Stability And Hopf Branch of A FAST TCP/RED Network Congestion Model with Feedback Control

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***Abstract* — This paper mainly investigated aFAST TCP/RED network congestion model with a hybrid controller by using the control and bifurcation theory and discussed the effect of the communication delay on the stability. It is found that the hybrid controller can effectively delay the generation of Hopf branch and increase the stability of wireless network.Besides, the linear stability of the model and the local Hopf bifurcation are studied and we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. At last, some numerical simulation results are confirmed that the feasibility of the theoretical analysis.**

Keywords — *Hopf bifurcation,Stability,Normal form theory,Center manifold theorem,Congestion control*

I. INTRODUCTION

In recent years, with the rapid development of science and technology,the Internet congestion control becomes a serious problem in practical application.When the required resources exceed the network transmission capacity, it will cause congestion, which may lead to the loss of information and even the destruction of the whole system[1-3].Therefore, more and more people pay attention to the stability and dynamic characteristics of wireless network congestion control system.Many congestion control mechanisms are developed to avoid the system congestion and collapse[4-7].TCP and AQM are central to these congestion control mechanisms[8-10].According to the characteristic of differential equation dynamical system and the related theory of cybernetics，network propagation can be regarded as a nonlinear dynamic model with time-delay feedback regulation.

The FAST TCP system model is given in literature[11]，the specific model is as follows：

$$\begin{cases} \dot{w}(t)=\gamma(\dfrac{\alpha}{d+q(t)}-\dfrac{q(t)}{(d+q(t))^2}w(t), \\ \dot{p}(t)=\dfrac{1}{c}(\dfrac{w(t-\tau^f)}{d+p(t-R)}-c). \end{cases} \quad (1)$$

where $w(t)$ indicates the average value of the source side congestion window size, c is the queue capacity , $p(t)$ represents the queuing delay,and $q(t)=p(t-\tau^b)$ is the queuing delay observed by the source.It is should be noted that the congestion window value and queue delay value are both non-negative. The round trip time $R(t)=\tau^f+\tau^b$, where τ^f is the forward delay from source to link and τ^b is the backward delay in the feedback path from link to source, and d represents the constant round trip propagation time defined as the minimum achievable round trip delay. Thus, $R(t)=d+p(t-\tau^b)$. The parameter α ($\alpha>0$) is the number of the packets that each source attempts to maintain in the network

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buffers at equilibrium point; γ is the source control parameter with $\gamma \in (0,1]$.The congestion window $w(t)$ and the queuing delay $p(t)$ are non-negative.In literature [12], the author applied dynamics theory to analyze the local stability of the FAST TCP model.In literature[13],Liu used pulse control method to control the branching problem. We can take communication delay as a branching parameter and only considered the network topology of single source and single link.The following model is proposed.

$$\begin{cases} \dot{w}(t) = \gamma\left(\dfrac{\alpha}{d + p(t-R)} - \dfrac{p(t-R)}{(d+p(t-R))^2} w(t)\right), \\ \dot{p}(t) = \dfrac{1}{c}\left(\dfrac{w(t)}{d + p(t-R)} - c\right). \end{cases} \tag{2}$$

First, let the equilibrium point of model (2) be (w_0, p_0), so it satisfies the following equation:

$$\gamma\left(\frac{\alpha}{d + p_0} - \frac{p_0}{(d + p_0)^2} w_0\right) = 0 ; \frac{1}{c}\left(\frac{w_0}{d + p_0} - c\right) = 0$$

That is

$$w_0 = \alpha R/(R-d), \qquad \alpha / c = p_0 = R - d.$$

We can draw the following conclusion:For the no-control system (2),when

$$\omega_0 = \sqrt{\frac{b^2 - a^2 + \sqrt{(b^2 - a^2)^2 + 4c^2}}{2}}, R_0 = \frac{1}{\omega_0}\arctan\left[\frac{b\omega_0^2 + ac}{(c-ab)\omega_0}\right],$$

where $a = \dfrac{\gamma p_0}{R^2}$, $b = \dfrac{1}{R}$, $c = \dfrac{\gamma}{R^2}$

If $R < R_0$,the system is locally asymptotically stable at the equilibrium point.

If $R > R_0$,the system is unstable at the equilibrium point.

If $R = R_0$,the system generates Hopf bifurcation near the equilibrium point, resulting in periodic solution.

In this article, to delay the Hopf branch, we added a hybrid controller $(1-k)(w(t)-w_0)$ to the FAST TCP network congestion model,then model (2) becomes

$$\begin{cases} \dot{w}(t) = k\gamma\left(\dfrac{\alpha}{d + p(t-R)} - \dfrac{p(t-R)}{(d+p(t-R))^2} w(t)\right) + (1-k)(w(t)-w_0), \\ \dot{p}(t) = \dfrac{1}{c}\left(\dfrac{w(t)}{d + p(t-R)} - c\right). \end{cases} \tag{3}$$

Where k is the hybrid control parameter, appropriate control parameters can be selected to delay or even eliminate the generation of Hopf branches.

II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

In this section, we focus on the problems of the Hopf bifurcation and stability for the system(3).It is clear that the controlled system has the same equilibrium point as the system (2)

$$w_0 = \alpha R/(R-d), \qquad \alpha / c = p_0 = R - d. \tag{4}$$

Let $x_1(t) = w(t) - w_0$, $x_2(t) = p(t) - p_0$,then the linearized approximation equation corresponding to model (3) at the equilibrium point (w_0, p_0) is:

$$\begin{cases} \dot{x}_1(t) = a_1 x_1(t) + a_2 x_2(t-R), \\ \dot{x}_2(t) = b_1 x_1(t) + b_2 x_2(t-R). \end{cases} \tag{5}$$

where $a_1 = -\frac{k\gamma p_0}{R^2} + 1 - k$, $a_2 = -\frac{k\gamma\alpha d}{p_0 R^2}$, $b_1 = \frac{1}{Rc}$, $b_2 = -\frac{1}{R}$

The correspoding characteristic equation of system (3) is as follows.

$$\lambda^2 - a_1\lambda - b_2\lambda e^{-\lambda R} + (a_1 b_2 - b_1 a_2)e^{-\lambda R} = 0. \tag{6}$$

Obviously, because the roots of the characteristic equation (6) have a continuous dependence on the parameters λ , when the delay R gradually approaches zero, equation (6) has no negative real part roots.That is,the system is stable.When the delay R in equation (6) increases slowly and slightly larger than zero, as long as it is small enough, the system will remain stable.That is, there is a critical time delay $R = R_0$ so that the real part of the characteristic root of the characteristic equation (6) less than zero is true.We have the following lemma.

