

A new PDE-based time dependent model for image restoration

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Abstract

In this paper, we present a new time-dependent model for image restoration. This model constructed by evolving the Euler-Lagrange equations of the optimization problem. We propose to apply prior smoothness on the solution image and then denoise it by minimizing the total variation norm of the estimated solution. The main idea is to apply a priori smoothness to the solution image. 2D numerical experimental results by explicit numerical schemes are discussed.

Key words: Total variation norm, image restoration, Lagrange's multiplier

I. Introduction

The total variation (TV) deblurring and denoising models are based on a variational problem with constraints using the TV norms as a nonlinear non-differentiable functional. The formulation of these models was first given by Rudin, Osher and Fatemi in [12] for denoising case and Rudin and Osher in [13] for the denoising and deblurring case. In spite of the fact that the variational problem is convex, the Euler Lagrange's equation is non-linear and ill-conditioned. Linear semi-implicit fixed point procedures devised by Vogel and Oman [19] and interior-point primal-dual implicit quadratic methods by Chain, Golub and Mulet [3], were introduced to solve the models. Those models give good results when treating pure denoising problem, but the models become highly ill-conditioned for the deblurring and denoising case.

In this paper, we have used Improvement in the signal quality (ISNR) to measure the goodness of restored image:

$$\text{ISNR} = 10 \log_{10} \left(\frac{\sum_{i,j}^n [u_{ij} - (u_0)_{ij}]^2}{\sum_{i,j}^n [u_{ij} - (u_{\text{new}})_{ij}]^2} \right) \text{dB}, \quad (1.2)$$

Where u_{new} is the restored image. That is, the value of ISNR is larger, the restored image is better.

In this paper, we present a new time-dependent model for image restoration. We have tested our algorithm on various types of images and found our model better than the previously known models. To quantify results, the experimental values in terms of ISNR are given in Figure 2 and Table 1.

II. Total variation based restoration algorithms

Image restoration is a fundamental problem in both image processing and computer vision with numerous application. Given in [12, 13] a blurry and noisy image $u_0 : \Omega \rightarrow \mathbb{R}$ is given by

$$u_0 = (k * u)(x, y) + n(x, y), \quad (2.1)$$

where Ω is a bounded open set in \mathbb{R}^2 , u_0 is the observed image, u is the original image, k is the PSF (point spread function) and also called blur kernel ($*$ stands for convolution) and n is additive white noise assumed to be close to Gaussian. The values $n(i, j)$ of n at the pixels (i, j) are independent random variables, each with a Gaussian distribution of zero mean and variance σ^2 .

In ref. [13] gave another models

$$u_0 = [(k * u)(x, y)][n(x, y)], \quad (2.2)$$

and

$$u_0 = (k * u)(x, y) + (u(x, y))(n(x, y)), \quad (2.3)$$

where n is multiplicative noise. Equation (2.2) and (2.3) have been treated in [22] and [13] respectively, for example, using homomorphic filtering, i.e., basically taking the log of u_0 and treating it as a problem involving additive noise, then filtering and applying the exponential. This is not appropriate for equation (2.3). We present below our restoration algorithms for (2.2) and (2.3). The total variation based image denoising model, which is based on the constrained minimization problem appeared in [12], is as follows:

$$\text{Minimize } \int_{\Omega} |\nabla u| dx dy = \int_{\Omega} \sqrt{u_x^2 + u_y^2} dx dy, \quad (2.4)$$

subject to constraints

$$\int_{\Omega} u dx dy = \int_{\Omega} u_0 dx dy, \quad (2.5)$$

$$\int_{\Omega} \frac{1}{2} (k * u - u_0)^2 dx dy = \sigma^2 \quad (2.6)$$

The first constraint corresponds to the assumption that the noise has zero mean and the second constraint uses a priori information that the standard deviation of the noise $n(x, y)$ is σ .

The Euler-Lagrange equation, see the references [12, 21, 23], by

$$0 = -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + \lambda k * (k * u - u_0), \quad (2.7)$$

in Ω , with $\partial u / \partial n = 0$ on the boundary of the domain.

Since (2.7) is not well defined at points where $\nabla u = 0$, due to the presence of the term $1/|\nabla u|$, it is common to slightly perturb the TV algorithm to become

$$\int_{\Omega} |\nabla u|_{\beta} dx dy = \int_{\Omega} \sqrt{u_x^2 + u_y^2 + \beta} dx dy \quad (2.8)$$

where β is a small positive parameter [4].

