Illuminating the Applications of Partitions' Theory: Ramanujan's Congruences and *Morowah* Numbers

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Abstract: *In this paper, we explore some notable applications of Partitions' Theory. Specifically, we highlight the specific contributions of Ramanujancongruences related to the partition function* ***p(n)****, and we describe what we call Morowah numbers based on the idea of prime partitions. We also generate potential sieves and partition filters using computational rudiments.*

Keywords: *Partition Theory, Integer partitions, Ramanujan'scongruences, Modular arithmetic, Digital root.*

I. Introduction

Attempts to delineate the genesis of mathematical thought can be traced back as early as the Stone Age. As a primary human endeavor, mathematics has evolved widely over time from rudimentary ideas modulating and explaining real life phenomenon to more sophisticated and abstract concepts beyond the realm of perceptual experiences. Arguably then, the mathematics we use today could be in many ways quite different from the mathematics of 500 years ago. Nonetheless, casting a wider net on the history of mathematics and considering the impressive contributions of numerous civilizations to the edifice of mathematical knowledge (Chahine&Kinuthia, 2013), it can be clearly discerned that mathematical traditions are rooted in antiquity (Josef, 2012). A large body of mathematics consist of facts that be described much like any other natural phenomenon. Such facts that make up most of the applications of mathematics are the most likely to survive changes of style and interest. The theory of partitions is one of the very few branches of mathematics whose applications are found wherever discrete objects are counted, whether in the atomic and molecular studies of matter, in the theory of numbers, or in the combinatorial problems across disciplines.

In this paper, we explore some notable applications of Partitions' Theory, highlighting the specific contributions of Ramanujancongruencesrelated to the partition function ***p(n)***, and generating new numbers using computational rudiments based on the idea of prime partitions.

II. Ramanujan's Congruences

Broadly, the word "partition" has numerous meanings in mathematics. Specifically, integer partition can be simply defined as a way to split integers into sum of its integer parts. Leonard Euler, in 1782 who coined and proved several significant partition theorems and laid the foundation of the Theory of Partitions, concluded that: *Every number has as many integer partitions into odd parts as into distinct parts* (Andrews, 2004, p. 3).By definition, a partition of a positive integer ***'n'*** is a non-increasing sequence of positive integers, called parts, whose sum equals ***n***. Generally, we characterize partitions as the number of ways in which a given number can be expressed as a sum of positive integers. For example, $\rho(4) = 5$ signifies the five different ways we can express the number 4. Therefore, the partitions of the number 4 are:

4,
3+1,
2+2,
2+1+1,
1+1+1+1.

However, the theory of partitions flourished significantly with the contributions of many other prominent mathematicians like SrinivasaRamanujan, an Indian mathematician who in 1913, established several partition identities and three *fundamental* congruences. In his own words, Ramanujan (1919) explained, "It appears that there are no equally simple properties for any moduli involving primes other than these three".For the partition function ***p(n)***, the three congruences are stated as follows: for $n \geq 0$,

$p(5n+4) = 0 (\bmod 5)$,

p(7n+4) = 0(mod 7),
p(11n+6) = 0(mod 11).

The study of Ramanujancongruences is an interesting and popular research topic of number theory. Furthermore, the two Rogers-Ramanujan identities provide the most fascinating concepts in the history of partitions. The Rogers-Ramanujan identities stated in terms of partitions were presented in two corollaries (Andrews, 2004):

Corollary 1 (The first Rogers-Ramanujan identity): *The partitions of an integer* ***n*** *in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into part ≡ 1 or 4 (modulo 5).*

Corollary 2 (The second Rogers-Ramanujan identity): *The partitions of an integer* ***n*** *in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into part ≡ 2 or 3 (modulo 5).*

Because of its significant applications in different areas like probability and particle physics (especially in quantum field theory), the theory of partitions has become one of the richest research areas of mathematics in recent times. Principled by Ramanujan'scongruences on ρ(n), arithmetic properties of many other partition functions such as t-core partition, Frobenius partition, broken k diamond partition, k dots bracelet partitions, rigorous and exciting prospects for ground breaking innovative theoretical research emerged informing new advancements in fields such as crystallography and string theory.An in-depth and more systematic examination of partition theory is fundamental in advancing mathematics and physics to new frontiers.

Perhaps the most intriguing aspect of Ramanujan's mathematical creations remains his method, which was discretely isolated from the works of other mathematicians trained in the conventional, deductive axiomatic methods of proof. What mattered most were the results of his mathematical realizations, which he felt no strong compulsion to test whether these were true. In this regards, a fundamental question raised by his works is whether any features of his Indian culture might have contributed to his flow of creative works in mathematics.

III. Morowah Numbers

The second application of partitions is the generation of a new set of numbers, we call "*Morowah*" numbers, as coined by the first author of this paper. Chahine and Morowah (2019) defined a *Morowah*number as an integer N wherethe sum of its digital roots $S_d(N)=a^r$ and the sum of the prime digital roots $S_p(N) = r^a$, and where a,r are natural numbers. We list several examples of *Morowah*numbersin Table 1.

