On the Solution of an Inverse Eigenvalue Problem by Newton's Method on a Fibre Bundle with Structure Group $SO(n)$

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Abstract *– In this paper a Newton-type algorithm is used to generate non-singular symmetric matrices of rank one, using a singular symmetric matrix of the same rank as an initial matrix for the iteration. In particular, numerical computations are performed with two different diagonal matrices which are in the neighbourhood of the eigenvalues of the initial singular symmetric matrix to construct a three by three and a four by four non-singular symmetric matrices to illustrate our result.*

Keywords – *Singular Symmetric Matrix, Non-Singular Symmetric Matrices, Diagonal Matrix, Eigenvalue.*

I. INTRODUCTION

Inverse eigenvalue problems are usually concerned with the reconstruction of a physical system from prescribed spectral data. The main objective of an inverse eigenvalue problem thus reduces to the construction of a physical system that maintains a certain specific structure as well as that of the given spectral property. Inverse eigenvalue problems arise in a remarkable variety of applications in both engineering and science [1]. Several authors [2]–[5] have used different methods including analytic, algebraic and numerical ones to solve inverse eigenvalue problems, particularly of various types of non-singular symmetric matrices. In [6] the construction of a 2×2 non-singular symmetric matrix was considered.

The following notations will be employed in the research: the set of real invertible $n \times n$ matrices forms a group under multiplication, denoted by $GL(n, R)$. The subset of $GL(n, R)$ consisting of those matrices having determinant $+1$ is a subgroup of $GL(n, R)$, denoted by $SL(n, R)$. The set of real $n \times n$ orthogonal matrices form a group under multiplication, denoted by $O(n)$. The subset of $O(n)$ consisting of those matrices having determinant $+1$ is a subgroup of $O(n)$ denoted by $SO(n)$.

Definition 1.1: The group $GL(n, R)$ is called the general linear group, and its subgroup $SL(n, R)$ is called the special linear group. The group $O(n)$ of orthogonal matrices is called the orthogonal group, and its subgroup $SO(n)$ is called the special orthogonal group(or group of rotations). The vector space of real $n \times n$ matrices with null trace is denoted by $sl(n, R)$, and the vector space of real $n \times n$ skew symmetric matrices is defined by $so(n)$ [7].

The focus of this paper is to construct certain non-singular symmetric matrices of order three and four using an arbitrary set of eigenvalues. The iteration will be based on newton type numerical technique, initialized by a singular symmetric matrix for solving the inverse eigenvalue problem on a fibre bundle with structure group $SO(n)$.

II. PRELIMINARIES

In order to present the main results of this research in a concise way, it is useful to give some preliminary results by [6] which play a fundamental role throughout the rest of the research.

Early work on inverse eigenvalue problem using the idea close to the notation of tangent bundles have tended to be somehow theoretical in nature. In particular, the initial matrix for the iteration is not given and therefore the special orthogonal matrix Q could not be directly determined and consequently the skew-symmetric matrix K. The algorithm described here is more practical in that respect and uses information for an initial singular symmetric matrix to generate a non-singular symmetric matrices.

Given an initial singular symmetric matrix, we obtain the matrices Q and K, which are used as initial guessed values for the iteration process. Following from these process, the singular symmetric matrix converges to a non-singular symmetric matrix of the same dimension. The chosen fibre bundle is an affine space consisting of a base manifold comprising the set of $n \times n$ symmetric matrices with each fibre called isospectral surface [4]. This is made up of a class of symmetric matrices with the same eigenvalues. The fibre is acted on by the structural group $SO(n, R)$. The family of normal subgroups being dealt with are $SO(3)$ and $SO(4)$ with corresponding Lie algebras as $so(3)$ and $so(4)$ respectively. These algebras are $n \times n$ skew-symmetric matrices with null trace and act on the fibres of the associated tangent bundle. We denote the isospectral surface by $M(\Lambda)$. A tangent vector to a point in the fibre $X \in M(\Lambda)$ is of the form $T(X) = XK - KX$. This is the Lie bracket of the Lie algebra $so(n)$ [4]. We let $M(\Lambda) = \{Q\Lambda Q^t | Q \in SO(n)\}$, where Λ contains the set of distinct eigenvalues. The equation to be solved is given by $X = Q\Lambda Q^t$ and the tangent vector arising is $XK - KX$. A Newton-type method employed is as follows. At the isospectral surface, $M(\Lambda)$, we have