Rudin and Osher [13] for the denoising and deblurring model is given by

$$u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - \lambda k * (k * u - u_0), \quad (2.9)$$

for $t > 0$, $x, y \in \Omega$, $\partial u / \partial n = 0$ on the boundary of the domain.

The first constraint (2.5) is dropped because it is automatically enforced by the evolution procedure, i.e., the mean of $u(x, y, 0)$ is the same as that of $u_0(x, y)$, see references [12, 13]. As t increases, a restoration version of the image is realized. The projection step is the gradient projection method [11] which just amounts to updating $\lambda(t)$ so that (2.6) remains true in time. This follows [21] if we define

$$\lambda(t) = \frac{\int_{\Omega} \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \cdot k * (k * u - u_0) dx dy}{\int_{\Omega} (k * (k * u - u_0))^2 dx dy} \quad (2.10)$$

We thus have a dynamic procedure for restoring the image. As $t \rightarrow \infty$, the steady state solution is the desired restoration.

Next we consider restoration involving multiplicative noise [23]. We are given an image $u_0(x, y)$ where

$$u_0 = u n. \quad (2.11)$$

The unknown function $u(x,y)$ is the image we wish to restore, and n is Gaussian white noise.

We are given the following information

$$\int_{\Omega} n = 1, \quad (2.12)$$

$$\int_{\Omega} (n - 1)^2 = \sigma^2, \quad (2.13)$$

where the equation (2.12) has mean one and equation (2.13) given variance.

Thus the constrained optimization algorithm is (2.4) subject to the following constraints

$$\int \frac{u_0}{u} = 1 \quad (2.14)$$

$$\frac{1}{2} \int \left(\frac{u_0}{u} - 1 \right)^2 = \frac{\sigma^2}{2} = \frac{1}{2} \int \left(\left(\frac{u_0}{u} \right)^2 - 1 \right) \quad (2.15)$$

The gradient projection method leads to

$$u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - \lambda \frac{u_0^2}{u^3} + \mu \frac{u_0}{u^2}. \quad (2.16)$$

Next we consider images which are both blurry and noisy. The model (2.2) is as follows. We are given an image $u_0(x, y)$ for which

$$u_0 = (k * u)n. \quad (2.17)$$

The noise n is as above (2.12) and (2.14) are still satisfied.

The constraints are given by

$$\int \frac{u_0}{k * u} = 1. \quad (2.18)$$

$$\int \left(\frac{u_0}{k * u} \right)^2 = \frac{\sigma^2}{2} = \int \left(\frac{u_0}{k * u} \right)^2 - 1. \quad (2.19)$$

The gradient projection method leads to

$$u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - \lambda k * \left(\frac{u_0}{(k * u)^2} \right) \left(\frac{u_0}{(k * u)^2} - 1 \right) + \mu k * \left(\frac{u_0}{(k * u)^2} \right). \quad (2.20)$$

with $u(x,y,0)$ given as initial data (the original blurred and noisy image u_0 is taken as initial guess) and homogeneous Neumann boundary conditions $\partial u / \partial n = 0$.

Applying a priori smoothness on the solution image, our new time dependent model becomes,

$$u_t = \nabla \cdot \left(\frac{\nabla G_{\sigma} * u}{|\nabla G_{\sigma} * u|} \right) - \lambda k * \left(\frac{u_0}{(k * G_{\sigma} * u)^2} \right) \left(\frac{u_0}{(k * G_{\sigma} * u)^2} - 1 \right) + \mu k * \left(\frac{u_0}{(k * G_{\sigma} * u)^2} \right). \quad (2.21)$$

with $u(x,y,0)$ given as initial data (the original blurred and noisy image u_0 is taken as an initial guess) and homogeneous Neumann boundary conditions $\partial u / \partial n = 0$. It should be noticed that (2.21) only replaces u in (2.20) by its estimate $G_{\sigma} * u$.

Within [16] noticed that the convolution of the signal with Gaussians at each scale was equivalent to solving the heat equation with the signal as an initial datum. The term $(G_{\sigma} * \nabla u)(x, y, t) = (\nabla G_{\sigma} * u)(x, y, t)$, which appears inside the divergence term of (2.21), is simply the gradient of the solution at time σ of the heat equation with $u(x, y, 0)$ as an initial datum. Then the restoration analysis associated with u_0 consists in solving the problem,

$$\partial u(x, y, t) / \partial t = \Delta u(x, y, t), \quad u(x, y, 0) = u_0(x, y).$$

The solution of this equation at time t is given by

$$u(x, y, t) = G_\sigma * u_0,$$

where G_σ is the Gaussian function.