TABLE 1

Examples of Morowah Numbers

N	$S_d(N)$	$S_p(N)$
$18= 2 \times 3^2$	$9 = 3^2$	$8 = 2^3$
$11977 = 7 \times 29 \times 59$	$25= 5^2$	$32 =2^5$
$26\,978 = 2 \times 7 \times 41 \times 47$	$32=2^5$	$25=5^2$
$406\,318\,734 = 2 \times 3^3 \times 17 \times 499 \times 887$	$36 = 6^2$	$64= 2^6$
$998\,299\,990 = 2 \times 5 \times 3823 \times 26113$	$64= 2^6$	$36 = 6^2$
$919\,999\,999\,800 = 2^2 \times 3^2 \times 5 \times 7^2 \times 10\,430\,839$	$81 = 3^4$	$64 = 4^3$
$99299\,998\,000 = 2^4 \times 5^3 \times 7 \times 79 \times 89\,783$	$64 = 4^3$	$81= 3^4$
$9\,491\,899 \times 10^{12} = 2^{12} \times 5^{12} \times 17 \times 281 \times 1987$	$49 = 7^2$	$128 = 2^7$
… … … ...... … ... ... … … …	... … …	... ... …

The Sieve for Morowah Numbers

We delineate a potential sieve that will help identify*Morowah* numbersacross a select range of a and r values, namely a =2,3,4,5,6,7,8 and r = 2,3,4,5,6. By applying our proposed definition, we extract and thereby identify Morowah numbers in 11 illustrative cases as follows:

Case 1.For$a$= 3 and r=2, then $S_d(N) = 3^2 = 9$and $S_p(N) = 2^3 = 8$.

In this case, we note that there is a unique *Morowah*number, namely $N = 18 = 2 \times 3^2$where$S_d(N) = 3^2 = 9$ and$S_p(N) = 2^3 = 8$. Since$\rho(9) = 9$ and $\rho(8) = 8$, then the prime partitions of8 must have a product P, whose digital root $\rho(P) = 9$. To illustrate, we list the prime partitions of 8 in Table 2.

Table 2

Prime Partitions of Number 8.

	$S_p(N)$	P	$\rho(P)$	$S_d(N) \in E$
(2,2,2,2)	8	16	7	E={7;16;25;34;...;9x+7;...}
(2,2,4)	8	16	7	E= {7;16;25;34; ... ;9x+7; ...}
(3,5)	8	15	6	E= {6;15;24;33; ... ;9x+6; ...}
(4,4)	8	16	7	E = {7; 16;25;34; ... ;9x + 7; ...
(2,3,3)	8	18	9	E = {9; 18; 27; 36; ... ; 9x; ... }

As shown in Table 2, $N = 18$ is the only*Morowah*number of this type. Note that, $S_d(N) = 16 = 4^2 = 24$;$S_d(N) = 81 = 9^2 = 3^4$; $S_d(N) = 256 = 16^2 = 4^4 = 2^8$;$S_d(N) = 625 = 25^2 = 5^4$;etc.

Case 2. For $a = 2$ and $r = 3$, $S_d(N) = 2^3 = 8$ and$S_p(N) = 3^2 = 9$. In Table 3, we demonstrate the prime partitions of 9 as follows:

Table 3

Prime Partitions of Number 9.

	$S_p(N)$	P	$\rho(P)$	$S_d(N) \in$E
(2,7)	9	14	5	E = {5;14;23;32; ... ;9x+5; ...}
(4,5)	9	20	2	E = {2;11; 20; 29; ... ; 9x + 2; ... }
(3,3,3)	9	27	9	E = {9;18;27;36; ... ;9x; ... }
(2,2,2,3)	9	24	6	E= {6;15;24;33; ... ;9x+6; ...}
(2,4,3)	9	24	6	E= {6;15;24;33; ... ;9x+6; ...}

Since none of the products P yields a digital root $\rho(P) = 8$; we deduce that there is no*Morowah*number having the following property:

$S_d(N)=2^3 = 8$ and$S_p(N)=3^2=9$.

Case 3· For $a = 4$ and $r=2$, $S_d(N)= 4^2=16$ and$S_p(N) = 2^4 = 16$,whichmeans that N is a *Smith* number, but such *Smith* number does not exist (Chahine &Morowah, 2019).

Case 4. For $a = 2$ and $r=5$, $S_d(N)= 2^5 =32$ and$S_p(N) = 5^2 =25$.

Knowing that the digital root of 32 is 5, thus we have to find all the prime partitions of 25[disregarding the prime 3] that yield the product P of digital root $\rho(P)$= 5. For instance, 25 = 4 + 7 + 14, the primes in (p,q,r) have $S_d(p) = 4$, $S_d(q)= 7$ and$S_d(r) = 14$; their product $P = 4 \times 7 \times 14 = 392$ and $\rho(392) = 5$.

Consequently, all the integers of the form $N = p \times q \times r$ have$S_p(N) = 4+7+14 = 25 = 5^2$ and $S_d(N)$ = 5; 14; 23; 32; 41; ... ;9x+ 5; ....

Our sieve then has to pick out the integers N whose $S_d(N) = 32 = 2^5$, corresponding to$x = 3$.

Table 4 lists the prime partitions of number 25 that generate *Morowah*numbers with examples and where $S_d(N) = 32 = 2^5$ and $S_d(N)=25 =5^2$.

