Holographic Dark Energy Model in Brans - Dicke Theory of Gravitation

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Abstract:- *In this paper, we have investigated spatially homogeneous anisotropic Axially Symmetric universe filled with two minimally interacting fields, matter and holographic dark energy components in Brans-Dicke theory of gravitation. Exact solutions of field equations are obtained using the fact that scalar expansion is proportional to the shear scalar and constant deceleration parameter. Some physical and kinematical properties of the model are also discussed.*

Keywords:- *Axially Symmetric Universe, Deceleration parameter. Holographic dark energy. Brans-Dicke theory.*

I. INTRODUCTION

Recent observation of the luminosity of type Ia supernovae indicate (Bachall et al. [1]; Perlmutter et al. [2]) an accelerated expansion of the universe and the surveys of clusters of galaxies show that the density of matter is very much less than the critical density. This observation leads to a new type of matter which violate the strong energy condition i.e., $\rho + 2p < 0$. The matter (fluid) content responsible for such a condition to be satisfied at a certain stage of evaluation of the universe is referred to as dark energy (Sahni and Starobinsky [3]; Peebles and Ratra [4]; Padmanabhan [5]; Copeland et al. [6]). This mysterious fluid is believed to dominate over the matter content of the Universe by 70% and to have enough negative pressure as to drive present day acceleration. Most of the dark energy models involve one or more scalar fields with various actions and with or without a scalar field potential (Maor and Brustein [7]; Cardenas and Campo[8]; Ferreira and Joyce [9]).

In recent times, considerable interest has been stimulated in explaining the observed dark energy by the holographic dark energy model (Enqvist et al. [10]; Zhang [11]; Pavon and Zimdahl [12]). An approach to the problem of dark energy arises from the holographic principle stated in the first paragraph. For an effective field theory in a box size L with UV cut off $\wedge_c$, the entropy $L^3 \wedge_c{}^3$. The non-extensive scaling postulated by Bekenstein suggested that quantum theory breaks down in large volume (Zhang [11]). To reconcile this breakdown, Cohen et al. [13] pointed out that in quantum field theory a short distance (UV) cut-off is related to a long distance (IR) cut-off due to the limit set by forming a black hole. Taking the whole universe into account the largest IR cut-off L is chosen by saturating the inequality so that we get the holographic dark energy density as $\rho_\wedge = 3C^2 M_p^2 L^{-2}$ (Zhang [11]) where c is a numerical constant and $M_p = \frac{1}{\sqrt{8\pi G}}$ is the reduced Plank mass. On the basis of the holographic principle proposed by Fischler and Susskind[14] several others have studied holographic model for dark energy (Gong [15]). Employment of Friedman equation (Setare [16]) $\rho_\wedge = 3M_p^2 H^2$ where ρ is the total energy density and taking $L = H^{-1}$ one can find $\rho_m = 3(1-C^2) M_p^2 H^2$. Thus either ρ_m or $\rho_\wedge$ behaves like H^2.

The effects of anisotropy in the universe can be studied in the framework of an anisotropic BI model. Reddy et al. [17] analyzed homogeneous and axially symmetric BI models in BD theory and found that the deceleration parameter is negative, leading to an accelerated expansion of the universe. Setare [18] studied the HDE model with non-flat FRW metric in BD cosmology and found that the EoS parameter demonstrates a phantom-like region and crosses the phantom divide line. Kumar and Singh [19] used exact solutions describing BI cosmological models to study the cosmic evolution in a scalar-tensor theory. Setare and Vanegas [20] investigated an interacting HDE model and discussed cosmological implications. Sharif and Kausar [21,22] examined the dynamical behavior of a Bianchi universe with anisotropic fluid in f (R) gravity. Sharif and Waheed [23] studied the evolution of a BI model in BD theory, using isotropic, anisotropic, as well as magnetized anisotropic fluid, and found that the latter may attain isotropy to the universe. Milan and Singh [24] discussed an HDE model with infrared cutoff as a future event horizon, as well as a logarithmic form of BD scalar field for the FRWuniverse in BD theory. Felegary et al. [25] studied the dynamics of an interacting HDE

model in BD cosmology as regards the future event horizon cutoff, as well as its Hubble-horizon counterpart, and discussed the coincidence problem.

