Lie Nilpotent Group Algebras: A Survey

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1 Introduction

Let R be any ring with unity. Then R can be treated as a Lie ring under the Lie multiplication $[x, y] = xy - yx; x, y \in R$. This ring is denoted by $L(R)$ and is called associated Lie ring of R.

If $R = KG$ is the group algebra of a group G over a field K, then $L(KG)$ is called the associated Lie algebra of the group algebra KG. Let $KG^{[1]} = KG$ and for $n > 1$, the nth lower Lie power $KG^{[n]}$ of KG is the associative ideal generated by all the Lie commutators $[x_1, x_2, ..., x_n]$, where $x_1, x_2, ..., x_n \in KG$. By induction, we define the nth strong Lie power $KG^{(n)}$ of KG as the associative ideal generated by all the Lie commutators $[x, y]$, where $KG^{(1)} = KG$, $x \in KG^{(n-1)}$ and $y \in KG$. Group algebra KG is called Lie nilpotent (strongly Lie nilpotent) if $KG^{[n]} = 0$ ($KG^{(n)} = 0$) for some $n \in N$ and the least non negative integer n such that $KG^{[n]} = 0$ ($KG^{(n)} = 0$) is called the Lie nilpotency index (strong Lie nilpotency index) of KG and is denoted by $t_L(KG)$ ($t^L(KG)$). It is easy to see that $t_L(KG) \leq t^L(KG)$. In [29], Gupta and Levin have given an example of an algebra of characteristic 2 which is Lie nilpotent of index 3 but not strongly Lie nilpotent. For any group algebra KG the augmentation ideal $\Delta(G)$ of KG is nilpotent if and only if $Char(K) = p > 0$ and G is a finite p group. It is well known that if $|G| = p^n$, then $1 + n(p-1) \leq t(G) \leq p^n$, where $t(G)$ denotes the nilpotency index of $\Delta(G)$, with equality on the left and right hand side iff G is elementary abelian and cyclic, respectively.

Let KG be the group algebra and let m be a positive integer. The mth lower Lie dimension subgroup of G is the normal subgroup of G defned as

$$D_{[m]}(G) := G \cap (1 + KG^{[m]})$$

whereas the mth upper Lie dimension subgroup of G is the normal subgroup of G defned as

$$D_{(m)}(G) := G \cap (1 + KG^{(m)})$$

By [9, p.48,Theorem 2.8], we have

$$D_{(m+1),K}(G) = \begin{cases} G & \text{if } m = 0 \\ G' & \text{if } m = 1 \\ (D_{(m),K}(G), G)(D_{(\lceil \frac{m}{p} \rceil + 1),K}(G))^p & \text{if } m \geq 2 \end{cases}$$

Here$\lceil \frac{m}{p} \rceil$ is the upper integral part of $\frac{m}{p}$. These subgroups play an important role in the computation of strong Lie nilpotency indices.

If KG is Lie nilpotent such that $|G'| = p^n$, then according to Jennings' theory [16],

$$t^L(KG) = 2 + (p-1)\left(\sum_{m\geq 1} md_{(m+1)}\right) \tag{1}$$

where $p^{d_{(m)}} = [D_{(m),K}(G) : D_{(m+1),K}(G)]$. It is easy to see that $\sum_{m\geq 2} d_{(m)} = n$.
The aim of this article is to survey results on Lie nilpotent group algebras.

2 Results on Lie Nilpotent Group Algebras

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$.
In 1973, Passi, Passman and Sehgal [22] classified Lie nilpotent group algebras. They proved that for a non commutative group algebra KG the following are equivalent

- KG is Lie nilpotent;
- KG is strongly Lie nilpotent;
- $Char(K) = p > 0$, G is nilpotent and its commutator subgroup G' is a finite p group.

In [19, 9], it is proved that if KG is Lie nilpotent then $t_L(KG) \leq t^L(KG) \leq |G| + 1$. Moreover, according to [1], if $Char(K) \geq 5$, then $t_L(KG) = t^L(KG)$. But the question when $t_L(KG) = t^L(KG)$ for $Char(K) = 2, 3$ is still open. Using the program packages GAP and LAGUNA (see [5, 9]), A.Konovalov [27] verified that $t_L(KG) = t^L(KG)$ for all $2-$groups of order at most 256 and $Char(K) = 2$. Several important results on this topic were obtained in [5].