Lemma 1. *For the system (3), assume that* $\omega_0 R_0 < \frac{\pi}{2}$ *is satisfied.Then equation(6) has a pair of purely imaginary roots* $\lambda = \pm i\omega_0$ *when* $R = R_0$ *, where*

$$\omega_0 = \sqrt{\frac{b_2^{\ 2} - a_1^{\ 2} + \sqrt{(b_2^{\ 2} - a_1^{\ 2})^2 + 4(a_1 b_2 - b_1 a_2)^2}}{2}} \tag{7}$$

$$R_0 = \frac{1}{\omega_0}\arctan[\frac{b_2\omega_0^{\ 2} + a_1(a_1 b_2 - b_1 a_2)}{a_2 b_1 \omega_0}] \tag{8}$$

Proof.First, we assume that $\lambda = i\omega(\omega > 0)$ is a root of the characteristic equation (6), then it satisfies the following equation

$$-\omega^2 - i\omega a_1 - i\omega b_2 e^{-i\omega R} + (a_1 b_2 - b_1 a_2)e^{-i\omega R} = 0. \tag{9}$$

That is
$$-\omega^2 - i\omega a_1 + (a_1 b_2 - b_1 a_2 - i\omega b_2)(\cos\omega R - i\sin\omega R) = 0. \tag{10}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega^2 + (a_1 b_2 - b_1 a_2)\cos\omega R - \omega b_2 \sin\omega R = 0, \\ a_1\omega + (a_1 b_2 - b_1 a_2)\sin\omega R + \omega b_2 \cos\omega R = 0. \end{cases} \tag{11}$$

From (11) we obtain

$$\begin{aligned} &\omega^4 + a_1^{\ 2}\omega^2 = (a_1 b_2 - b_1 a_2)^2 + b_2^{\ 2}\omega^2 \\ &\omega^4 + (a_1^{\ 2} - b_2^{\ 2})\omega^2 - (a_1 b_2 - b_1 a_2)^2 = 0 \end{aligned} \tag{12}$$

So, we can get

$$\omega = \sqrt{\frac{b_2^{\ 2} - a_1^{\ 2} + \sqrt{(b_2^{\ 2} - a_1^{\ 2})^2 + 4(a_1 b_2 - b_1 a_2)^2}}{2}} \tag{13}$$

$$R=\frac{1}{\omega}\arctan[\frac{b_2\omega_0{}^2+a_1(a_1b_2-b_1a_2)}{a_2b_1\omega_0}+k\pi],k=0,1,2,\cdots \qquad (14)$$

Obviously, set $k=0$, then

$$\omega_0=\sqrt{\frac{b_2{}^2-a_1{}^2+\sqrt{(b_2{}^2-a_1{}^2)^4+4(a_1b_2-b_1a_2)^2}}{2}}\ ,\quad R_0=\frac{1}{\omega_0}\arctan[\frac{b_2\omega_0{}^2+a_1(a_1b_2-b_1a_2)}{a_2b_1\omega_0}]$$

As a result, when $R=R_0$, the characteristic equation (6) have a pair of purely imaginary root. This completes the proof.

Lemma 2.Let $\lambda(R)=\alpha(R)+i\omega(R)$ be the root of (6) with $\alpha(R_0)=0$ and $\omega(R_0)=\omega_0$ then we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{dR_0})^{-1}\big|_{R=R_0}>0$ is satisfied.

Proof. Multiply both sides of the characteristic equation (6) by $e^{\lambda R}$, and you get

$$(\lambda^2-a_1\lambda)e^{\lambda R}-b_2\lambda+(a_1b_2-b_1a_2)=0. \qquad (15)$$

By differentiating both sides of equation (15) with regard to R and applying the implicit function theorem, we have :

$$\frac{d\lambda}{dR}=\frac{(a_1\lambda-\lambda^2)\lambda e^{\lambda R}}{(2\lambda-a_1)e^{\lambda R}+(\lambda^2-a_1\lambda)e^{\lambda R}R-b_2},$$

Therefore,

$$(\frac{d\lambda}{dR})^{-1}=\frac{(2\lambda-a_1)e^{\lambda R}-b_2}{(a_1\lambda-\lambda^2)\lambda e^{\lambda R}}-\frac{R}{\lambda},$$

It can be obtained from equation (15) that

$$e^{\lambda R}=\frac{a_1b_2-b_1a_2-b_2\lambda}{a_1\lambda-\lambda^2}$$

So

$$\begin{aligned}\mathrm{Re}(\frac{d\lambda}{dR})^{-1}|_{\lambda=i\omega_0}&=\mathrm{Re}[\frac{2\lambda-a_1}{(a_1\lambda-\lambda^2)\lambda}]_{\lambda=i\omega_0}-\mathrm{Re}[\frac{b_2}{\lambda(a_1b_2-b_1a_2-b_2\lambda)}]_{\lambda=i\omega_0},\\&=\frac{2\omega_0{}^2+a_1{}^2}{a_1{}^2\omega_0{}^2+\omega_0{}^4}-\frac{b_2{}^2}{b_2{}^2\omega_0{}^2+(a_1b_2-b_1a_2)^2}\\&=\frac{b_2{}^2\omega_0{}^4+(2\omega_0{}^2+a_1{}^2)(a_1b_2-b_1a_2)^2}{(a_1{}^2\omega_0{}^2+\omega_0{}^4)[b_2{}^2\omega_0{}^2+(a_1b_2-b_1a_2)^2]}\end{aligned}$$

Obviously, $\mathrm{Re}(\frac{d\lambda}{dR_0})^{-1}\big|_{R=R_0}>0$,the proof is completed.