In particular, Setare [26] studied holographic dark energy model in Brans – Dicke theory. Sheykhi [27] studied interacting holographic dark energy models in Brans-Dicke (1961) theory of gravitation. Setare and Vanegas [28] have discussed the cosmological dynamics of interacting holographic dark energy model. Sarkar and Mahanta [29] have investigated holographic dark energy in Bianchi type-I space – time with constant deceleration parameter. Das and Mammon [30] studied holographic dark energy models in Brans-Dicke (1961) theory of gravitation. Sarkar [31] has discussed the evolution of holographic dark energy model in Bianchi type – I universe with linearly varying deceleration parameter and established a correspondence with generalized Chapligin gas models of the universe. Also holographic scalar field dark energy models are studied by many authors. For instance, Very recently Kiran et al. [32, 33] studied minimally interacting holographic dark energy models in Bianchi type-V space time in the scalar-tensor theories of gravitation proposed by Brans and Dicke (1961) and Saez and Ballester (1986). Also kadam [34] studied Bianchi type-V cosmological model for minimally interacting holographic dark energy in Seaz-Ballester theory of gravitation. In a recent paper, Kumar and Singh [35] have revisited these papers and have proposed a logarithmic form of BD scalar field to discuss new agegraphic dark energy model in BD theory.

This paper is organized as follows: In Sec 2, we present the field equations of HDE model in BD theory. Section 3 deals solutions of field equations. In Sec 4, gives the physical discussion. Section 5 gives the conclusion of the work.

II. METRIC AND FIELD EQUATIONS

We consider the Axially symmetric space time described by the metric

$$ds^2 = dt^2 - A^2\left(d\chi^2 + f^2 d\psi^2\right) - B^2 dz^2 \tag{1}$$

where A, B, C are functions of cosmic time t only.

Brans-Dicke (1961) field equations for the combined scalar and tensor fields are

$$R_{ij} - \frac{1}{2} g_{ij} R = -\omega\phi^{-2}\left(\phi_{,i}\,\phi_{,j} - \frac{1}{2} g_{ij}\phi_{,k}\,\phi^{,k}\right) - \phi^{-1}\left(\phi_{i,j} - g_{ij}\phi^{,k}_{,k}\right) - 8\pi\left(T_{ij}^{'} + \bar{T}_{ij}\right) \tag{2}$$

Where $T_{ij}^{'}$ and $\bar{T}_{ij}$ are energy – momentum tensor for matter and holographic Ricci dark energy resp. which are defined as

$$T_{ij}^{'} = \rho_m u_i u_j \tag{3}$$

And

$$\bar{T}_{ij} = \left(\rho_\lambda + p_\lambda\right) u_i u_j + g_{ij} p_\lambda \tag{4}$$

where ρ_m is matter the energy density and ρ_λ is the holographic dark energy and p_λ is the pressure of the holographic dark energy.

Also, the energy conservation equation is

$$T^{ij}_{;j} + \bar{T}^{ij}_{;j} = 0 \tag{5}$$

In a commoving coordinate system, the field equations (2) and (3) for the metric (1) with the help of Eqns.(4) and (5) can be, explicitly, written as

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = -8\pi\, p_\lambda \phi^{-1} \tag{6}$$

$$2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 - \frac{f_{11}}{A^2 f} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + 2\frac{\dot{\phi}}{\phi}\frac{\dot{A}}{A} = -8\pi\, p_\lambda \phi^{-1} \tag{7}$$

$$\left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{f_{11}}{A^2 f} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\dot{\phi}}{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 8\pi\phi^{-1}\left(\rho_\lambda + \rho_m\right) \tag{8}$$

$$\ddot{\phi} + \dot{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \frac{8\pi}{3+2\omega}\left(\rho_m + \rho_\lambda - 3p_\lambda\right) \tag{9}$$

where an overhead dot indicates differentiation with respect to t.