$$Q(A_i)\Lambda Q^t(A_i) = X_{i+1}, \qquad i = 1, 2, ..$$

Where A_i is a singular symmetric matrix, Q with columns which are the normalized eigenvectors of the matrix A_i and Λ a diagonal matrix which is similar to the matrix X_{i+1} and therefore have the same eigenvalues. Linearising iteratively at the tangent space of the Lie group which is the Lie algebra, we obtain the following equation,

$$X_{i+1} + X_{i+1}K - KX_{i+1} = A_{i+1}, \qquad i = 1, 2, ..$$

where K is a skew symmetric matrix which is given by

$$K = \frac{1}{2}(Q - Q^t)$$

Theorem 2.1: The exponential map

$$exp : so(n) \to SO(n)$$

is well-defined.

Proof

First we need to prove that if A is skew symmetric matrix, then e^A is a rotation matrix. For this, first check that

$$(e^A)^T = e^{A^T}.$$

Then, since $A^T = -A$, we get

$$(e^A)^T = e^{A^T} = e^{-A},$$

and so

$$(e^A)^T e^A = e^{-A}e^A = e^{-A+A} = e^{0_n} = I_n,$$

and similarly,

$$e^A(e^A)^T = I_n,$$

showing that e^A is orthogonal. Also,

$$det(e^A) = e^{tr(A)},$$

and since A is real skew symmetric, $tr(A) = 0$, and so $det(e^A) = +1$ [7].

Lemma 2.2: The exponential map

$$exp : so(3) \to SO(3)$$

is given by

$$e^A = \cos\theta I_3 + \frac{\sin\theta}{\theta}A + \frac{(1-\cos\theta)}{\theta^2}B,$$

or equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta}A + \frac{(1-\cos\theta)}{\theta^2}A^2$$

if $\theta \neq 0$, with $e^{\theta_3} = I_3$.

Proof
First, prove that

$$A^2 = -\theta^2 + B,$$
$$AB = BA = 0.$$

From the above, it can be deduced that

$$A^3 = -\theta^2 A,$$

and for any $k \geq 0$, we obtain the following recurrence relation

$$A^{4k+1} = \theta^{4k} A,$$
$$A^{4k+2} = \theta^{4k} A^2,$$
$$A^{4k+3} = -\theta^{4k+2} A,$$
$$A^{4k+4} = -\theta^{4k+2} A^2.$$

Finally, we prove the desired result by writing the powers for e^A and regrouping terms so that the power series for cosine and sine show up.

We state the next lemma without proof. The lemma makes a single proposition that every symmetric matrix A is of the form e^B [7].

Lemma 2.3: For every symmetric matrix B, the matrix e^B is symmetric positive definite. For every symmetric positive definite matrix A, there is a unique symmetric matrix B such that $A = e^B$.

Lemma 2.4: A non-singular symmetric matrix can be generated using a singular symmetric matrix as initial matrix in the following algorithm [4]:

$$X_i = Q_i(A_i)\Lambda Q_i^t(A_i) \qquad i = 1, 2, \cdots$$

and

$$A_i = X_i + X_i K_i - K_i X_i$$

The authors in [6] started with an initial 2×2 singular symmetric matrix of the form: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ with the eigenvalues as $\lambda_1 = 5, \lambda_2 = 0$. The normalized eigenvectors are the column vectors of the matrix $Q_1 = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$. Let $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

Step 1: The iterate X_1 is obtained as:

$$X_1 = Q_1 \Lambda Q_1^t = \begin{bmatrix} -0.200 & 1.5999 \\ 1.5999 & 2.1999 \end{bmatrix}$$

Step 2: The skew-symmetric matrix $K = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & -0.8944 \\ 0.8944 & 0 \end{bmatrix}$

This implies that

$$A_1 = X_1 + X_1 K - K X_1 = \begin{bmatrix} 2.6619 & 3.7463 \\ 3.7463 & -0.6620 \end{bmatrix}$$

The normalized eigenvectors of the matrix A_1 are the column space vectors of

$$Q_2 = \begin{bmatrix} 0.8383 & -0.5452 \\ 0.5452 & 0.8383 \end{bmatrix}$$

This is used for the next iteration X_2.

Step 3: Now $X_2 = \begin{bmatrix} 1.8110 & 1.8282 \\ 1.8282 & 0.1890 \end{bmatrix}$ with eigenvalues 3 and -1.

The authors in [6] concluded that the non-singular symmetric matrix X_2 has the same eigenvalues as the diagonal matrix Λ.