Lemma 3. For equation (6), when $R\in[0,R_0)$, all of its roots have negative real parts. The equilibrium

(w_0, p_0) is locally asymptotically stable, and system (3) produces a Hopf bifurcation at the equilibrium (w_0, p_0) when $R = R_0$.

By applying the Hopf bifurcation theorem for delayed differential equation and the three lemmas [14-15], we have the following results.

Theorem 1. For system (3), the following conclusions are true:
If $R < R_0$, the equilibrium point is asymptotically uniformly stable.
If $R = R_0$, model (3) generates Hopf branch at the equilibrium point (w_0, p_0).
If $R > R_0$, model (3) is unstable at the equilibrium point (w_0, p_0).

III. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In the analysis in the above section, we have obtained the conditions for the system to generate Hopf branch. In this section, by using the normal form theory and the center manifold theorem introduced in [16-17], we discuss the Hopf bifurcation direction and bifurcated periodic solution stability of model (3).

First, we consider the Taylor expansion of model (3) at the equilibrium point (w_0, p_0),

$$\begin{cases} \dot{x}_1(t) = a_1 x_1(t) + a_2 x_2(t-R) + a_3 x_1(t) x_2(t-R) + a_4 x_2{}^2(t-R) + \\ \qquad a_5 x_1(t) x_2{}^2(t-R) + a_6 x_2{}^3(t-R) + \cdots, \\ \dot{x}_2(t) = b_1 x_1(t) + b_2 x_2(t-R) + b_3 x_1(t) x_2(t-R) + b_4 x_2{}^2(t-R) + \\ \qquad b_5 x_1(t) x_2{}^2(t-R) + b_6 x_2{}^3(t-R) + \cdots. \end{cases} \tag{16}$$

Where $a_1 = -\dfrac{k\gamma\alpha}{cR^2} + 1 - k$, $a_2 = -\dfrac{k\gamma cd}{R^2}$, $a_3 = -\dfrac{k\gamma(\alpha - cd)}{cR^3}$, $a_4 = -\dfrac{2k\gamma cd}{R^3}$, $a_5 = \dfrac{k\gamma(2cd-\alpha)}{cR^4}$,
$a_6 = -\dfrac{3k\gamma cd}{R^4}$, $b_1 = \dfrac{1}{Rc}$, $b_2 = -\dfrac{1}{R}$, $b_3 = -\dfrac{1}{R^2 c}$, $b_4 = \dfrac{1}{R^2}$, $b_5 = \dfrac{1}{R^3 c}$, $b_4 = -\dfrac{1}{R^3}$

let $R = R_0 + \mu$, $u(t) = (x_1(t), x_2(t))^T$ and $u_t(\theta) = u(t+\theta), \theta \in [-R, 0)$, clearly, $\mu = 0$ is model(3) generates Hopf branches at R_0. Then the model (3) is equivalent to the following Functional Differential Equation (FDE) system

$$\dot{u}(t) = L_\mu + F(u_t, \mu). \tag{17}$$

$$L_\mu \varphi = B_1 \varphi(0) + B_2 \varphi(-R) \tag{18}$$

and

$$F(\mu, \varphi) = \begin{pmatrix} a_3 x_1(t) x_2(t-R) + a_4 x_2{}^2(t-R) + a_5 x_1(t) x_2{}^2(t-R) + a_6 x_2{}^3(t-R) + \cdots \\ b_3 x_1(t) x_2(t-R) + b_4 x_2{}^2(t-R) + b_5 x_1(t) x_2{}^2(t-R) + b_6 x_2{}^3(t-R) + \cdots \end{pmatrix}. \tag{19}$$

Where L_μ is the one family of bounded linear operator in $C([-R, 0], R^2) \to R^2$ and

$$\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^T \in C[-R, 0], B_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & a_2 \\ 0 & b_2 \end{pmatrix}.$$

Bythe Riesz representation theorem, there exists a bounded variation function $\eta(\theta, \mu) : [-R, 0] \to R^{2\times 2}$, such that

$$L_\mu \varphi = \int_{-R}^{0} d\eta(\theta,\mu)\varphi(\theta), \varphi \in C. \tag{20}$$

In fact, we can choose

$$\eta(\theta,\mu) = B_1\delta(\theta) + B_2\delta(\theta+R). \tag{21}$$

where $\delta(\theta)$ is a Delta function. For $\varphi \in C([-R,0])$, the operators A and R are defined as follow

$$A(\mu)\varphi(\theta) = \begin{cases} \dfrac{d(\varphi(\theta))}{d\theta}, & \theta \in [-R,0), \\ \int_{-R}^{0} d(\eta(\theta,\mu)\varphi(\theta)), & \theta = 0. \end{cases} \tag{22}$$

$$R(\mu)\varphi(\theta) = \begin{cases} 0, & \theta \in [-R,0), \\ F(\mu,\varphi), & \theta = 0. \end{cases} \tag{23}$$

Hence the equation (17) can be written as the following form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t \tag{24}$$

For $\psi \in C[0,R]$, we define the adjoint operator $A^*(0)$ of $A(0)$ as

$$A(\mu) * \psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,R], \\ \int_{-R}^{0} d(\eta^T(t,0)\psi(-t)), & s = 0. \end{cases} \tag{25}$$

For $\varphi(\theta) \in C[-R,0)$ and $\psi \in C[0,R]$, we define a Bilinear form

$$<\psi,\varphi> = \overline{\psi}^T(0)\varphi(0) - \int_{\theta=-R}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi. \tag{26}$$

where $\eta(\theta) = \eta(\theta,0)$

Lemma 4.The eigenvectors $q(\theta) = Ve^{i\omega_0\theta}$ and $q^*(s) = DV^*e^{-i\omega_0 s}$ are respectively the eigenvectors correspondingto the eigenvalues $i\omega_0$ and $-i\omega_0$ of $A(0)$ and $A^*(0)$, and

$$<q^*,q>=1, \quad <q^*,\overline{q}>=0,$$

where $V = (1, \dfrac{b_1}{i\omega_0 - b_2e^{-i\omega_0 R_0}})^T, V^* = (-\dfrac{b_1}{a_1 + i\omega_0})^T, \overline{D} = [\overline{V}^{*T}V - R_0e^{i\omega_0\theta}\overline{V}^{*T}B_2V]$.