The functional dependence of the metric together with (9) and (10) imply that

$$\frac{f_{11}}{f} = k^2, \qquad k^2 = \text{ constant} \tag{10}$$

If $k = 0$ then $f(\chi) = cons\tan t, \quad 0 < \chi < \alpha$. This constant can be made equal to 1 by suitably choosing units for ψ. Thus we shall have $f(\chi) = \chi$ resulting in the flat model of the universe (Hawking and Ellis 1976 [36]).

Now the field equations (6) – (9) gives us the following independent equations

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = -8\pi\, p_\lambda \phi^{-1} \tag{11}$$

$$2\frac{\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + 2\frac{\dot{\phi}}{\phi}\frac{\dot{A}}{A} = -8\pi\, p_\lambda \phi^{-1} \tag{12}$$

$$\left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\dot{\phi}}{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 8\pi\phi^{-1}(\rho_\lambda + \rho_m) \tag{13}$$

$$\ddot{\phi} + \dot{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \frac{8\pi}{3+2\omega}(\rho_m + \rho_\lambda - 3p_\lambda) \tag{14}$$

The energy conservation equation gives

$$\dot{\rho}_m + \rho_m\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \dot{\rho}_\lambda + \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)(1+\omega_\lambda)\rho_\lambda = 0 \tag{15}$$

Here we are considering the minimally interacting matter and holographic dark energy components. Hence both the components conserve separately, so that we have (Sarkar 2014a,2014b)

$$\dot{\rho}_m + \rho_m\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 \tag{16}$$

$$\dot{\rho}_\lambda + \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)(1+\omega_\lambda)\rho_\lambda = 0 \tag{17}$$

Where $p_\lambda = \omega_\lambda \rho_\lambda$ is barotropic equation of state parameter for holographic dark energy.

The expression for physical and general parameters to be used in solving Saez-Ballester field equations for the metric (1) are as the follows:

The average scale factor $a(t)$ is given by

$$a(t) = \left(A^2 B\right)^{\frac{1}{3}} \tag{18}$$

The spatial volume V is given by

$$V = a^3 = A^2 B \tag{19}$$

The directional Hubble parameter, respectively, are

$$H_1 = \frac{\dot{A}}{A} = H_2, \quad H_3 = \frac{\dot{B}}{B}, \tag{20}$$

The average Hubble parameter is

$$H = \frac{1}{3}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \frac{1}{3}\frac{\dot{V}}{V} = \frac{\dot{a}}{a} \tag{21}$$

The dynamical scalar expansion θ for the space time given by Eq. (1) and the shear scalar σ^2 are

$$\theta = \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \tag{22}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{3}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 - 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right] \tag{23}$$

The average anisotropy parameter is

$$A_h = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \tag{24}$$

where $H_i (i = 1,2,3)$ represent the directional Hubble parameters.

III. SOLUTIONS OF FIELD EQUATIONS

i) We assume that expansion scalar (θ) in the model is proportional to shear scalar (σ). This condition leads to

$$A = B^m \tag{25}$$

where m is proportionality constant. The motive behind assuming condition (i) is explained with reference to Throne [37], the observation of the velocity red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropy today within ≈ 30 percent (Kantowski and Sachs [38]; Kristian and Sachs [39]) To put more precisely, red-shift studies place the limit $\frac{\sigma}{H} \leq 0.3$ on the ratio of shear to Hubble's parameter in the neighborhood of our galaxy today. Collin et al. [40] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfied that the condition $\frac{\sigma}{\theta}$ is constant.

ii) we consider that $T + \bar{T} = 3p_\lambda - \rho_m - \rho_\lambda = 0$ (26)

which physically corresponds the vanishing of trace of both matter and dark energy tensors.

iii) Variation of Hubble's parameter proposed by Berman (1983) which yields constant deceleration parameter models of the universe defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \text{ constant} \tag{27}$$

Which admits the solution

$$a(t) = (ct + d)^{\frac{1}{1+q}} \tag{28}$$

Where $c \neq 0$ and d are constants of integration.