Shalev [25], started the study of Lie nilpotent group algebras having maximal Lie nilpotency index $|G'| + 1$ and proved that if G is a finite p group and $Char(K) \geq 5$, then $t_L(KG)$ is maximal if and only if G' is cyclic.
After that Bovdi and Spinelli [26] completed the characterization by proving the following results

Theorem 2.1. *[26] Let KG be a Lie nilpotent group algebra of a group G over a field K with characteristic $p > 0$. Then $t_L(KG) = |G'| + 1$ if and only if one of the following conditions holds:*

1. *G' is cyclic;*
2. *$p = 2, G' \cong C_2 \times C_2$ and $\gamma_3(G) \neq 1$.*

Moreover, $t_L(KG) = |G'| + 1$ *if and only if* $t^L(KG) = |G'| + 1$.

Working in the same direction Bovdi, Juhasz and Spinelli [3], classified Lie nilpotent group algebras having almost maximal Lie nilpotency index i.e. $t^L(KG) = |G'| - p + 2$. They proved that

Theorem 2.2. *Let* KG *be a Lie nilpotent group algebra over a field* K *of positive characteristic* p. *Then* KG *has upper almost maximal Lie nilpotency index i.e.* $t^L(KG) = |G'| - p + 2$ *if and only if one of the following conditions holds:*

1. $p = 2$ *and* G' *is a group of one of the following types*
   (a) G' *is noncyclic of order* 4 *and* $cl(G) = 2$ *;*
   (b) $G' \cong C_4 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2$ *and* $cl(G) = 4$*;*
   (c) G' *is elementary abelian of order* 8 *and* $cl(G) = 4$ *;*
2. $p = 3$ *and* G' *is elementary abelian of order* 9 *and* $cl(G) = 3$.

Continuing this in 2006 [2], Bovdi proved that

Theorem 2.3. *Let* KG *be a Lie nilpotent group algebra over a field* K *of positive characteristic* p. *Then* KG *has lower almost maximal Lie nilpotency index if and only if one of the following conditions holds:*

1. $p = 2$ *and* G' *is a group of one of the following types*
   (a) G' *is noncyclic of order* 4 *and* $cl(G) = 2$ *;*
   (b) $G' \cong C_4 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2$ *and* $cl(G) = 4$*;*
   (c) G' *is elementary abelian of order* 8 *and* $cl(G) = 4$ *;*
2. $p = 3$ *and* G' *is elementary abelian of order* 9 *and* $cl(G) = 3$.

Hence, in the same paper it is also proved that the group algebra KG has lower almost maximal Lie nilpotency index if and only if it has upper almost maximal Lie nilpotency index.

In 2010 [7], Bovdi and Srivastava focused on Lie nilpotent group algebras with $t^L(KG) = |G'| - 2p + 3, |G'| - 3p + 4$ and $|G'| - 4p + 5$. In the same paper they also proved that for $k = |G'| - 2p + 3, |G'| - 3p + 4$, and $|G'| - 4p + 5$, $t^L(KG) = k$ if and only if $t_L(KG) = k$.

Continuing this study, Sharma, Sahai and Siwach [11, 14] have obtained classification for Lie nilpotent group algebras KG with $t^L(KG) = |G'| - 5p + 6$, $|G'| - 6p + 7$, $|G'| - 7p + 8$, $|G'| - 8p + 9$, $|G'| - 9p + 10$, $|G'| - 10p + 11$, $|G'| - 11p + 12$, $|G'| - 12p + 13$ and $|G'| - 13p + 14$.

On the other hand, Sharma and Bist in [19] began to study the question when do the Lie nilpotent group algebras KG have minimal Lie nilpotency index $p + 1$ i.e. $t_{(L)}(KG) = p + 1$. In the same paper they also proved that $t_L(KG) = p + 1$ if and only if $t^L(KG) = p + 1$.