proof. $\pm i\omega_0$ are the eigenvalues of $A(0)$, so they are also the eigenvalues of $A^*(0)$ .In order to determine the standard form of the operator $A(0)$,we assume that the eigenvectors $q(\theta)$ and $q_1^*(s)$ are respectively the eigenvectors correspondingto the eigenvalues $i\omega_0$ and $-i\omega_0$ of $A(0)$ and $A^*(0)$ .We can obtain

$$\begin{cases} A(0)q(\theta) = i\omega_0 q(\theta) \\ A^*(0)q_1^*(s) = i\omega_0 q_1^*(s) \end{cases} \tag{27}$$

From (20) and (22), (27) can be written as

$$\frac{dq(\theta)}{d\theta} = i\omega_0 q(\theta), \qquad \theta \in [-R,0). \qquad (28)$$
$$L_0 q(0) = i\omega_0 q(0), \qquad \theta = 0.$$

therefore $q(\theta) = q(0)e^{i\omega_0\theta}, \qquad \theta \in [-R,0].$

Where $q(0) = (q_1(0), q_2(0))^T \in C^2$ is a constant vector, obtained from (18), (27)

$$B_1 q(0) + B_2 e^{-i\omega_0 R_0} q(0) = i\omega_0 I q(0)$$

By direct calculate, we get $q(0) = \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{b_1}{i\omega_0 - b_2 e^{-i\omega_0 R_0}} \end{pmatrix}$

let $V = q(0)^T = (1, \dfrac{b_1}{i\omega_0 - b_2 e^{-i\omega_0 R_0}})$,

then $q(\theta) = Ve^{i\omega_0\theta}$

For non-zero vectors $q_1^*(s), s \in [0, R]$, we have $B_1^T q_1^*(0) + B_2^T e^{-i\omega_0 R_0} q_1^*(0) = i\omega_0 I q_1^*(0)$

Similarly $q_1^*(0) = \begin{pmatrix} \rho_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{b_1}{i\omega_0 + a_1} \\ 1 \end{pmatrix}$, let $V^* = q_1^*(0)^T = (-\dfrac{b_1}{i\omega_0 + a_1}, 1)$

then $q_1^*(s) = V^* e^{-i\omega_0 s}$,we make $q^*(s) = DV^* e^{-i\omega_0 s}$,

Now let's prove that $< q^*, q >= 1$ and $< q^*, q >= 1$, from equation (26), we get

$$\begin{aligned}
\langle q^*, q \rangle &= \bar{q}^{*T} q(0) - \int_{\theta=-R_0}^{0} \int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi - \theta) d\eta(\theta) q(\xi) d\xi. \\
&= \bar{D}[\bar{V}^{*T} V - \int_{\theta=-R_0}^{0} \int_{\xi=0}^{\theta} \bar{V}^{*T} e^{-i\omega_0(\xi-\theta)} d\eta(\theta) V e^{i\omega_0\xi} d\xi\} \\
&= \bar{D}[\bar{V}^{*T} V - \int_{\theta=-R_0}^{0} \bar{V}^{*T} [d\eta(\theta)] \theta e^{-i\omega_0\theta} V] \\
&= \bar{D}[\bar{V}^{*T} V - R_0 e^{i\omega_0\theta} \bar{V}^{*T} B_2 V].
\end{aligned} \qquad (29)$$

Let $\bar{D} = [\bar{V}^{*T} V - \tau_0 e^{i\omega_0\theta} \bar{V}^{*T} B_2 V]^{-1}$, we can get $< q^*, q >= 1$ .By $\langle \psi, A\varphi \rangle = \langle A^* \psi, \varphi \rangle$, we obtain

$$-i\omega_0 \langle q^*, \bar{q} \rangle = \langle q^*, A\bar{q} \rangle = \langle A^* q^*, \bar{q} \rangle = \langle -i\omega_0 q^*, \bar{q} \rangle = i\omega_0 \langle q^*, \bar{q} \rangle. \qquad (30)$$

So $< q^*, \bar{q} >= 0$. The proof is completed.

Next, we will use the same notations as in Hassard et al.[18], we firstcompute the coordinates to describe the centermanifold C_0 at $\mu = 0$. Define

$$z(t) = < q^*, u_t > \tag{31}$$

and
$$W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{32}$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t), \overline{z}(t), \theta) \tag{33}$$

Where $W(z,\overline{z},\theta) = W_{20}(\theta)\dfrac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\dfrac{\overline{z}^2}{2} + \cdots$.

For the central epidemic C_0, z and $\overline{z}$ respectively represent the local coordinates of the central epidemic in the direction of q and q^*. Note that W is real if u_t is real, therefore we onlyreal solutions. Since $\mu = 0$, it is easy to see that

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{\mu}_t \rangle = \langle q^*, (A(0) + R(0))\mu_t \rangle \\ &= < q^*, A\mu_t > + < q^*, R\mu_t > \\ &= i\omega_0 z + \overline{q}^{*T} f_0(z, \overline{z}).\end{aligned} \tag{34}$$

Let
$$\dot{z}(t) = i\omega_0 z + g(z, \overline{z}), \tag{35}$$

Where
$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{36}$$

from (24) and (36), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}\dot{q} = \begin{cases} AW - 2\,\mathrm{Re}\,\overline{q}^{*T}(0) f_0(z,\overline{z})q(\theta), & \theta \in [-R_0, 0), \\ AW - 2\,\mathrm{Re}\{\overline{q}^{*T}(0) f_0(z,\overline{z})q(\theta)\} + f_0(z,\overline{z}), & \theta = 0. \end{cases} \tag{37}$$

Which can be rewritten as

$$\dot{W} = AW + H(z, \overline{z}, \theta) \tag{38}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots \tag{39}$$