This equation implies that the condition for spatial expansion of the universe is $1 + q > 0$.

Now from equations (18), (25), (27) we obtain the expressions for metric potentials as

$$A = (ct + d)^{\frac{3m}{(1+q)(2m+1)}} \tag{29}$$

$$B = (ct + d)^{\frac{3}{(1+q)(2m+1)}} \tag{30}$$

Also from equations (14), (26) and (28) we obtain the expression for Brans-Dicke scalar field as

$$\phi = \left(\frac{\phi_0}{c}\right)\left(\frac{1+q}{q-2}\right)(ct + d)^{\frac{q-2}{1+q}} + k \tag{31}$$

Where k is a constant of integration.

Now through a proper choice of coordinates and constants (choosing $c = 1$ and $d = 0$), the metric (1) with the help of equation (29), (30) can be written as

$$ds^2 = dt^2 - t^{\frac{6m}{(1+q)(2m+1)}}\left(d\chi^2 + f^2 d\psi^2\right) - t^{\frac{6}{(1+q)(2m+1)}} dz^2 \tag{32}$$

And scalar field in the model is

$$\phi = \phi_0 \left(\frac{1+q}{q-2} \right) t^{\frac{q-2}{1+q}} \tag{33}$$

Where the constant k is omitted.

IV. PHYICAL DISCUSSION

Eq. (32) along with (33) represents minimally interacting Axially symmetric universe holographic dark energy model in Brans-Dicke scalar-tensor theory with the following physical and geometrical parameters .

Spatial volume in the model is

$$V = t^{\frac{3}{1+q}} \tag{34}$$

The average Hubble parameter is

$$H = \frac{1}{(1+q)t} \tag{35}$$

The scalar expansion is

$$\theta = \frac{3}{(1+q)t} \tag{36}$$

The shear scalar is

$$\sigma^2 = \frac{3(m-1)^2}{(2m+1)^2 (q+1)^2 t^2} \tag{37}$$

The average anisotropy parameter is

$$A_m = \frac{2(m-1)^2}{(2m+1)^2} \tag{38}$$

From Eqns. (12), (29) and (33) the holographic pressure in the model is

$$8\pi\phi^{-1} p_\lambda = \frac{9m(m+2)}{(1+q)^2 (2m+1)^2 t^2} - \frac{\omega}{2} \frac{(q-2)^2}{(1+q)^2 t^2} \tag{39}$$

where ϕ is given by equation (33)