Shalev[25], investigated Lie nilpotent group algebras whose Lie nilpotency indices are next lower, namely $2p$ and $3p-1$ for $p \geq 5$. This classification was completed by Sahai [23]. She has completed this characterization by classifying group algebras which are Lie nilpotent having Lie nilpotency indices $2p$, $3p-1$, $4p-2$ for $p = 2, 3$ also. She proved the following results:

Theorem 2.4. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p \geq 0$. Then the following are equivalent:*

- $t_L(KG) = 2p$;
- $t^L(KG) = 2p$;
- $G' = C_p \times C_p$, $\gamma_3(G) = 1$.

Theorem 2.5. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p \geq 0$. Then $t^L(KG) = 3p-1$ if and only if one of the following conditions holds:*

- $G \cong C_p \times C_p \times C_p$, $\gamma_3(G) = 1$;
- $G \cong C_p \times C_p$, $\gamma_3(G) \cong C_p$, $\gamma_4(G) = 1$;
- $p = 2$, $G' \cong C_4$.

Moreover, $t^L(KG) = 3p-1$ if and only if $t_L(KG) = 3p-1$.

Theorem 2.6. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p \geq 0$. Then $t^L(KG) = 4p-2$ if and only if one of the following conditions holds:*

- $G \cong C_p \times C_p \times C_p \times C_p$, $\gamma_3(G) = 1$;
- $G \cong C_p \times C_p \times C_p$, $\gamma_3(G) \cong C_p$, $\gamma_4(G) = 1$;
- $p = 3$, $G \cong C_9$;
- $p = 2$, $G \cong C_4 \times C_2, \gamma_3(G) = G'^2$, $\gamma_4(G) = 1$.

Moreover, $t^L(KG) = 4p-2$ if and only if $t_L(KG) = 4p-2$.

Sahai and Sharan extended this work and in [21, 28] classified group algebras which are Lie nilpotent with $t^L(KG) = 5p-3, 6p-4, 7p-5, 8p-6$, and $9p-7$.

On the other side, Chandra and Sahai [8] characterized group algebras which are strongly Lie nilpotent of index atmost 8 i.e. $t^L(KG) \leq 8$. It is easy to see that if $t^L(KG) = 2$, then G is abelian. So they started with $t^L(KG) = 3$.

Theorem 2.7. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p > 0$. Then $t^L(KG) = 3$ if and only if $p = 2$, $G' \cong C_2$ and $\gamma_3(G) = 1$.*

Moreover, if $t^L(KG) = 3$, then $t_L(KG) = 3$.

Theorem 2.8. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p > 0$. Then $t^L(KG) = 4$ if and only if one of the following conditions holds:*

- $p = 3$, $G' \cong C_3$ *and* $\gamma_3(G) = 1$;
- $p = 2$, $G' \cong C_2 \times C_2$ *and* $\gamma_3(G) = 1$.

Moreover, if $t^L(KG) = 4$, then $t_L(KG) = 4$.

Theorem 2.9. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p > 0$. Then $t^L(KG) = 5$ if and only if one of the following conditions holds:*

- $p = 2$, $G' \cong C_4$ *and* $\gamma_4(G) = 1$;
- $p = 2$, $G' \cong C_2 \times C_2$, $\gamma_3(G) \neq 1$ *and* $\gamma_4(G) = 1$;
- $p = 2$, $G' \cong C_2 \times C_2 \times C_2$, *and* $\gamma_3(G) = 1$.

Moreover, if $t^L(KG) = 5$, then $t_L(KG) = 5$.

Theorem 2.10. *Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p > 0$. Then $t^L(KG) = 6$ if and only if one of the following conditions holds:*

- $p = 5$, $G' \cong C_5$ *and* $\gamma_3(G) = 1$;
- $p = 3$, $G' \cong C_3 \times C_3$, $\gamma_3(G) = 1$;
- $p = 2$ *and* G' *is a group of one of the following types:*
  1. $G' \cong C_2 \times C_2 \times C_2, \gamma_3(G)$ *cyclic and* $\gamma_4(G) = 1$;
  2. $G' \cong C_4 \times C_2$, $G'^2 = \gamma_3(G)$ *and* $\gamma_4(G) = 1$;
  3. $G' \cong C_2 \times C_2 \times C_2 \times C_2$ *and* $\gamma_3(G) = 1$.