On the other hand, on C_0,

$$\dot{W} = W_z \dot{z} + W_{\overline{z}} \dot{\overline{z}} \tag{40}$$

Using (34) and (36) to replace W_z and $\dot{z}$ and their conjugates by their power series expansions, we obtain

$$\dot{W} = i\omega_0 W_{20}(\theta)z^2 - i\omega_0 W_{02}(\theta)\overline{z}^2 + \cdots. \tag{41}$$

Comparing the coefficients of the above equation with those of (39) and (41), we get

$$\begin{cases}(A-2i\omega_0)W_{20}(\theta)=-H_{20}(\theta),\\ AW_{11}(\theta)=-H_{11}(\theta),\\ (A+2i\omega_0)W_{02}(\theta)=-H_{02}(\theta).\end{cases} \tag{42}$$

Notice that $u_t=u(t+\theta)=W(z(t),\bar{z}(t),\theta)+zq+\overline{zq}$ and $q(\theta)=(1,\rho_1)^T e^{i\omega_0\theta}$, we get

$$u_t=\begin{pmatrix}x_1(t+\theta)\\ x_2(t+\theta)\end{pmatrix}=\begin{pmatrix}W^{(1)}(z,\bar{z},\theta)\\ W^{(2)}(z,\bar{z},\theta)\end{pmatrix}+z\begin{pmatrix}1\\ \rho_1\end{pmatrix}e^{i\omega_0\theta}+\bar{z}\begin{pmatrix}1\\ \bar{\rho}_1\end{pmatrix}e^{-i\omega_0\theta}. \tag{43}$$

so

$$x_1(t+\theta)=W^{(1)}(z,\bar{z},\theta)+ze^{i\omega_0\theta}+\bar{z}\,e^{-i\omega_0\theta}$$
$$=ze^{i\omega_0\theta}+\bar{z}\,e^{-i\omega_0\theta}+W_{20}^{(1)}(\theta)\frac{z^2}{2}+W_{11}^{(1)}(\theta)\,z\,\bar{z}+W_{02}^{(1)}(\theta)\frac{\bar{z}^2}{2}+\cdots$$

$$x_2(t+\theta)=W^{(2)}(z,\bar{z},\theta)+z\,\rho_1\,e^{i\omega_0\theta}+\bar{z}\bar{\rho}_1\,e^{-i\omega_0\theta}$$
$$=z\,\rho_1\,e^{i\omega_0\theta}+\bar{z}\bar{\rho}_1\,e^{-i\omega_0\theta}+W_{20}^{(2)}(\theta)\frac{z^2}{2}+W_{11}^{(2)}(\theta)\,z\,\bar{z}+W_{02}^{(2)}(\theta)\frac{\bar{z}^2}{2}+\cdots$$

Obviously,

$$\varphi_1(0)=z+\bar{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+\cdots$$

$$\varphi_2(-R_0)=z\rho_1e^{-i\omega_0R_0}+\bar{z}\,\bar{\rho}_1e^{i\omega_0R_0}+W_{20}^{(2)}(-R_0)\frac{z^2}{2}+W_{11}^{(2)}(-R_0)z\bar{z}+W_{02}^{(2)}(-R_0)\frac{\bar{z}^2}{2}+\cdots$$

$$\varphi_1(0)\varphi_2(-R_0)=\rho_1e^{-i\omega_0R_0}z^2+(\bar{\rho}_1e^{-i\omega_0R_0}+\rho_1e^{-i\omega_0R_0})z\bar{z}+\bar{\rho}_1e^{i\omega_0R_0}\bar{z}^2$$
$$+[W_{11}{}^{(2)}(-R_0)+\frac{1}{2}W_{20}{}^{(2)}(-R_0)+\frac{1}{2}\bar{\rho}_1e^{i\omega_0R_0}W_{20}{}^{(1)}(0)+\rho_1e^{-i\omega_0R_0}W_{11}{}^{(1)}(0)]z^2\bar{z}+\cdots$$

$$\varphi_2{}^2(-R_0)=\rho_1{}^2e^{-2i\omega_0R_0}z^2+\bar{\rho}_1{}^2e^{2i\omega_0R_0}\bar{z}^2+2\rho_1\bar{\rho}_1z\bar{z}+$$
$$[\bar{\rho}_1e^{i\omega_0R_0}W_{20}{}^{(2)}(-R_0)+2\rho_1e^{-i\omega_0R_0}W_{11}{}^{(2)}(-R_0)]z^2\bar{z}+\cdots$$

$$\varphi_1(0)\varphi_2{}^2(-R_0)=(\rho_1{}^2e^{-2i\omega_0R_0}+2\rho_1\bar{\rho}_1)z^2\bar{z}+\cdots$$

$$\varphi_2{}^3(-R_0)=3\rho_1{}^2\bar{\rho}_1e^{-i\omega_0R_0}z^2\bar{z}+\cdots$$

From the (35) and (36), we obtain

$$f_0(z,\bar{z})=\begin{pmatrix}K_1z^2+K_2z\bar{z}+K_3\bar{z}^2+K_4z^2\bar{z}\\ K_5z^2+K_6z\bar{z}+K_7\bar{z}^2+K_8z^2\bar{z}\end{pmatrix}$$

where

$$K_1=a_3\rho_1e^{-i\omega_0R_0}+a_4\rho_1{}^2e^{-2i\omega_0R_0}$$

$$K_2 = a_3(\bar{\rho}_1 e^{i\omega_0 R_0} + \rho_1 e^{-i\omega_0 R_0}) + 2a_4 \rho_1 \bar{\rho}_1$$

$$K_3 = a_3 \bar{\rho}_1 e^{i\omega_0 R_0} + a_4 \bar{\rho}_1{}^2 e^{2i\omega_0 R_0}$$

