum_{l=0}^{p-1} e^{v_1(l)}\right)^2 < p(r_2-D_2)^M \sum_{l=0}^{p-1} e^{v_1(l)} + pD_2^M \sum_{l=0}^{p-1} e^{v_1(l)}$$

$$\sum_{l=0}^{p-1} e^{v_2(l)} \le \sum_{l=0}^{p-1} e^{v_1(l)} < \frac{p(r_2-D_2)^M + pD_2^M}{a_{22}^m}. \tag{2.11}$$

Multiplying by the third equation of system (2.3) by $e^{u_2(l)}$ we get

$$\sum_{l=0}^{p-1} b_{11}(l)e^{2u_2(l)} < (s_1-D_1)^M \sum_{l=0}^{p-1} e^{u_2(l)} + D_1^M \sum_{l=0}^{p-1} e^{u_1(l)}$$

$$b_{11}^m\left(\sum_{l=0}^{p-1} e^{u_2(l)}\right)^2 < p(s_1-D_1)^M \sum_{l=0}^{p-1} e^{u_2(l)} + pD_1^M \sum_{l=0}^{p-1} e^{u_1(l)}. \tag{2.12}$$

If $\sum\limits_{l=0}^{p-1} e^{u_1(l)} \le \sum\limits_{l=0}^{p-1} e^{u_2(l)}$, then it follows from (2.12) we obtain

$$b_{11}^m\left(\sum_{l=0}^{p-1} e^{u_2(l)}\right)^2 < p(s_1-D_1)^M \sum_{l=0}^{p-1} e^{u_2(l)} + pD_1^M \sum_{l=0}^{p-1} e^{u_2(l)}$$

$$\sum_{l=0}^{p-1} e^{u_1(l)} \le \sum_{l=0}^{p-1} e^{u_2(l)} < \frac{p(s_1-D_1)^M + pD_1^M}{b_{11}^m}. \tag{2.13}$$

Multiplying by the fourth equation of system (2.3) by $e^{v_2(l)}$ we get

$$\sum_{l=0}^{p-1} b_{22}(l)e^{2v_2(l)} < (s_2 - D_2)^M \sum_{l=0}^{p-1} e^{v_2(l)} + D_2^M \sum_{l=0}^{p-1} e^{v_1(l)}$$

$$b_{22}^m \left(\sum_{l=0}^{p-1} e^{v_2(l)} \right)^2 < p(s_2 - D_2)^M \sum_{l=0}^{p-1} e^{v_2(l)} + pD_2^M \sum_{l=0}^{p-1} e^{v_1(l)}. \tag{2.14}$$

If $\sum_{l=0}^{p-1} e^{v_1(l)} \leq \sum_{l=0}^{p-1} e^{v_2(l)}$, then it follows from (2.14) we obtain

$$b_{22}^m \left(\sum_{l=0}^{p-1} e^{v_2(l)} \right)^2 < p(s_2 - D_2)^M \sum_{l=0}^{p-1} e^{v_2(l)} + pD_2^M \sum_{l=0}^{p-1} e^{v_2(l)}$$

$$\sum_{l=0}^{p-1} e^{v_1(l)} \leq \sum_{l=0}^{p-1} e^{v_2(l)} < \frac{p(s_2 - D_2)^M + pD_2^M}{b_{22}^m}. \tag{2.15}$$

From (2.4), (2.13) and (2.15) we obtain

$$\sum_{l=0}^{p-1} \Big| u_1(l+1) - u_1(l) \Big| < 2a_{11}^M \Big[\frac{p(s_1 - D_1)^M + pD_1^M}{b_{11}^m} \Big] + 2a_{12}^M \Big[\frac{p(s_2 - D_2)^M + pD_2^M}{b_{22}^m} \Big] := N_1. \tag{2.16}$$

From (2.5), (2.13) and (2.15) we obtain

$$\sum_{l=0}^{p-1} \Big| v_1(l+1) - v_1(l) \Big| < 2a_{21}^M \Big[\frac{p(s_1 - D_1)^M + pD_1^M}{b_{11}^m} \Big] + 2a_{22}^M \Big[\frac{p(s_2 - D_2)^M + pD_2^M}{b_{22}^m} \Big] := N_2. \tag{2.17}$$