Table 4
Prime Partitions of Number 25

$(2^2, 5, 16)$	$56948 = 2^2$ x 23 x 619;	87 494 = 2 x 11 x 41 x 97;
	47 795 = 5 x 11^2x 79;	1 286 942 = 2 x23 x101 x 277;
	17 800 745 = 5 x 101^2x 349;	2 038 685 = 5 x 11x101x 367; ...
(4,5,16)	594 347 = 13 x 131 x 349;	276 737 = 31 x 79 x 113;
	1 169 771 = 41x103 x 277;	2 654 591= 23 x 211 x 547; ...
$(2^2, 7, 14)$	39 956 = 2^2 x 7 x 1427;	157 982 = 2 x 11 x 43 x 167;
	49 973 = 7 x 11^2 x 59;	337 946 = 2 x7 x 101 x 239;
	73 253 381 = 43 x 101^2x 167;	458 843 = 7 x 11 x59x101; ...
(4,7,14)	163 787 = 13 x 43 x 293;	75 299 = 7 x 31 x 347;
	370 697 = 59 x 61 x 103;	2 776 127 = 59 x 211 x 223; ...
(5,7,13)	276 791 = 41 x 43 x 157;	270 779 = 23 x 61 x 193;
	154 985 = 5 x139x223;	52 997 = 7 x 67 x 113; ...
(5,10,10)	181 697 = 19 x 73 x 131;	192 659 = 37 x41 x 127;
	273 677 = 23 x 73 x 163;	88 835 = 5 x109x163; ...
(2,5,7,11)	86 198 = 2 x7 x 47 x 131;	68 585 = 11 x 5 x 43 x 29;
	2 556 815 = 5 x 61 x 83 x 101;	... ...
(2,7,8,8)	691 862 = 2 x 53 x61 x107;	304 997 = 7 x 11 x 17x 233;
	21 893 063 = 43x71^2 x 101;	... ...

Case 5. For a=5 and r=2 $S_d(N)$= 5^2 =25 and $S_p(N)$ = 2^5 =32.
We note that since the digital root of 25 is 7, then the prime partitions of 32 [disregarding the prime 3] that yield a product P with digital root $\rho(P)$ = 7 are:

(2,5,25)	9 970 = 2 x 5 x 997;	78 154 = 2 x 23 x 1699;	54 835 = 11 x 5 x 997;
	1 212 739 = 11x41x2 689;	903 445 = 101x5x1 789;	13 191 307 = 101x131x997; ....
(2,7,23)	8 386 = 2x7x599 ;	419 551 = 11 x 43 x 887;	4 910 317 = 61 x 101 x 797
(2, 14, 16)	32 686 = 2 x 59 x 277 ;	62 953 = 11 x 59 x 97 ;	6 606 511 = 101 x 149 x 439
(5 ,7, 20)	16 765 = 5 x 7 x 479 ;	384 721 = 23 x 43 x 389;	1 423 069 = 41 x 61 x 569; …
(5, 11, 16)	63 655 = 5 x 29 x 439;	104 857 = 23 x 47 x 97;	942 631 = 41 x 83 x 277
(7, 8, 17)	55 573 = 7 x 17 x 467 ;	407 941 = 43 x 53 x 179; …	
	853 207 = 61 x 71 x 197	2 892 103 = 151x 107 x 179; …	
(7, 11, 14)	11 977 = 7 x 29 x 59 ;	119 239 = 43 x 47 x 59 ;	845 521 = 61 x 83 x 167;…

$(2^6, 5^2, 10)$	289 600 = 2^6 x 5^2 x 181;	... . … ... … …
(25,2,5,5, 10)	2 955 040 = 2^5x 5x11x 23 x 73;	... ... … … …
$(24,2^2,5,5,10)$	133 272 304 = 2^4 x 11^2 x 23 x 41 x 73	... ... … … …
(24,4,52, 10)	98 800 = 2^4 x 5^2 x 13 x 19;	... ... … … … … …

$(2^3,2,4,5,5,10)$	16 708 120 = 2^3 x5 x 11 x 13 x 23 x127;	
$(2, 2, 42, 5^2, 10)$	1 766 050 = 2 x 11 x 13^2 x 5^2 x 19;	... ...
$(4^3, 5^2, 10)$	104 3 575 = 13^3 x 5^2 x 19;	... ...

(24, 5^2, 7, 7)	270630052= 2^2x7 x11^2x 23^2x 151;	739 600 = 2^4x 5^2x43^2; ...
(2^3, 2, 5^2, 7^2)	2 281 048 = 2^3x7^2x11 x 23^2;	… …
(2, 2, 4, 5^2, 7,7)	51 701 650= 2x5^2x 7 x 11 x 13x 1033;	……

We remark thatin all of the above prime partitions and $S_J(N) = 9x + 7$, the function of the sieve is to select all numbers N whose $S_J(N) = 25 = 5^2$ corresponding to $x = 2$. If we assign for our sieve the function $x = 1$, then it selects the numbers N whose$S_J(N) = 16$, thus generating two *Smith* numbers, such that $S_p(N) = 2 \text{ x } S_J(N) = 2 \text{ x } 16 = 32$.

Case 6.For $a = 6$ and r=2$S_J(N) = 6^2 = 36$ and $S_p(N) = 2^6 = 64$.

The digital root of 36 is 9, then the prime partitions of 36 must yield a product $P$with digital root $\rho(P) = 7$.

Obviously, the prime 3 must be one of these primes, raised, at least,to the power two. The Table below lists some of the very few prime partitions of 64.