Here, we extend the dimension of this non-singular matrix to $3 \le n \le 4, n \in I$. We show that, any singular symmetric matrix has a corresponding non-singular symmetric matrix having the same eigenvalues as an assumed diagonal matrix of the same dimension.

III. MAIN RESULT

In this section we construct 3×3 and 4×4 non-singular symmetric matrices using an initial singular symmetric matrices of the same dimension by direct iterative approach.

A. 3×3 *matrices illustration*

We begin with an initial 3×3 singular symmetric matrix of the form $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$ whose eigenvalues are $\lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 0$. The normalized eigenvectors of A are the column vectors of the matrix

$$Q_1 = \begin{bmatrix} -0.5774 & 0.7071 & -0.4082 \\ 0.5774 & 0.7071 & 0.4082 \\ -0.5774 & 0 & 0.8165 \end{bmatrix}$$

Illustration 1 :

Let $\Lambda_1 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a matrix whose diagonal entries are in the neighbourhood of the eigenvalues of matrix A.

The matrix X_1 is obtained as follows

$$X_1 = Q_1 \Lambda_1 Q_1^t = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

with eigenvalues -1, 1, and 4. For the skew-symmetric matrix we have

$$K_1 = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & 0.0649 & 0.0846 \\ -0.0649 & 0 & 0.2041 \\ -0.0846 & -0.2041 & 0 \end{bmatrix}$$

Now the iteration A_1 is obtained as follows

$$A_1 = X_1 + X_1K_1 - K_1X_1 = \begin{bmatrix} 1.0904 & -2.1196 & 0.5721 \\ -2.1196 & 1.1487 & -1.3083 \\ 0.5721 & -1.3083 & 1.7609 \end{bmatrix}$$

The normalized eigenvectors of the matrix A_1 are the column vectors of

$$Q_2 = \begin{bmatrix} 0.6519 & -0.5064 & 0.5645 \\ 0.7306 & 0.2198 & -0.6465 \\ 0.2033 & 0.338 & 0.5132 \end{bmatrix}$$

This is used for the next iteration. Thus, we solve for X_2 as:

$$X_2 = Q_2\Lambda_1Q_2^t = \begin{bmatrix} 1.7620 & 1.6514 & 1.2420 \\ 1.6514 & 2.5045 & 0.0790 \\ 1.2420 & 0.0790 & -0.2666 \end{bmatrix}$$

with eigenvalues -1, 1, and 4.

As final step we obtain

$$K_2 = \frac{1}{2}(Q_2 - Q_2^t) = \begin{bmatrix} 0 & -0.0185 & 0.1806 \\ 0.6185 & 0 & -0.7402 \\ -0.1805 & 0.7402 & 0 \end{bmatrix}$$

$$A_2 = X_2 + X_2K_2 - K_2X_2 = \begin{bmatrix} 3.3560 & 0.30156 & 0.4349 \\ 3.0156 & 0.5788 & -2.4420 \\ 0.4349 & -2.4420 & 0.0652 \end{bmatrix}$$

Therefore

$$Q_3 = \begin{bmatrix} 0.3721 & -0.7870 & 0.4921 \\ -0.7076 & -0.5836 & -0.3984 \\ -0.6007 & 0.1999 & 0.7740 \end{bmatrix} \text{ and } X_3 = Q_3\Lambda_1Q_3^t = \begin{bmatrix} 0.1766 & -1.7085 & -0.3559 \\ -1.7085 & 1.8207 & 1.5086 \\ -0.3559 & 1.5086 & 2.0027 \end{bmatrix}$$

with eigenvalues -1, 1, and 4.

Illustration 2:

Let $\Lambda_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a matrix whose diagonal entries are in the neighbourhood of the eigenvalues of matrix A. Then the matrix X_1 is obtained as follows

$$X_1 = Q_1 \Lambda_2 Q_1^t = \begin{bmatrix} 2.3336 & -1.3336 & 1.3337 \\ -1.3336 & 2.3336 & -1.3337 \\ 1.3337 & -1.3337 & 2.3336 \end{bmatrix}$$

with eigenvalues 0.9999, 1.0000, 5.0009.
The skew-symmetric matrix K_1 is computed as

$$K_1 = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & 0.0648 & 0.0846 \\ -0.0648 & 0 & 0.2041 \\ -0.0846 & -0.2041 & 0 \end{bmatrix}$$