From (2.6), (2.9) and (2.11) we obtain

$$\sum_{l=0}^{p-1} \Big| u_2(l+1) - u_2(l) \Big| < 2b_{11}^M \Big[\frac{p(r_1 - D_1)^M + pD_1^M}{a_{11}^m} \Big] + 2b_{12}^M \Big[\frac{p(r_2 - D_2)^M + pD_2^M}{a_{22}^m} \Big] := N_3. \tag{2.18}$$

From (2.7), (2.9) and (2.11) we obtain

$$\sum_{l=0}^{p-1} \Big| v_2(l+1) - v_2(l) \Big| < 2b_{21}^M \Big[\frac{p(r_1 - D_1)^M + pD_1^M}{a_{11}^m} \Big] + 2b_{22}^M \Big[\frac{p(r_2 - D_2)^M + pD_2^M}{a_{22}^m} \Big] := N_4. \tag{2.19}$$

Since $(u_1(l), v_1(l), u_2(l), v_2(l))^T \in Z$, there exists $\alpha_i, \beta_i \in I_p$ such that

$$u_i(\alpha_i) = \max_{l \in I_p} u_i(l), \;\; u_i(\beta_i) = \min_{l \in I_p} u_i(l), \;\; v_i(\alpha_i) = \max_{l \in I_p} v_i(l), \;\; v_i(\beta_i) = \min_{l \in I_p} v_i(l), \;\; i = 1, 2. \tag{2.20}$$

From (2.13) and (2.20) becomes

$$e^{u_1(\beta_1)} < \frac{(s_1 - D_1)^M + D_1^M}{b_{11}^m} := B_{11}$$

$$u_1(\beta_1) < \log B_{11}. \tag{2.21}$$

From (2.15) and (2.20) becomes

$$e^{v_1(\beta_1)} < \frac{(s_2 - D_2)^M + D_2^M}{b_{22}^m} := B_{12}$$

$$v_1(\beta_1) < \log B_{12}. \tag{2.22}$$

From (2.9) and (2.20) becomes

$$e^{u_2(\beta_2)} < \frac{(r_1 - D_1)^M + D_1^M}{a_{11}^m} := B_{21}$$

$$u_2(\beta_2) < \log B_{21}. \tag{2.23}$$

From (2.11) and (2.20) becomes

$$e^{v_2(\beta_2)} < \frac{(r_2 - D_2)^M + D_2^M}{a_{22}^m} := B_{22}$$

$$v_2(\beta_2) < \log B_{22}. \tag{2.24}$$

From Lemma 2.2, (2.21), (2.22), (2.23) and (2.24) we obtain

$$\begin{cases} u_1(l) \leq u_1(\beta_1) + \sum_{l=0}^{p-1} |u_1(l+1) - u_1(l)| < \log B_{11} + N_1, \\ v_1(l) \leq v_1(\beta_1) + \sum_{l=0}^{p-1} |v_1(l+1) - v_1(l)| < \log B_{12} + N_2, \\ u_2(l) \leq u_2(\beta_2) + \sum_{l=0}^{p-1} |u_2(l+1) - u_2(l)| < \log B_{21} + N_3, \\ v_2(l) \leq v_2(\beta_2) + \sum_{l=0}^{p-1} |v_2(l+1) - v_2(l)| < \log B_{22} + N_4. \end{cases} \tag{2.25}$$

From the first equation of system (2.3) and (2.20) becomes

$$e^{u_1(\alpha_1)} > \frac{\overline{(r_1 - D_1)} b_{22}^m - a_{12}^M[(s_2 - D_2)^M + D_2^M]}{\overline{a_{11}} b_{22}^m} := A_{11}$$

$$u_1(\alpha_1) > \log A_{11}. \tag{2.26}$$

From the second equation of system (2.3) and (2.20) becomes

$$e^{v_1(\alpha_1)} > \frac{\overline{(r_2 - D_2)} b_{11}^m - a_{21}^M[(s_1 - D_1)^M + D_1^M]}{\overline{a_{22}} b_{11}^m} := A_{12}$$