$$K_4 = a_3[W_{11}{}^{(2)}(-R_0) + \frac{1}{2}W_{20}{}^{(2)}(-R_0) + \frac{1}{2}\bar{\rho}_1 e^{i\omega_0 R_0} W_{20}{}^{(1)}(0) + \rho_1 e^{-i\omega_0 R_0} W_{11}{}^{(1)}(0)] + a_4[\bar{\rho}_1 e^{i\omega_0 R_0} W_{20}{}^{(2)}(-R_0) + 2\rho_1 e^{-i\omega_0 R_0} W_{11}{}^{(2)}(-R_0)] + a_5(\rho_1{}^2 e^{-2i\omega_0 R_0} + 2\rho_1 \bar{\rho}_1) + 3a_6 \rho_1{}^2 \bar{\rho}_1 e^{-i\omega_0 R_0}$$

$$K_5 = b_3 \rho_1 e^{-i\omega_0 R_0} + b_4 \rho_1{}^2 e^{-2i\omega_0 R_0}$$

$$K_6 = b_3(\bar{\rho}_1 e^{i\omega_0 R_0} + \rho_1 e^{-i\omega_0 R_0}) + 2b_4 \rho_1 \bar{\rho}_1$$

$$K_7 = b_3 \bar{\rho}_1 e^{i\omega_0 R_0} + b_4 \bar{\rho}_1{}^2 e^{2i\omega_0 R_0}$$

$$K_8 = b_3[W_{11}{}^{(2)}(-R_0) + \frac{1}{2}W_{20}{}^{(2)}(-R_0) + \frac{1}{2}\bar{\rho}_1 e^{i\omega_0 R_0} W_{20}{}^{(1)}(0) + \rho_1 e^{-i\omega_0 R_0} W_{11}{}^{(1)}(0)] + b_4[\bar{\rho}_1 e^{i\omega_0 R_0} W_{20}{}^{(2)}(-R_0) + 2\rho_1 e^{-i\omega_0 R_0} W_{11}{}^{(2)}(-R_0)] + b_5(\rho_1{}^2 e^{-2i\omega_0 R_0} + 2\rho_1 \bar{\rho}_1) + 3b_6 \rho_1{}^2 \bar{\rho}_1 e^{-i\omega_0 R_0}$$

From $\bar{q}^{*T}(0) = \bar{D}(\bar{\rho}_2, 1)$, we obtain

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^{*T}(0) f_0(z,\bar{z}) \\ &= \bar{D}(\bar{\rho}_2, 1)\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2 \bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7 \bar{z}^2 + K_8 z^2 \bar{z} \end{pmatrix} \\ &= \bar{D}[(\bar{\rho}_2 K_1 + K_5) z^2 + (\bar{\rho}_2 K_2 + K_6) z\bar{z} + (\bar{\rho}_2 K_3 + K_7)\bar{z}^2 + (\bar{\rho}_2 K_4 + K_8) z^2 \bar{z}] \end{aligned}$$

Comparing the cofficients of the above equation with those in (36), we get

$$\begin{aligned} g_{20} &= 2\bar{D}(\bar{\rho}_2 K_1 + K_5), g_{11} = \bar{D}(\bar{\rho}_2 K_2 + K_6), \\ g_{02} &= 2\bar{D}(\bar{\rho}_2 K_3 + K_7), g_{20} = 2\bar{D}(\bar{\rho}_2 K_4 + K_8). \end{aligned} \tag{44}$$

In order to determine the value of g_{21}, we also need to compute the values of $W_{20}(\theta)$ and $W_{11}(\theta)$, from $\theta \in [-R_0, 0)$, we obtain

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(\theta)] \\ &= -(g_{20}(\theta)\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(\theta) \\ &\quad -(\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(\theta). \end{aligned} \tag{45}$$

Comparing the coefficients with (39), we gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20} q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11} q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{46}$$

When $\theta = 0$, we have

$$H(z,\bar{z},0) = -2\operatorname{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z})q(0)] + f_0(z,\bar{z})$$
$$= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots)q(0)$$
$$-(\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(0) + \begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7\bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix}.$$

Comparing the coefficients with (45), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\begin{pmatrix} K_1 \\ K_5 \end{pmatrix},$$
$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} K_2 \\ K_6 \end{pmatrix}. \tag{47}$$

Using (42), (46), we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2. \tag{48}$$

Where $E_1 = (E_1^{(1)}, E_1^2) \in R^2$ and $E_2 = (E_2^{(1)}, E_2^2) \in R^2$ are two two-dimensional vectors.

From the definition of $A(0)$ and (42) , we have

$$\int_{-R_0}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0 W_{20}(0) - H_{20}(0)$$

$$\int_{-R_0}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0)$$

and

$$(i\omega_0 I - \int_{-R_0}^{0} e^{i\omega_0\theta} d\eta(\theta))q(0) = 0$$

$$(-i\omega_0 I - \int_{-R_0}^{0} e^{-i\omega_0\theta} d\eta(\theta))\bar{q}(0) = 0.$$

Hence, we can get

$$(2i\omega_0 I - \int_{-R_0}^{0} e^{2i\omega_0} d\eta(\theta))E_1 = 2\begin{pmatrix} K_1 \\ K_5 \end{pmatrix}$$

$$(\int_{-R_0}^{0} d\eta(\theta))E_2 = -\begin{pmatrix} K_2 \\ K_6 \end{pmatrix}$$

Therefore, we have

$$\begin{cases} \begin{pmatrix} i2\omega_0 - a_1 & -a_2 e^{-2i\omega_0 R_0} \\ -b_1 & 2i\omega_0 - b_2 e^{-2i\omega_0 R_0} \end{pmatrix}\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = 2\begin{pmatrix} K_1 \\ K_5 \end{pmatrix} \\ \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = -\begin{pmatrix} K_2 \\ K_6 \end{pmatrix} \end{cases} \tag{49}$$

By calculation we can get

$$E_1^{(1)} = \frac{(2a_1K_5 - 2a_2K_1)e^{-i2\omega_0 R_0} + i4\omega_0 K_1}{(a_1b_2 - a_2b_1 - i2\omega_0 b_2)e^{-i2\omega_0 R_0} - 4\omega_0^2 - i2\omega_0 a_1}$$
$$E_1^{(2)} = \frac{2b_1K_1 - 2a_1K_5 + i4\omega_0 K_5}{(a_1b_2 - a_2b_1 - i2\omega_0 b_2)e^{-i2\omega_0 R_0} - 4\omega_0^2 - i2\omega_0 a_1}. \quad (50)$$