(3^3, 55)	105 299 703 = 3^3 x 3 899 989; …	(3^3 , 5, 50)	7 515 001 665 = 3^3 x 5 x 55 666 679;
(3^4 , 52)	160 371 819 = 3^4 x 1 979 899; …		5 079 900 123 = 3^3 x 41 x 4 588 889
(3^6 ,46)	217 970 271 = 3^6 x 298 999; …		6 396 012 423 = 3^3 x 41 x 5 777 789

(2, 3^3, 8,22,23)	2 404 834 326 = 2 x 3^3 x 71 x 787 x 797;	274 391118 = 2 x 3^3 x 17 x 499 x 599; .
(2^3, 3^3, 4, 22, 23)	30907641204 = 2^2x 3^3x 11 x 31x 859 x 977;	839 314 008 = 2^3x 3^3x13 x 499 x 599; ..
(2,34,7,17,26)	3350472741 = 3^4x 7x 11 x 269x 1 997;	190 649 214 = 2 x3^4 x7 x 89 x 1 889; ..
(2^5, 34, 10,29)	2 104 085 808 = 2^4 x 3^4x 11 x 37x 3 989;	382 561 056 = 2^5x3^4x37 x 3989; ..
(2^3, 34, 4, 10,29)	12 341 173 284 = 2^2x 3^4x11 x 31 x19 x 5 879;	480007944=13 x 19 x 23 x 34 x 2 999; ..,
(2, 34, 4^2 , 10,29)	7 244 330 418 = 2 x 3^4 x 13 x 31x37 x2 999;	… … … … … … … …
(2^4, 3^2, 5,11,14,20)	2 415 194 640 = 2^4x 3^2x 5 x 47 x 149 x 479;	… … … … … … … …

Notice that in each of the above partitions,$S_p(N) = 2^6 = 64$ and$S_J(N) = 9; 18;27;36;45; ... ;9x;...$, the function of the sieve is to filter out the numbers Nwhose $S_J(N) = 36$, corresponding to $x = 4$.The sieve can pick out the numbers N whose $S_J(N) = 81 = 3^4$, corresponding to$x = 9$, thus we obtain the case of$a$=3and r= 4 where $S_J(N) = 81 = 3^4$ and $S_p(N) = 64 = 4^3$.

Case 7. For a= 2 and r= 6 where $S_J(N) = 2^6 = 64$ and $S_p(N) = 6^2 = 36$. Considering that the digital root of 64 is 1, then the prime partitions of 64 must yield a product $P$with digital root $\rho(P) = 1$.
In the Table below, we list very few of such prime partitions of 36:

(2^2 , 2, 5, 25)	15 077 499 877 = 11^2 x 101 x 311 x 3967;	26 297 657 749 = 11 x 101^2 x 131 x 1 789; …
(2, 4, 5, 25)	398 998 990 = 2 x 13 x 5 x 3 069 223;	76 369 980 538 = 2 x 1 789 x2 111 x10 111;
	48 896 236 279 = 11 x 1301x 1699 x 2011;	22 979 416 879 = 101 x 211 x 401 x 2 689; …
(2,5,7,22)	999 788 770 = 2 x 5 114 001 x 499;	674 073 656 794 = 2 x 1 579 x 3 111 x 10 113 x;
	12 687 949 747 = 11 x 23x 499 x 100 501;	95 860 866 187 = 101 x 41 x 769 x 30103;…
(2,5,10, 19)	379 999 990 = 2 x 5 x 37 x 1 027 027	307 457 780 698 = 2 x 3 011 x 3 169 x 16 111;
	284 641 738 939 = 101 x 311 x 6 211 x 1 459;	229 259 987 209 = 11 x 1 103 x 1 693 x 11 161;…
(2, 5, 13, 16)	885 799 990 = 2 x 5 x 1 093 x 81 043	88 185 790 099 = 11 x 1031 x 2083 x 3733;
	153 669 279 718= 2 x 4001 x 3019 x 6361;	42 156 598 969 = 101 x 131 x1 249 x 2 551; …
(2,5,5,7,17)	6 896 756 494 = 2 x 23 x 89 x 401 x 4 201;	84 896 619 490 = 2 x 5 x 7 451 x 1 1033;
	13 779 875 197 = 11 x 41 x 131 x 1 303 x 179	78 484 623 985 = 5 x 311 x 421 x 101 x 1 187
(2,5,7,11,11)	7 997 573 674 = 2 x 47 x 401 x 331 x 641;	99 409 494 547 = 11 x 263 x 311 x 313 x 353;
	878 999 590 = 2 x 5 x 7 x 1 019 x 12 323	20 694 199 969 = 47 x 101 x 113 x 223 x 137;…

In each of the above partitions,$S_d(N) = 1;10;19;28;37; 46; ... ;9x+1$;etc.The function of the sieve is to pick out the numbers N whose $S_d(N) = 64$ corresponding to x=7.

Case 8. In case a= 4 and r=3 where $S_d(N)$= 4^3=64 and$S_p(N)$ = 3^4=81. The digital root of 64 is 1, then the prime partitions of 81 must yield a product Pwithdigitalroot$\rho(P)$ =l.Thefollowingsaretheprimepartitionsof 81,thatgeneratenumbers$N$whose$S_d(N)$= 1;10;19; ... ;55;64;73; ... ;9x+1; etc. The sieve picks out those numbers whose $S_d(N)$ = 4^3 = 64, corresponding to x = 7 as follows:

(2^3, 23, 52)	48 185 539 768 = 2^3 x 7 529 x 799 999;	240 974 263 684 = 2^2 x 7^2 x 11 x 5477 x 20407;
	1 776 350 695 852 = 2^2 x 101 x 4 397 x 999 979;	1 886 607 945 442 = 2 x 11^2 x 1 949 x 3 999 949;
	9 993 020 587 363 = 11^3 x 1 877 x 3 999 949;	… … … … … … … … … … … … … … …
(2, 4, 23, 52)	1 927 339 990 822 = 2 x 599 x 2 011 x 799 999 ;	40 654 405 649 719 = 101 x 103 x 977 x3999 949 ;
	1 849 798 153 423 = 11 x 211 x 797 x 999 979;	… … … … … … … … … … … … … … …
(2^3, 25, 50)	36 667 138 888 = 2^3 x 7 639 x 599 999;	346 273 088 788 = 2^2 x 11 x 7 873 x 999 599;
	2 613 368 847 484 = 2^2 x 101 x 6 469 x 999 959	1 415 932 666 786 = 2 x 11^2 x 5 857 x 998 969;
	11 818 672 862 293 = 11^2 x 101 x 997 x 969 989;	… … … … … … … … … … … … … … … …
(2, 4, 25, 50)	9 075653 627 806 = 2 x 1 699 x3001x 889 997;	3 638 726891 173 = 11 x 211x 1 987 x 788 999;
	1784582753 473=13x101 x 1699 x799979;	… … … … … … … … … … … … … … … …
(2^3 , 32,43)	399 480 419 656=2^3x 49937 x 999961;	1 966006665964=2^2 x11x157x 6367x 44699;
	16 959604 262 284=2^2 x 101x41999x 999529;	10 862 408 698 273 =11^2 x101x 9887x89 899;
	694 957 232 926 = 2 x 11^2x 35 897 x 79 999 ;	… … … … … … … … … … …
(2, 4, 32, 43)	1 788 415 259 482 = 2 x 31 x 28 859 x 999 529;	47 545 323 975 523 = 11 x 103 x 41999 x 999 169;
	34 859 342 992 123 =13 x 101 28 859 x 919 969;	… … … … … … … … … … … … … … … …
(2^3, 34, 41)	597 830586328=2^3 x 74779 x 999329;.	2967256198324=2^2 x 11x67489 x999239;
	22 471 664 907 484=2^2x13x 1783 x 3719 x 65171;	12729939068242=2 x11^2x56599 x929899;
	108 516 281 594 347 = 11^2x 101 x 29599x299993;	... … … … … … … … … … … … … …
(2, 4, 34, 41)	14 708905119 586=2 x 2 011 x 36 979 x 98897;	4 997 161 682 623 = 11 x 103 x 49 669 x 88 799;
	5462 765663 851=31x101 x 17 989 x 96 989;	... … … … … … … … … … … … … … … …

Case 9. For a = 3 and r=4, we have $S_d(N)$ = 3^4=81and $S_p(N)$ = 4^3 =64.
The digital root of 81 is 9, then the prime partitions of 64 must yield a product Pwithdigitalroot$\rho(P)$ =9. From a listing of prime partitions of 64 that generate numbers N whose$S_d(N)$ = 9;18;27; ... ;63;72;81; ...;9x; etc., the sieve picks out those numbers whose $S_d(N)$ = 81 corresponding to x = 9 as follows:

(3^2 , 58)	9 799999929=3^2x1088 888881;
(2, 3^3, 8,22,23)	263 758696195194=2 x 3^3 x 503 x 82 561x117 617;
	14792377179987=3^3 x 11 x233 x 9833 x 21739;
	8797 681737 477=3^3x71 x 101 x 6691 x 6 791; … … … … … … … …
(2,34, 7, 17,26)	3564394777 245 474=2 x 3^4 x 4 003 x 59723 x 92 033;
	2806366999689=3^4x11x 313 x 1 997 x 5039;
	19196488 732779=3^4x101 x241 x 3491 x 2789;... … … … … …
(2^2, 3^2, 5^2, 13,31)	729 999999900=2^2 x 3^2 x 5^2x 283 x 2866 117;
	9685881358668=2^2x3^2x 41x 113x 3109 x18 679;
	15533967863 898=2 x3^2x11 x 23 x131x3343x7789;
	64 797 608 9628 45 = 3^2 x 5 x 11^2x 311 x 2029 x18 859 ;
	52 798759 037 874 = 2 x3^2 x 23 x 101 x 401 x 409 x 7 699;
	17558964677925=3^2x 5^2x 11x101 x6997 x 10039;... … … … …