$$v_1(\alpha_1) > \log A_{12}. \tag{2.27}$$

From the third equation of system (2.3) and (2.20) becomes

$$e^{u_2(\alpha_2)} > \frac{\overline{(s_1 - D_1)} a_{22}^m - b_{12}^M[(r_2 - D_2)^M + D_2^M]}{\overline{b_{11}} a_{22}^m} := A_{21}$$

$$u_2(\alpha_2) > \log A_{21}. \tag{2.28}$$

From the fourth equation of system (2.3)and (2.20) becomes

$$e^{v_2(\alpha_2)} > \frac{\overline{(s_2 - D_2)} a_{11}^m - b_{21}^M[(r_1 - D_1)^M + D_1^M]}{\overline{b_{22}} a_{11}^m} := A_{22}$$

$$v_2(\alpha_2) > \log A_{22}. \tag{2.29}$$

From Lemma 2.2, (2.26), (2.27), (2.28) and (2.29) we obtain

$$\begin{cases} u_1(l) \geq u_1(\alpha_1) - \sum_{l=0}^{p-1} |u_1(l+1) - u_1(l)| > \log A_{11} - N_1, \\ v_1(l) \geq v_1(\alpha_1) - \sum_{l=0}^{p-1} |v_1(l+1) - v_1(l)| > \log A_{12} - N_2, \\ u_2(l) \geq u_2(\alpha_2) - \sum_{l=0}^{p-1} |u_2(l+1) - u_2(l)| > \log A_{21} - N_3, \\ v_2(l) \geq v_2(\alpha_2) - \sum_{l=0}^{p-1} |v_2(l+1) - v_2(l)| > \log A_{22} - N_4. \end{cases} \tag{2.30}$$

It follows from (2.25) and (2.30) we get

$$\begin{cases} \max_{l \in I_p} |u_1(l)| < \max \left\{ |\log A_{11}| + N_1, |\log B_{11}| + N_1 \right\} := S_1, \\ \max_{l \in I_p} |v_1(l)| < \max \left\{ |\log A_{12}| + N_2, |\log B_{12}| + N_2 \right\} := S_2, \\ \max_{l \in I_p} |u_2(l)| < \max \left\{ |\log A_{21}| + N_3, |\log B_{21}| + N_3 \right\} := S_3, \\ \max_{l \in I_p} |u_2(l)| < \max \left\{ |\log A_{22}| + N_4, |\log B_{22}| + N_4 \right\} := S_4. \end{cases}$$

Clearly, S_1, S_2, S_3 and S_4 are independent of λ. Denote $S = S_1 + S_2 + S_3 + S_4 + S_0$, where S_0 is sufficiently large such that each solution $(u_1^*, v_1^*, u_2^*, v_2^*)$ of the system of algebraic equations

$$\begin{cases} \overline{(r_1 - D_1)} - \overline{a_{11}}e^{u_1} - \overline{a_{12}}e^{v_1} + \overline{D_1}e^{u_2-u_1} = 0, \\ \overline{(r_2 - D_2)} - \overline{a_{21}}e^{u_1} - \overline{a_{22}}e^{v_1} + \overline{D_2}e^{v_2-v_1} = 0, \\ \overline{(s_1 - D_1)} - \overline{b_{11}}e^{u_2} - \overline{b_{12}}e^{v_2} + \overline{D_1}e^{u_1-u_2} = 0, \\ \overline{(s_2 - D_2)} - \overline{b_{21}}e^{u_2} - \overline{b_{22}}e^{v_2} + \overline{D_2}e^{v_1-v_2} = 0, \end{cases}$$

satisfies,

$$\|(u_1^*, v_1^*, u_2^*, v_2^*)^T\| = |u_1^*| + |v_1^*| + |u_2^*| + |v_2^*| < S$$

and

$$\max|u_1(l)| + \max|v_1(l)| + \max|u_2(l)| + \max|v_2(l)| < S.$$

We now take,

$$\Omega = \Big\{(u_1(l), v_1(l), u_2(l), v_2(l))^T \in Z : \|(u_1, v_1, u_2, v_1)^T\| < S\Big\}.$$