Now the iteration A_1 is obtained as follows

$$A_1 = X_1 + X_1 K_1 - K_1 X_1 = \begin{bmatrix} 2.2809 & -1.4930 & 1.1480 \\ -1.4930 & 2.7050 & -1.3600 \\ 1.1480 & -1.3600 & 2.0149 \end{bmatrix}$$

The normalized eigenvectors of the matrix A_1 are the column vectors of

$$Q_2 = \begin{bmatrix} -0.1664 & -0.8096 & 0.5630 \\ -0.6897 & -0.3125 & -0.6532 \\ -0.7047 & 0.4970 & 0.5063 \end{bmatrix}$$

This is used for the next iteration. Here, we solve for X_2 as: $X_2 = Q_2 \Lambda_2 Q_2^t = \begin{bmatrix} 1.1108 & 0.4591 & 0.4691 \\ 0.4591 & 2.9026 & 1.9442 \\ 0.4691 & 1.9442 & 2.9867 \end{bmatrix}$ with eigenvalues 5, 1, and 1.

As a final step we have

$$K_2 = \frac{1}{2}(Q_2 - Q_2^t) = \begin{bmatrix} 0 & -0.0599 & 0.6338 \\ 0.0599 & 0 & -0.5751 \\ -0.6338 & 0.5751 & 0 \end{bmatrix}$$

$$A_2 = X_2 + X_2K_2 - K_2X_2 = \begin{bmatrix} 0.5711 & -0.3960 & -0.8674 \\ -0.3960 & 5.0837 & 2.2554 \\ -0.8674 & 2.2554 & 1.3452 \end{bmatrix}$$

Therefore

$$Q_3 = \begin{bmatrix} -0.1287 & 0.7530 & 0.6454 \\ 0.8917 & 0.3726 & -0.2569 \\ 0.4339 & -0.5424 & 0.7194 \end{bmatrix} \text{ and } X_3 = Q_3\Lambda_2Q_3^t = \begin{bmatrix} 1.0663 & -0.4591 & -0.2234 \\ -0.4591 & 4.1808 & 1.5476 \\ -0.2234 & 1.5476 & 1.7529 \end{bmatrix}$$

with eigenvalues 1, 5, and 1.

It is observed that the iterates X_1, X_2, X_3 in all the cases are non-singular symmetric matrices whose eigenvalues are respectively the same as their assumed diagonal matrices Λ_1 andΛ_2.

B. 4×4 matrices illustration

Finally we construct a 4×4 non-singular symmetric matrix with an initial singular symmetric matrix and with a different diagonal matrices which are in the neighbourhood of the eigenvalues of the A below

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & 4 & -2 & -4 \\ -1 & -2 & 1 & 2 \\ -2 & -4 & 2 & 4 \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = 10, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$.

The normalized eigenvectors of A forms the column vectors of the matrix

$$Q_1 = \begin{bmatrix} 0.1108 & 0.8944 & -0.2962 & -0.3162 \\ 0.2217 & -0.4472 & -0.5923 & -0.6325 \\ 0.9485 & 0 & 0.0173 & 0.3162 \\ -0.1972 & 0 & -0.7491 & 0.6325 \end{bmatrix}$$

Illustration 3:

Assume the diagonal matrix $\Lambda_1 = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

The first iteration X_1 is obtained as follows

$$X_1 = Q_1 \Lambda_1 Q_1^t = \begin{bmatrix} 0.8860 & -0.2280 & 0.9359 & 0.2470 \\ -0.2280 & 0.5439 & 1.8717 & 0.4940 \\ 0.9359 & 1.8717 & 7.0979 & -1.7093 \\ 0.2470 & 0.4940 & -1.7093 & 0.4722 \end{bmatrix}$$

with eigenvalues -1, 1, 1, and 8.

The skew-symmetric matrix K is

$$K_1 = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & 0.3364 & -0.6223 & -0.0595 \\ -0.3364 & 0 & -0.2962 & -0.3162 \\ 0.6223 & 0.2962 & 0 & 0.5327 \\ 0.0595 & 0.3162 & -0.5327 & 0 \end{bmatrix}$$

The matrix A_1 is obtained below

$$A_1 = X_1 + X_1K_1 - K_1X_1 = \begin{bmatrix} 2.2337 & 1.4366 & 4.0065 & -0.4370 \\ 1.4366 & 1.8117 & 3.4659 & 1.0588 \\ 4.0065 & 3.4659 & 6.6452 & 0.8723 \\ -0.4370 & 1.0588 & 0.8723 & -1.6906 \end{bmatrix}$$