And

$$E_2^{(1)} = \frac{b_2K_1 - a_2K_6}{a_2b_1 - a_1b_2},$$
$$E_2^{(2)} = \frac{a_1K_6 - b_1K_2}{a_2b_1 - a_1b_2}. \quad (51)$$

Based on the above analysis, next we determine several important values of the Hopf periodic solution properties at the critical value R_0 [19]:

$$C_1(0) = \frac{i}{2\omega_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \mu_2 = -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(0)\}}, \beta_2 = 2\,\mathrm{Re}\{C_1(0)\},$$

$$T_2 = -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2(\mathrm{Im}\{\lambda'(0)\})}{\omega_0}. \quad (52)$$

Theorem 2. In the case of system (3), the conclusion holds [20]

a) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical . If $\mu_2 < 0$, the Hopf bifurcation is subcritical .

b) β_2 determines the stability of the bifurcating periodic solution. If $\beta_2 < 0$, the bifurcating periodic solutions is stable; if $\beta_2 > 0$, the bifurcating periodic solutions is unstable.

c) The period of the bifurcating periodic solution is decided by the parameter T_2. If $T_2 > 0(< 0)$, the period increases(decreases).

IV. NUMERICAL SIMULATION

In this section, we present numerical results to confirm the analytical predictions obtained in the previous section. For system (3) , We take the parameters:

$$\alpha = 1100; \quad \gamma = 0.8; \quad C = 50000 \ packet/s$$

According to the previous analysis, for the original system (2), we can draw the relation diagram of critical time delay R_0 and R (see Fig1).When $R < 0.07236$, $R < R_0$ shows that the original system is stable, that is to say, When $R_C = 0.07236$ the Hopf branch is generated. If we choose $R = 0.072$, we get that

$$W_0 = 3600; \ p_0 = 0.022; a = 3.3951; b = 13.8889; c = 154.321;$$

$$\omega_0 = 16.422; \ R_0 = 0.0718$$

Because of $R = 0.072 > R_0$,the equilibrium point (w_0, p_0) of the system (2) loses its stability and the system is unstable(see Fig 2).Next, verify the control effect and select the above parameters again.By selecting an appropriate hybrid control factor $k = 0.8$,we can calculate the critical value $R_C = 0.081$ of R in system (3) .When $R = 0.072$,we get that $W_0 = 3600;\ p_0 = 0.022$ .Because of $R = 0.072 < R_0$,the equilibrium point (w_0, p_0) of the system (3) is asymptotically stable which proved by numerical simulations (see Fig 3.).But as R continues to increase,such as $R = 0.1 > R_0$ .At this point, the FAST TCP network congestion model with the hybrid controller added still generates Hopf branches.The system loses stability and produces limit cycles (see Fig 4.).Therefore, the Hopf branch can be delayed by selecting a suitable hybrid control factor.

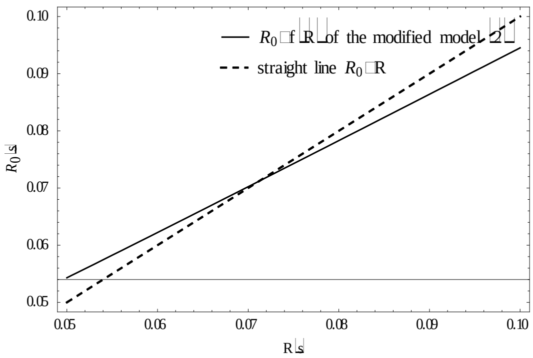

Figure 1 Relationship curve between R_0 and R

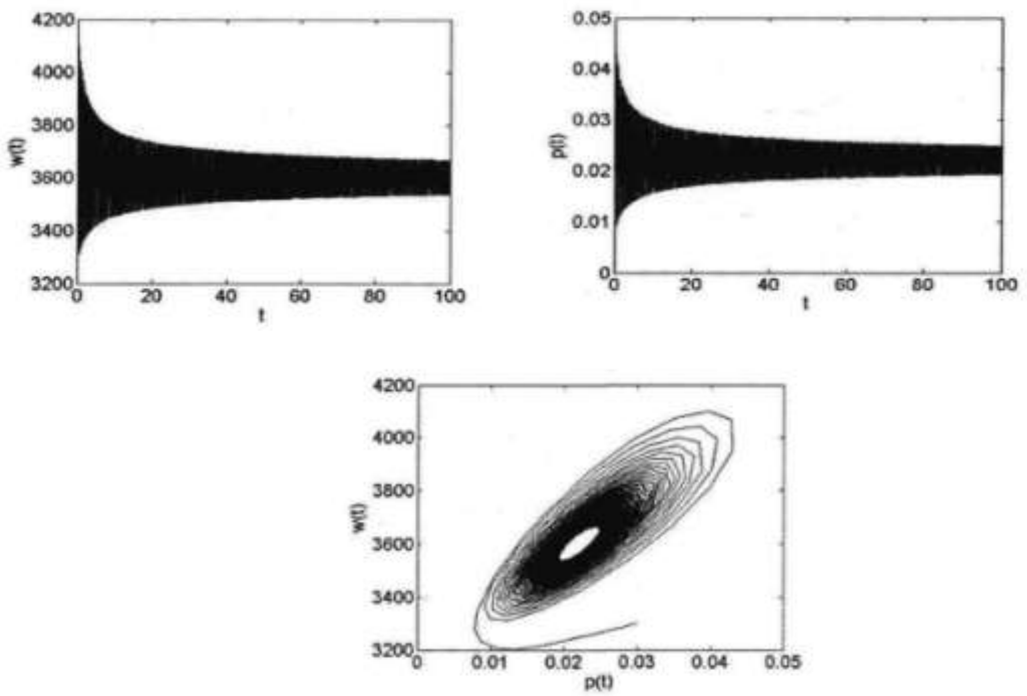

Figure 2 State and Phase plot of $w(t)$ and $\text{p}(t)$ with $R = 0.072$.