This satisfies the condition (I) in Lemma 2.1. If $z \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}^4$, then z is a constant vector in $\mathbb{R}^4$ with $\|z\| = S$ satisfying

$$QN\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \overline{(r_1 - D_1)} - \overline{a_{11}}e^{u_1} - \overline{a_{12}}e^{v_1} + \overline{D_1}e^{u_2-u_1} \\ \overline{(r_2 - D_2)} - \overline{a_{21}}e^{u_1} - \overline{a_{22}}e^{v_1} + \overline{D_2}e^{v_2-v_1} \\ \overline{(s_1 - D_1)} - \overline{b_{11}}e^{u_2} - \overline{b_{12}}e^{v_2} + \overline{D_1}e^{u_1-u_2} \\ \overline{(s_2 - D_2)} - \overline{b_{21}}e^{u_2} - \overline{b_{22}}e^{v_2} + \overline{D_2}e^{v_1-v_2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therfore the condition (II) is satisfied in Lemma 2.1.
In order to prove that the condition (III) in Lemma 2.1 holds.
Consider a homotopy

$$B_\mu((u_1, v_1, u_2, v_2)^T) = \mu JQN((u_1, v_1, u_2, v_2)^T) + (1-\mu)\rho((u_1, v_1, u_2, v_2)^T), \ \mu \in [0,1].$$

By a direct computation and the invariance property of homotopy, one has

$$\deg(JQN(u_1, v_1, u_2, v_2)^T, \Omega \cap \mathrm{Ker}L, (0,0,0,0)^T) = \deg(\rho(u_1, v_1, u_2, v_2)^T, \Omega \cap \mathrm{Ker}L, (0,0,0,0)^T) \neq 0.$$

We have proved that Ω verifies all the requirements in Lemma 2.1. Then, we get that equations (2.1) have at least one periodic solution $(u_1(l), v_1(l), u_2(l), v_2(l))^T$ with period p in $\mathrm{Dom}L \cap \Omega$, which implies that systen (1.1) has at least one positive periodic solution $(e^{u_1(l)}, e^{v_1(l)}, e^{u_2(l)}, e^{v_2(l)})^T$ with period p. □

Example 2.1. *Consider the following two species discrete Lotka-Volterra system with diffusion*

$$\begin{cases} x_1(l+1) = x_1(l)e^{\left[(1.3+0.02\cos(\frac{\pi l}{2})) - (1.81+0.01\sin(\frac{\pi l}{2}))x_1(l) - (0.05+0.01\sin(\frac{\pi l}{2}))y_1(l) + (1+0.02\cos(\frac{\pi l}{2}))\left(\frac{x_2(l)}{x_1(l)} - 1\right)\right]} \\ y_1(l+1) = y_1(l)e^{\left[(0.8+0.02\sin(\frac{\pi l}{2})) - (0.6+0.02\cos(\frac{\pi l}{2}))x_1(l) - (2+0.01\sin(\frac{\pi l}{2}))y_1(l) + (1+0.03\sin(\frac{\pi l}{2}))\left(\frac{y_2(l)}{y_1(l)} - 1\right)\right]} \\ x_2(l+1) = x_2(l)e^{\left[(0.86+0.03\sin(\frac{\pi l}{2})) - (0.5+0.1\cos(\frac{\pi l}{2}))x_2(l) - (3+0.01\sin(\frac{\pi l}{2}))y_2(l) + (1+0.02\cos(\frac{\pi l}{2}))\left(\frac{x_1(l)}{x_2(l)} - 1\right)\right]} \\ y_2(l+1) = y_2(l)e^{\left[(1.7+0.02\cos(\frac{\pi l}{2})) - (1.3+0.01\sin(\frac{\pi l}{2}))x_2(l) - (0.08+0.01\sin(\frac{\pi l}{2}))y_2(l) + (1+0.03\sin(\frac{\pi l}{2}))\left(\frac{y_1(l)}{y_2(l)} - 1\right)\right]} \end{cases} \quad (2.31)$$

has at least one 2π-periodic solution.

Proof. The assumptions (i), (ii) and (iii) are holds for the above system (2.31) from the main Theorem 2.1.
Thus, the Theorem 2.1 yields that the system (2.31) has at least one 2π-periodic solution. □

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