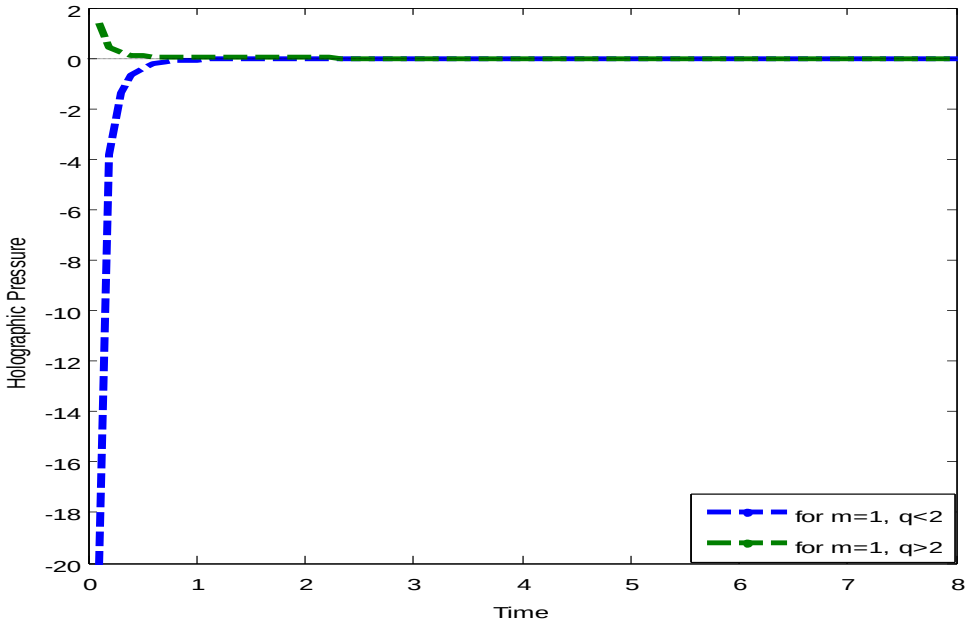

Fig-1- Holographic Pressure vs time

The energy density of dark matter is

$$\rho_m = \rho_0 t^{\frac{-3}{1+q}} \tag{40}$$

The holographic energy density in the model is

$$8\pi\phi^{-1}\rho_\lambda = \frac{9m^2 + 18m + 3(2m+1)^2(q-2)}{(2m+1)^2(1+q)^2 t^2} - \frac{\omega}{2}\frac{(q-2)^2}{(1+q)^2 t^2} - 8\pi\phi^{-1}\rho_0 t^{\frac{-3}{1+q}} \tag{41}$$