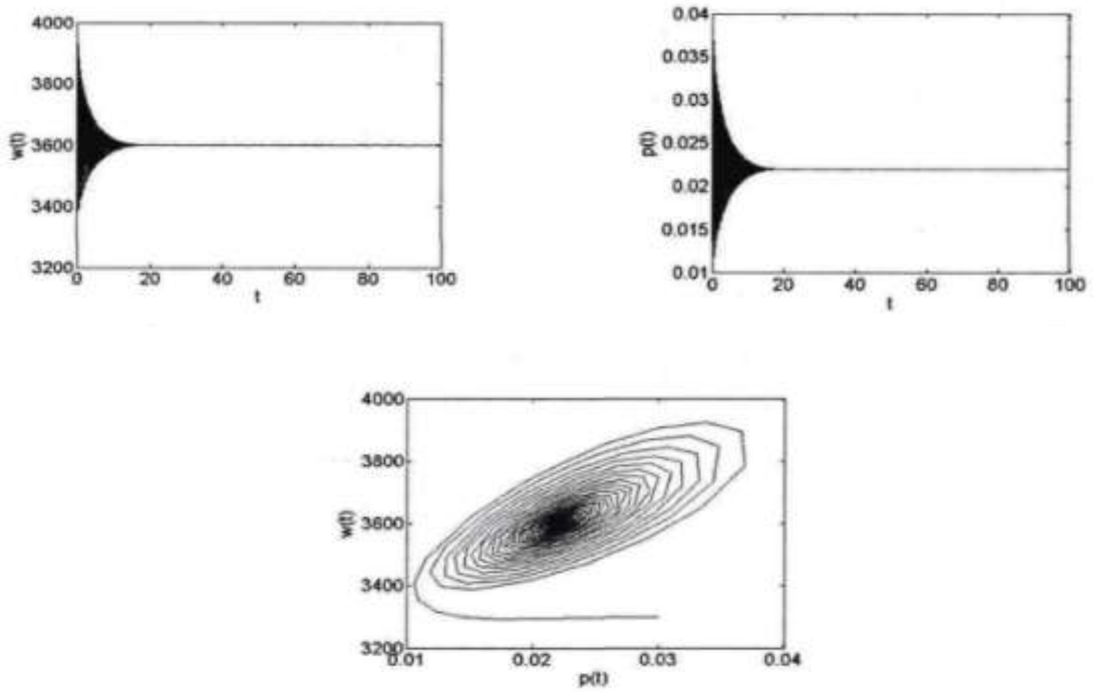

Figure 3 State and Phase plot of $w(t)$ and $\mathrm{p}(t)$ of the FAST TCP network congestion modelwith the hybrid controller with $R = 0.072$

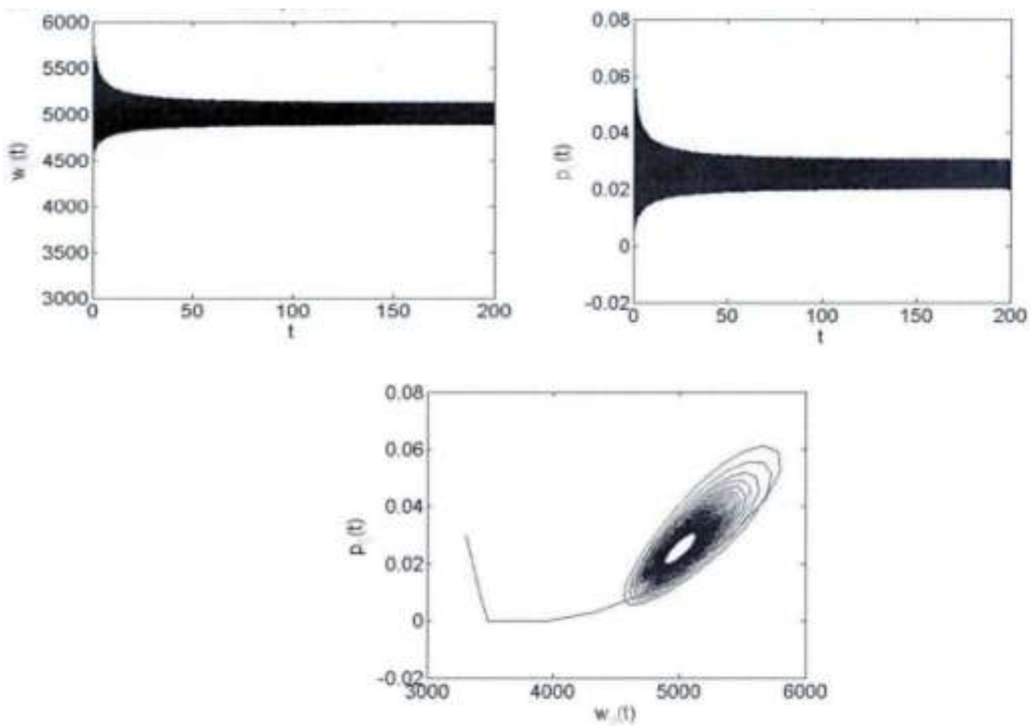

Figure 4 State and Phase plot of $w(t)$ and $\mathrm{p}(t)$ of the FAST TCP network congestion model with the hybrid controller with $R = 0.1$

V. CONCLUSIONS

Based on the FAST TCP system model, a FAST TCP network congestion model with hybrid controller is studied in this paper.By applying the control and bifurcation theory, the critical value of communication delay that keeps the original system stable can be obtained.When the time delay of the system exceeds this critical value, stability is lost at the equilibrium point and Hopf branch is generated.By choosing the communication delay as the bifurcation parameter, we can obtain the critical value of the communication delay that keeps the controlled system stable.When branch still take the above parameters, the controlled control system at the equilibrium point is stable. When time delay increases to a large, the system still will be blocked, even collapse. Numerical simulation results verify the validity of the theoretical analysis.So by choosing appropriate control parameters can effectively delay the Hopf branch production and increase the stability of the wireless network.

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