In order to preserve the notion of scale in the gradient estimate, it is convenient that this kernel G_σ depends on a scale parameter [5,6]. In fact, the function G_σ can be considered as “low-pass filter” or any smoothing kernel, i.e., a denoising technique is used before solving the nonlinear diffusion problem [1, 2].

We still write $G_\sigma * u$ as u . Let u_{ij}^n be the approximation to the value $u(x_i, y_j, t_n)$, where

$$x_i = i\Delta x, \quad y_j = j\Delta x, \quad i, j = 1, 2, \dots, N, \\ N\Delta x = 1, \quad t_n = n\Delta t, \quad n \geq 1,$$

where Δx , Δy and Δt are the spatial step sizes and the time step size respectively.

The explicit partial derivatives of models (2.20) and (2.21) can be expressed as:

$$u_t = \frac{u_{xx}(u_y^2 + \beta) - 2u_{xy}u_xu_y + u_{yy}(u_x^2 + \beta)}{(u_x^2 + u_y^2 + \beta)^{\frac{3}{2}}} - \lambda k * \left(\frac{u_0}{(k * u)^2} \right) \left(\frac{u_0}{(k * u)^2} - 1 \right) + \mu k * \left(\frac{u_0}{(k * u)^2} \right).$$

We define the derivative terms as,

$$u_{ij}^x = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x}, \quad u_{ij}^y = \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2\Delta x}, \\ u_{ij}^{xx} = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2}, \quad u_{ij}^{yy} = \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta x^2}, \\ u_{ij}^{xy} = \frac{u_{i+1,j+1}^n - u_{i-1,j+1}^n - u_{i+1,j-1}^n + u_{i-1,j-1}^n}{4\Delta x\Delta x}, \quad u_{ij}^t = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t}.$$

We let,

$$r_{ij}^n = u_{ij}^{xx}((u_{ij}^y)^2 + \beta) - 2u_{ij}^{xy}u_{ij}^xu_{ij}^y + u_{ij}^{yy}((u_{ij}^x)^2 + \beta), \quad (2.23)$$

and

$$p_{ij}^n = \sqrt{((u_{ij}^x)^2 + (u_{ij}^y)^2 + \beta)}. \quad (2.24)$$

Then (2.22) reads as follows:

$$u_{ij}^t = \frac{r_{ij}^n}{p_{ij}^n} - \lambda k * \left(\frac{u_0}{(k * u_{ij}^n)^2} \right) \left(\frac{u_0}{(k * u_{ij}^n)^2} - 1 \right) + \mu k * \left(\frac{u_0}{(k * u_{ij}^n)^2} \right), \quad (2.25)$$

with homogeneous Neumann boundary conditions.

III. Numerical implementation

We have used two gray scale images as shown in Figure 1. The pixel values of all images lie in interval $[0, 255]$. The Gaussian white noise is added by the normal imnoise function `imnoise(I1,'speckle', σ^2)` i.e., the mean M and variance σ^2 in Matlab. We first scale the intensities of the images into the range between zero and one before we begin our experiments. We have taken $\Delta t/\Delta x^2 < 0.5$ and Lagrange multiplier $= 0.85$ see reference [3,4,10].



Figure 1: Original Test Images used for different experiments (A) Lena: 256×256 , (B) Boat: 256×256 .



Figure 2: (a) Noisy Boat image $\sigma^2 = 0.002$, (b)-(c) corresponding denoised images by model (2.20) and (2.21) respectively, (d) Noisy Lena image $\sigma^2 = 0.002$, (e)-(f) corresponding denoised images by model (2.20) and (2.21) respectively.

Table 1. Results obtained by using models (2.20) and (2.21) applied to the images in Figures 2(a) and 3(a) with Gaussian white noise ($\sigma^2 = 0.002$).

Images	PSNR (Model-2.20)	Images	PSNR (Model-2.21)
Figure 2(c)	0.5406	Figure 2(d)	0.8609
Figure 2(e)	0.4504	Figure 2(f)	0.5511
No. of iterations	5	No. of iterations	5

IV. Concluding Remarks:

We have presented a new time-dependent model (2.21) to solve the nonlinear total variation problem for image restoration. The main idea is to apply a priori smoothness to the solution image. Nonlinear explicit schemes are used to discretize models (2.21) and (2.20). The model (2.21) gives larger ISNR values than that of model (2.20), at the same iteration numbers.

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