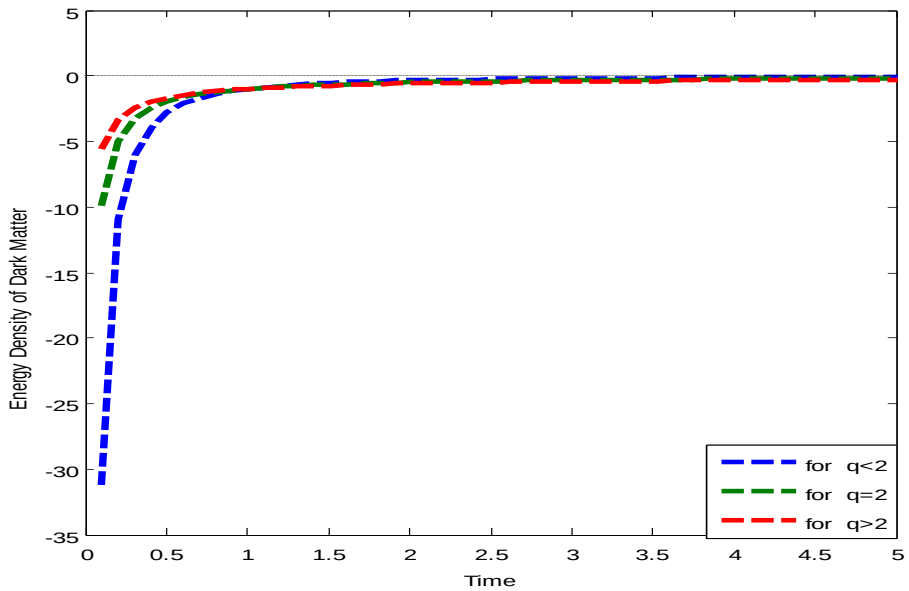

Fig-2 Energy Density of Dark Matter Vs Time

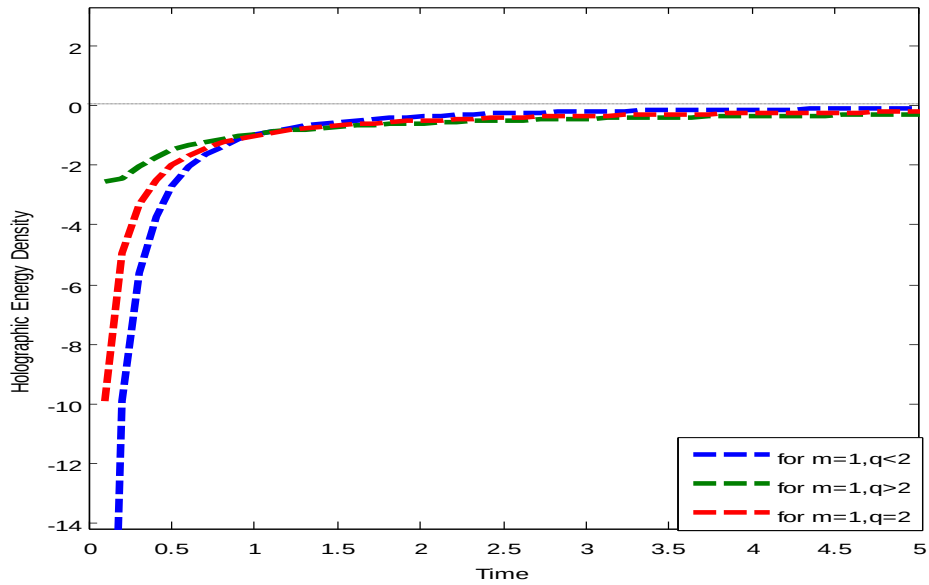

Fig-3- Holographic Energy Density Vs Time

Now by using barotropic equation of state parameter for holographic dark energy, we get the equation of state of parameter (EoS) as

$$\omega_\lambda = \frac{\dfrac{9m(m+2)}{(1+q)^2(2m+1)^2t^2} - \dfrac{\omega}{2}\dfrac{(q-2)^2}{(1+q)^2t^2}}{\dfrac{9m^2+18m+3(2m+1)^2(q-2)}{(2m+1)^2(1+q)^2t^2} - \dfrac{\omega}{2}\dfrac{(q-2)^2}{(1+q)^2t^2} - 8\pi\phi^{-1}\rho_0 t^{\frac{-3}{1+q}}} \tag{42}$$

which shows that ω_λ is a function of cosmic time t only.

The coincidence parameter is

$$r = \frac{9m^2+18m+3(2m+1)^2(q-2)}{8\pi\rho_0(1+q)(2m+1)^2t} - \frac{\omega(q-2)}{16\pi\rho_0(1+q)t} - 1 \tag{43}$$

The matter density parameter Ω_m and holographic energy density parameter Ω_λ are given by

$$\Omega_m = \frac{\rho_m}{3H^2} \text{ and } \Omega_\lambda = \frac{\rho_\lambda}{3H^2} \tag{44}$$

Using barotropic equation of state parameter for holographic dark energy, (35), (40) and (41) we get the overall density parameter as

$$\Omega = \Omega_m + \Omega_\lambda = \frac{t^{\frac{q-2}{1+q}}}{24\pi(q-2)}\left[\frac{9m^2+18m+3(2m+1)^2(q-2)}{(2m+1)^2(q+1)} - \frac{\omega}{2}\frac{(q-2)^2}{(q+1)}\right] \tag{45}$$

The cosmic jerk parameter is dimension less third order derivative of the scale factor with respect to the cosmic time defined as

$$j(t) = \frac{1}{H^3}\frac{\dddot{a}}{q} = q(1+2q) \tag{46}$$

V. CONCLUSIONS

In this paper we have investigated spatially homogeneous Axially symmetric universe minimally interacting holographic dark energy models in the Brans-Dicke scalar-tensor theory of gravitation. To determine exact solution of field equation we use

i) expansion scalar (θ) in the model is proportional to shear scalar (σ). ii) the trace of energy tensor of fluids vanishes and iii) special form of deceleration parameter proposed by Berman (1983). We have obtained the energy density of matter, holographic dark energy density, holographic pressure, equation of state (EoS) parameter, the scalar field in the model, total energy density in the universe discussed their physical behavior in each of the models. In this model jerk parameter is constant. Average anisotropic parameter $A_m \neq 0$ so our model is anisotropic except $m \neq 1$. Thus the model presented here is anisotropic, shearing except $m \neq 1$. The spatial volume increases with time and the physical parameters decrease and ultimately tend to zero as $T \to \infty$.