$(2^2, 3^2, 5^2, 19, 25)$	$199\ 999\ 989\ 900 = 2^2 \times 3^2 \times 5^2 \times 2\ 7\ 91 \times 79\ 621$; … … … … … … … …
	$2\ 591\ 929\ 789\ 668 = 2^2 \times 3^2 \times 41 \times 311 \times 757 \times 7\ 459$;
	$4\ 592\ 949\ 847\ 398 = 2 \times 3^2 \times 11 \times 23 \times 131 \times 883 \times 8\ 719$;
	$3793\ 877\ 088\ 894 = 2 \times 3^2 \times 23 \times 41 \times 101 \times 379 \times 5\ 839$; … … … … …
$(2^2, 3^4, 5^2, 10, 28)$	$99\ 999\ 999\ 900 = 2^2 \times 3^4 \times 5^2 \times 37 \times 333\ 667$; … … … … … … … … …
$(2, 3^4, 5, 7, 10, 28)$	$69999999930 = 2 \times 3^4 \times 5 \times 7 \times 37 \times 333667$ … … … … … … … … … …
	$2359\ 437\ 982\ 080\ 786 = 2 \times 3^4 \times 73 \times 311 \times 11113 \times 57\ 727$; … … … …
$(2, 3^5, 5^3, 10, 11^2)$	$39999969990 = 2 \times 3^5 \times 5 \times 23^2 \times 29^2 \times 37$; … … … … … … … … … …
$(4, 3^2, 5^2, 19, 25)$	$1\ 565\ 959\ 996\ 650\ 411 = 3^2 \times 2820401 \times 61691779$,… … …
$(4, 3^2, 5^2, 13, 31)$	$459\ 932\ 655\ 943791 = 3^3 \times 17034542812733$; … … … … … …
$(2^3, 3^2, 5^2, 7^2, 28)$	$919999999\ 800 = 2^3 \times 3^2 \times 5^2 \times 7^2 \times 10430839$;… … … … … … …
$(2^3, 3^3, 5^3, 34)$	$3999699999000 = 2^3 \times 3^3 \times 5^3 \times 148137037$; … … … … … … …
	$6939\ 999999\ 000 = 2^3 \times 3^3 \times 5^3 \times 257\ 037\ 037$;… … … … … …
$(2^4, 3^2, 5, 11, 14, 20)$	$88\ 774\ 476\ 378\ 480 = 2^4 \times 3^2 \times 5 \times 83 \times 34\ 283 \times 43\ 331$;
$(2^5, 3^5, 10, 29)$	$489643199913568 = 2^5 \times 3^5 \times 59393 \times 1060201$;… … … … … … …
$(2^6, 3^2, 5^2, 7, 13, 16)$	$8046973\ 298779200 = 2^6 \times 3^2 \times 5^2 \times 151 \times 51\ 241 \times 72223$;… …
$(2^4, 3^2, 4, 5, 13, 28)$	$97\ 999\ 999920 = 2^4 \times 3^2 \times 5 \times 31 \times 1\ 129 \times 3\ 889$; … … … … … … … …
$(2^6, 3^2, 5^6, 16)$	$9\ 199\ 999\ 989\ 000\ 000 = 2^6 \times 3^2 \times 5^6 \times 1\ 022\ 222\ 221$

Case 10. For a= *8* and *r*=2 then we deduce that $S_d(N) = 8^2 = 64$ and $S_p(N) = 2^8 = 256$. Here we consider three conditions:

(i) Generally, if $S_d(N) = 64$ and $S_p(N) = 25$, then $25 + [7 \times 33] = 25 + 231 = 256$. Consequently,
$S_d(10^{33} \times N) = 64 = 8^2$ and $S_p(10^{33} \times N) = 25 + 231 = 256 = 2^8$

To illustrate, for the prime partitions $(2^{33}, 5^{33}, 5, 20)$ having $S_d(N) = 8^2$ and $S_p(N) = 2^8$, we show the following examples:

$69\ 969\ 979 \times 10^{33} = 2^{33} \times 5^{33} \times 23 \times 3042173$;	$79\ 939\ 999 \times 10^{33} = 2^{33} \times 5^{33} \times 131 \times 610229$;
$89\ 399\ 899 \times 10^{33} = 2^{33} \times 5^{33} \times 2003 \times 44\ 633$;	$99\ 989\ 299 \times 10^{33} = 2^{33} \times 5^{33} \times 311 \times 321\ 509$; … … … …

ii) If $S_d(N) = 64$ and $S_p(N) = 32$, then $32 + [7 \times 32] = 32 + 224 = 256$.
Consequently, $S_d(10^{32} \times N) = 64 = 8^2$ and $S_p(10^{32} \times N) = 32 + 224 = 256 = 2^8$. For examples, see Table below:

$(2^{32}, 5^{32}, 4, 11, 17)$	$74\ 399\ 999 \times 10^{32} = 2^{32} \times 5^{32} \times 31 \times 47 \times 51\ 407$; … … … … … … …
$(2^{35}, 5^{32}, 2, 4, 7, 13)$	$87\ 898\ 888 \times 10^{32} = 2^{35} \times 5^{32} \times 11 \times 31 \times 7 \times 4603$; … … … … … … … …
$(2^{32}, 5^{32}, 5, 13, 14)$	$99982\ 999 \times 10^{32} = 2^{32} \times 5^{32} \times 131 \times 1\ 237 \times 617$; … … … … …

(iii) If $S_d(N) = 64$ and $S_p(N) = 39$, then $39 + [7 \times 31] = 39 + 217 = 256$.
Consequently, $S_d(10^{31} \times N) = 64 = 8^2$ and $S_p(10^{31} \times N) = 39 + 217 = 256 = 2^8$.

Here are some examples:

$(2^{31}, 5^{31}, 2^3, 2, 31)$	$78\ 988\ 888 \times 10^{31} = 2^{34} \times 5^{31} \times 11 \times 897\ 601$;
	$87\ 888988 \times 10^{31} = 2^{33} \times 5^{31} \times 11 \times 101 \times 19\ 777$;... … … … …
$(2^{31}, 5^{31}, 7, 16, 16)$	$83\ 999\ 989 \times 10^{31} = 2^{31} \times 5^{31} \times 61 \times 79 \times 17431$; ... … … … … …
$(2^{33}, 5^{31}, 7, 28)$	$88\ 887\ 988 \times 10^{31} = 2^{33} \times 5^{31} \times 7 \times 3\ 174\ 571$; ... … … … … … …
$(2^{31}, 5^{31}, 4, 5^3, 20)$	$89999299 \times 10^{31} = 2^{31} \times 5^{31} \times 13 \times 23^3 \times 569$;... … … … …
$(2^{31}, 5^{31}, 10^2, 19)$	$99\ 899\ 929 \times 10^{31} = 2^{31} \times 5^{31} \times 19 \times 163 \times 32\ 257$; ... … … … … …
$(2^{31}, 5^{31}, 10, 13, 16)$	$99\ 999793 \times 10^{31}$=231x 531x 19 x 14 341 x 367;... …… …
$(2^{32}, 5^{31}, 7, 11, 19)$	$99\ 929998 \times 10^{31} = 2^{32} \times 5^{31} \times 7 \times 29 \times 246\ 133$; ... … … … … …
$(2^{34}, 5^{31}, 11, 22)$	$329\ 998\ 888 \times 10^{31} = 2^{34} \times 5^{31} \times 29 \times 1422\ 409$; ... … … … … …
$(2^{33}, 5^{31}, 10, 25)$	$4575778588 \times 10^{31} = 2^{33} \times 5^{31} \times 19 \times 60\ 207613$;... ... ...

(iv) If $S_d(N)$ =64 and $S_p(N)$ = 46, then 46 + [7 x 30] = 46 + 210 =256.
Consequently, $S_d(10^{30} \times N)$= 64= 8^2 and $S_p(10^{30} \times N)$ = 46 + 210 = 256 = 2^8.Some examples include:

$(2^{35}, 5^{30}, 14, 22)$	$87\ 889\ 888 \times 10^{30} = 2^{35} \times 5^{30} \times 257 \times 10687$ … … … … …
$(2^{33}, 5^{30}, 4, 11^2, 14)$	$87\ 988\ 888 \times 10^{30} = 2^{33} \times 5^{30} \times 13 \times 47^2 \times 383$;… … … … … …
$(2^{33}, 5^{30}, 2, 5, 11, 22)$	$89878888 \times 10^{30} = 2^{33} \times 5^{30} \times 11 \times 41 \times 29 \times 859$; … … … … …
$(2^{35}, 5^{30}, 8, 11, 17)$	$16258888864 \times 10^{30} = 2^{35} \times 5^{30} \times 53 \times 1163 \times 8243$;... … … …

(v) If $S_d(N)$=64 and $S_p(N)$ = 53, then 53 + [7 x 29] = 53 + 203 =256.Consequently, $S_d(10^{29} \times N)$ = 64 $=8^2$ and $S_p(10^{29} \times N)$ = 53 + 203 = 256 = 2^8. A list of examples is shown below:

$(2^{29}, 5^{29}, 10, 20, 23)$	$82\ 999\ 999 \times 10^{29} = 2^{29} \times 5^{29} \times 19 \times 1\ 289 \times 3\ 389$; ... … …
	$99\ 939\ 997 \times 10^{29} = 2^{29} \times 5^{29} \times 37 \times 479 \times 5639$; … … …
$(2^{29}, 5^{30}, 2, 46)$	$4\ 557\ 778\ 885 \times 10^{29} = 2^{29} \times 5^{30} \times 11 \times 82\ 868707$; ... … … …
$(2^{31}, 5^{29}, 8, 19, 20)$	$988\ 988\ 923 \times 10^{29} = 2^{31} \times 5^{29} \times 1\ 601 \times 397 \times 389$;... … … …
$(2^{34}, 5^{29}, 2, 11, 13, 17)$	$5\ 455\ 777 \times 888 \times 10^{29} = 2^{34} \times 5^{29} \times 11 \times 29 \times 2\ 713 \times 197$; ... … …

(vi) If $S_d(N)$ =64 and $S_p(N)$ = 60, then 60 + [7 x 28] = 60 + 196 =256.Consequently, $S_d(10^{28} \times N)$ = 64= 8^2 and $S_p(10^{28} \times N)$ = 53 + 203 = 256 = 2^8. The following are some examples:

$(2^{29}, 5^{28}, 7, 16, 35)$	$99\ 299998 \times 10^{28} = 2^{29} \times 5^{28} \times 7 \times 79 \times 89783$; … …. …
$(2^{30}, 5^{28}, 10, 17, 29)$	$898898932 \times 10^{28} = 2^{30} \times 5^{28} \times 73 \times 89 \times 34\ 589$;… … …
$(2^{33}, 5^{28}, 2, 10, 16, 22)$	$26\ 248\ 888\ 864 \times 10^{28} = 2^{33} \times 5^{28} \times 11 \times 163 \times 79 \times 5791$; … … …

We proceed likewise with $S_p(N)$ = 67; 74; 81; 88; ….; 249; 256.

Finally, when $S_d(N)$ =64 and $S_p(N)$ = 256, we find the prime partitions of 256 whose product *P* has the digital root $\rho(P)$ = 1.