Moreover, if $t^L(KG) = 6$, then $t_L(KG) = 6$.

Results for $t^L(KG) = 7, 8$ can be seen in [8]. Also they have proved that for $k \leq 6$, $t^L(KG) = k$ if and only if $t_L(KG) = k$. Working in the same direction Siwach, Sharma and Sahai [12, 15] have classified group algebras KG which are strongly Lie nilpotent of index $9, 10, 11, 12$ and 13. Recently Bhatt, Chandra and Sahai [30] have classified Lie nilpotent group algebras with nilpotency index 14.

As we have discussed earlier that group algebras with $t_L(KG) \leq 6$, have already been discussed in [8]. In 2015, Sahai extended this work and characterized the Lie nilpotent group algebras with $t_L(KG) = 7$ or 8. She proved the following results:

Theorem 2.11. *[23] Let KG be a Lie nilpotent group algebra of a non-abelian group G over a field K of characteristic $p \geq 0$ such that $t_L(KG) = 7$ or 9. Then $p = 2$.*

Theorem 2.12. *[23]Let KG be a Lie nilpotent group algebra of a non-abelian group G over a field K of characteristic $p \geq 3$ such that $t_L(KG) = 8$. Then $t^L(KG) = 8$.*

Theorem 2.13. *[23] Let KG be a Lie nilpotent group algebra of a non-abelian group G over a field K of characteristic $p \geq 3$ such that $t_L(KG) = 10$. Then $t^L(KG) = 10$.*

After that Sahai and Sharan [24] carried forward this work and proved the following results

Theorem 2.14. *[24] Let G be a group and let K be a field of characteristic $p > 0$ such that KG is Lie nilpotent. Then $t_L(KG) = 7$ if and only if $p = 2$ and one of the following conditions hold:*

- $G' \cong C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) = 1$;
- $G' \cong C_4 \times C_2 \times C_2$, $\gamma_3(G) \subseteq G'^2$;
- $G' = C_2 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) \cong C_2$;
- $G' \cong C_4 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_2 \times C_2$, $\gamma_4(G) = 1$;
- $G' \cong C_4 \times C_2$, $\gamma_3(G) \cong C_2$, $G'^2 \cap \gamma_3(G) = 1$;
- $G' \cong C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2$, $\gamma_3(G) = 1$.

Moreover, $t_L(KG) = 7$ if and only if $t^L(KG) = 7$.

Theorem 2.15. *Let G be a group and let K be a field of characteristic $p \geq 0$ such that KG is Lie nilpotent.Then $t_L(KG) = 8$ if and only if one of the following conditions hold:*

- $p = 7, G' \cong C_7$.
- $p = 3$ *and* G' *is a group of one of the following types:*
  1. $G' \cong C_3 \times C_3 \times C_3$, $\gamma_3(G) = 1$;
  2. $G' \cong C_3 \times C_3$, $\gamma_3(G) \cong C_3$.
- $p = 2$ *and* G' *is a group of one of the following types:*
  1. $G' \cong C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) = 1$;
  2. $G' \cong C_4 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) \subseteq G'^2$;
  3. $G' \cong C_4 \times C_4$, $\gamma_3(G) \subseteq G'^2, \gamma_4(G) = 1$;
  4. $G' \cong C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) \cong C_2$;
  5. $G' \cong C_4 \times C_2 \times C_2$, $G'^2 \subseteq \gamma_3(G) \cong C_2 \times C_2, \gamma_4(G) = 1$;
  6. $G' \cong C_4 \times C_2 \times C_2$, $\gamma_3(G) \cong C_2, G'^2 \cap \gamma_3(G) = 1$;
  7. $G' \cong C_2 \times C_2 \times C_2 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2, \gamma_4(G) = 1$;

8. $G' \cong C_4 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2, \gamma_4(G) \cong C_2$;
9. $G' \cong C_2 \times C_2 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2, \gamma_4(G) \cong C_2$.

Moreover, $t_L(KG) = 8$ *if and only if* $t^L(KG) = 8$.

In this survey, we have seen that Lie nilpotent group algebras have been study extensively. A lot of work has been done in this area. But still there are much more to do. Some problems have been discussed in this survey.

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