REFERENCES

[1] Bachall, N.A.,et al.,: Revealing the State of the Uni., Science, 284(1999),1481-1988.
[2] Perlmutter, S:et al. Nature 391, 51 (1998).
[3] Sahni, V., Starobinsky, A.A.: Int. J. Mod. Phys. D, 9 (2000), 373-443.
[4] Peebles,P.J.E., Ratra, B.: Rev. Mod. Phys., 75 (2003), 559.
[5] Padmanabhan, T.: Phys. Rep., 380 (2003), 235-320.
[6] Copeland, E.J., Sami, M., Tsujikawa, S.,: Int. J. Mod. Phys. D,15 (2006), 1753-1935.
[7] Maor, I., Brustein, R.: Phys. Rev. D, 67 (2003), 103508.
[8] Cardenas, V.H., Campo, S.D. : Phys. Rev. D, 69 (2004), 083508.
[9] Ferreira, P.G., Joyce, M.: Phys. Rev. D, 58 (1998), 023503.
[10] K. Enqvist, S. Hannested, M.S.Sloth,:, J. Cosmol. Astropart. Phys., 2 (2005), 004.
[11] X.Zhang,: Int. J. Mod. Phys. D, 14 (2005),
[12] D.Pavon, W. Zimdahl,: Holographic Dark Energy and Present Cosmic Acceleration,(2005).
[13] A.G.Cohen, et al., : Phys. Rev. Lett., 82 (1999), 4971 .
[14] W.Fischler, L.Susskind, Holography and Cosmology, hep-th/9806039 (1998).
[15] Y.Gong,: Phys. Rev. D, 70 (2004), 064029.
[16] M.R.Setare, :Holographic Chaplygin gas model, Phys. Lett. B, 648 (2007), 329-332.
[17] Reddy, D.R.; Naidu, R.L.; Rao, V.U.: Int. J. Theor. Phys. **2007**, 46, 1443–1448.
[18] Setare, M.R. : Phys. Lett. B **2007**, 644, 99–103.
[19] Kumar, S.; Singh, C.P.: Int. J. Theor. Phys. **2008**, 47, 1722–1730.
[20] Setare, M.R.; Vagenas, E.C. : Int. J. Mod. Phys. D **2009**, 18, 147–157.
[21] Sharif, M.; Kausar, H.R.: Phys. Lett., B **2011**, 697, 1–6.
[22] Sharif, M.; Kausar, H.R. : Astrophys. Space Sci. **2011**, 332, 463–471.
[23] Sharif, M.; Waheed, S.: Eur. Phys. J. C **2012**, 72, 1876.
[24] Srivastava, M.; Singh, C.P.:. arXiv **2017**, arXiv:1706.06777.

[25] Felegary, F.; Darabi, F.; Setare, M.R.: Int. J. Mod. Phys. D **2018**, 27, 1850017.
[26] Setare, M.R.: Phys. Lett. B 644, 99 (2007)
[27] Sheykhi, A.: Physics Letters B, Volume 681, Issue 3, 2 Nov. 2009.
[28] Setare, M.R., Vanegas, E.C.: Int. J. Mod. Phys. D 18, 147 (2009)
[29] Sarkar, S; Mahanta, C.R.: Int. J. Theor. PhyS. 52, 1482 (2013).
[30] Das, S., Mammon, A.A.: Astrophys. Space Sci. 351, 651 (2014)
[31] Sarkar, S.: Astrophys. Space Sci. 349, 985 (2014)
[32] Kiran,M., et al. :Astrophys. Space Sci. 354,577 (2014a)
[33] Kiran,M., et al. :Astrophys. Space Sci. 356,407(2014b)
[34] Kadam V P: JETIR, NOV 2018, Vol. 5, issue-11.
[35] Kumar Pankaj and Sing C.P. : Astrophys. Space Sci. 362, 52 (2017).
[36] Hawhking, S.W., Ellis, G.F.R.: Large scale structure of space- time. Cambrig University press. London (1976)
[37] Throne, K.S.: Astrophys. J. 148,503 (1967)
[38] Kantowski, R., Sachs, R.K.: J. Math. Phys. 7, 433 (1966)
[39] Kristian, J., Sachs, R.K.: Astrophys. J. 143,379 (1966)
[40] Collins, C.B., Glast, E.N.,Wilkisons,D.A.: Gen. Relativ. Grav.12,805(1980)