Case 11· For a=7 and r=2, $S_d(N)$= 7^2=49 and $S_p(N)$ = 2^7 =128. Some examples are listed as follows:

(20,20,41,47)	4360061551 236331 =389x479x59 999x389 999;...
$(29^2, 35^2)$	3 212407132966201=2999^2x 18 899^2;...
(32^4)	5 726 342 542 105 201 =8 699^4
$(2^2, 5, 25, 47^2)$	49 989 505879 732 722 192 497 = 101^2 x 2 003 x 1 699 x 1199 999^2 ; …

(i) If $S_d(N)$ =49 and $S_p(N)$ = 23, then 23 + [7 x 15] = 23 + 105 =128.Consequently, $S_d(10^{15}$x$N)$ =49 =7^2 and $S_p(10^{15}$x$N)$= 23 + 105 = 128 = 2^7 . We show this condition with some examples:

$(2^{15},5^{15},4, 19)$	1 499989x10^{15}= 2^{15}x5^{15}x103x14 563;
	1899999x10^{15}=2^{15}x5^{15}x31x61129;...
$(2^{15},5^{15}, 7, 16)$	949 999x10^{15} =2^{15} x 5^{15} x 43 x 22 093;
	9 929299 x 10^{15} = 2^{15} x 5^{15} x 313 x 3 723; ...
$(2^{15}, 5^{15}, 2, 10, 11)$	9991 399 x 10^{15} = 2^{15} x 5^{15} x 11 x 31 321 x 29; ...
$(2^{15},5^{15}, 10, 13)$	1 984 999 x 10^{15} = 2^{15} x 5^{15} x 109 x18 211;
	4 998 919 x 10^{15} = 2^{15} x 5^{15} x 19 x 263101;
	9 198 949x 10^{15} =2^{15} x 5^{15} x 73 x 126 013;

(ii) If $S_d(N)$=49 and $S_p(N)$ = 30, then 30 + [7 x 14] = 30 + 98 =128.Consequently, $S_d(10^{14}$ x $N)$ = 49 =7^2 and $S_p(10^{14}$ x $N)$ = 30 + 98 = 128 = 2^7. To illustrate, we give the examples below:

$(2^{14}, 5^{14}, 2,7,7, 14)$	9999 913 x10^{14}= 2^{14}X 5^{14}x 11 x 7 x 61 x 2129; ...
$(2^{14},5^{14}, 2^2,28)$	9 149 899 x 10^{14}= 2^{14}X 5^{14}X 11^2 x 75619; ...
$(2^{15}, 5^{14}, 2^2, 10, 14)$	8 991 994x 10^{14}=2^{14}x 5^{14}x 2x 11^2 x 73x 509;
$(2^{15}, 5^{14}, 4,10,14)$	9 819 994x 10^{14}=2^{14}X5^{14}x 2 x31 x 1063 x 149;

(iii) If $S_d(N)$ =49and $S_p(N)$ = 37, then 37 + [7 x 13] = 37 + 91 =128.Consequently, $S_d(10^{13}$x $N)$ = 49 =7^2 and $S_p(10^{13}$ x $N)$ = 37 + 91 = 128 =2^7 . To demonstrate, we give these examples:

$(2^{13}, 5^{13}, 5, 5, 7, 7, 13)$	388 337 773 x 10^{13} = 2^{13} x 5^{13} x 23 x 41 x 43 x 61 x 157; ...
	530 198 599x10^{13}=2^{13}x5^{13}x23x113x7x151x193;…

(iv) If $S_d(N)$ =49and $S_p(N)$ = 44, then 44 + [7 x 12] = 44 + 84 =128.Consequently, $S_d(10^{12}$x $N)$= 49 =7^2 and S_p $(10^{12}$ x $N)$ = 44 + 84 = 128 =2^7.
For example:

$(2^{12}, 5^{12}, 2,1, 35)$	9 931 999 x 10^{12} = 2^{12} x 5^{12} x 11 x 7 x 128 987; ...
$(2^{12}, 5^{12}, 7,8,29)$	9 894 199 x 10^{12} = 2^{12} x 5^{12} x 7 x 53 x 26 669; ...
$(2^{12}, 5^{12}, 8,11,25)$	9491899 x 10^{12}=2^{12}x5^{12}x17 x 281 x 1 987;...
$(2^{12}, 5^{12}, 7, 14, 23)$	7777 777 x 10^{12} = 2^{12}x 5^{12}x 7 x 239 x4 649; …

$(2^{12}, 5^{12}, 7, 17,20)$	$4819999 \times 10^{12} = 2^{12} \times 5^{12} \times 43 \times 197 \times 569$; …
$(2^{12}, 5^{15}, 11,16, 17)$	$8\,999\,941 \times 10^{12} = 2^{12} \times 5^{12} \times 137 \times 367 \times 179$; ...

(v)If $S_d(N)$ =49 and $S_p(N)$=51, then 51 + [7 x 11] = 51 + 77 =128.Consequently, $S_d(10^{11} x N) = 49 = 7^2$ and $S_p(10^{11} x N)$= 51 + 77 = 128 = 2^7. An example is given below:

$(2^{11}, 5^{11}, 2, 7, 35)$	$9\,931\,999 \times 10^{11} = 2^{11} \times 5^{11} \times 11 \times 7 \times 128\,987$; ...

We proceed likewise for $S_p(N)$ = 58; 65; 72; 79; ...; 114; 121; 128.Finally, when $S_d(N)$=49 and $S_p(N)$ = 128, we find theprimepartitions of 128 whose product P has the digital root $\rho(P)$ = 4.

Employing the aforementioned sieve, we can find more *Morowah* numbers for the cases listed in this Table:

a	r	$S_d(N)$	$S_p(N)$	a	r	$S_d(N)$	$S_p(N)$
3	5	$3^5 = 243$	$5^3 = 125$	5	4	$5^4 = 625$	$4^5 = 1024$
5	3	$5^3 = 125$	$3^5 = 243$	3	7	$3^7 = 2187$	$7^3 = 343$
3	6	$3^6 = 729$	$6^3 = 216$	4	6	$4^6 = 4096$	$6^4 = 1296$
4	5	$4^5 = 1024$	$5^4 = 625$	…	…	......	......

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