Whose normalized eigenvectors forms the column vector of the matrix

$$Q_2 = \begin{bmatrix} 0.4443 & 0.6582 & -0.5396 & 0.2797 \\ 0.3948 & -0.6570 & -0.5979 & -0.2344 \\ 0.8007 & -0.0069 & 0.5856 & -0.1257 \\ 0.0747 & -0.3675 & 0.0915 & -0.9225 \end{bmatrix}$$

Which is used for the next step iteration

We obtain X_2 below

$$X_2 = Q_2 \Lambda_1 Q_2^t = \begin{bmatrix} 2.2250 & 1.3589 & 2.5605 & -0.2838 \\ 1.3589 & 1.9811 & 2.1540 & 0.6389 \\ 2.5605 & 2.1540 & 5.4567 & 0.6505 \\ -0.2838 & 0.6389 & 0.6505 & -0.6629 \end{bmatrix}$$

with eigenvalues -1, 1, 1, and 8.

Finally

$$K_2 = \frac{1}{2}(Q_2 - Q_2^t) = \begin{bmatrix} 0 & 0.1317 & -0.6702 & 0.1025 \\ -0.1317 & 0 & -0.2955 & 0.0665 \\ 0.6702 & 0.2955 & 0 & -0.1086 \\ -0.1025 & -0.0665 & 0.1086 & 0 \end{bmatrix}$$

$$A_2 = X_2 + X_2K_2 - K_2X_2 = \begin{bmatrix} 5.3573 & 3.5446 & 3.9436 & 0.1765 \\ 3.5446 & 3.5269 & 2.6335 & 0.8752 \\ 3.9436 & 2.6335 & 0.8931 & 0.3933 \\ 0.1765 & 0.8752 & 0.3933 & -0.7774 \end{bmatrix}$$

Therefore

$$Q_3 = \begin{bmatrix} -0.7133 & -0.5489 & 0.4353 & 0.0184 \\ -0.5385 & 0.7725 & 0.1051 & -0.3197 \\ -0.4432 & -0.1019 & -0.8639 & 0.2164 \\ -0.6684 & 0.3027 & 0.2304 & 0.9223 \end{bmatrix}; X_3 = Q_3\Lambda_1Q_3^t = \begin{bmatrix} 4.5613 & 2.7009 & 2.2052 & 0.3076 \\ 2.7009 & 2.8257 & 1.8092 & 0.8476 \\ 2.2052 & 1.8092 & 2.2815 & -0.1869 \\ 0.3076 & 0.8476 & -0.1869 & -0.6685 \end{bmatrix}$$

with eigenvalues -1, 1, 1, and 8.

Illustration 4:

Assume the diagonal matrix $\Lambda_2 = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & -1.3 \end{bmatrix}$

The first iteration X_1 is obtained as follows

$$X_1 = Q_1\Lambda_2Q_1^t = \begin{bmatrix} 1.1998 & -0.6003 & 0.8633 & 0.2179 \\ -0.6003 & 0.2994 & 1.7266 & 0.4359 \\ 0.8633 & 1.7266 & 6.1681 & -1.5758 \\ 0.2179 & 0.4359 & -1.5758 & 0.0327 \end{bmatrix}$$

with eigenvalues -1.3, 0.5, 1.5, and 7

The skew-symmetric matrix K_1 is

$$K_1 = \frac{1}{2}(Q_1 - Q_1^t) = \begin{bmatrix} 0 & 0.3364 & -0.6223 & -0.0595 \\ -0.3364 & 0 & -0.2962 & -0.3162 \\ 0.6223 & 0.2962 & 0 & 0.5327 \\ 0.0595 & 0.3162 & -0.5327 & 0 \end{bmatrix}$$

The matrix A_1 is obtained below

$$A_1 = X_1 + X_1K_1 - K_1X_1 = \begin{bmatrix} 2.7042 & 1.1277 & 3.3424 & -0.3291 \\ 1.1277 & 1.1939 & 3.3983 & 0.9136 \\ 3.3424 & 3.3983 & 5.7494 & 0.8301 \\ -0.3291 & 0.9136 & 0.8301 & -1.9475 \end{bmatrix}$$

Whose normalized eigenvectors forms the column vector of the matrix

$$Q_2 = \begin{bmatrix} 0.4555 & 0.8194 & -0.3092 & 0.1598 \\ 0.3971 & -0.4596 & -0.7571 & -0.2405 \\ 0.7931 & -0.2148 & 0.5665 & -0.0634 \\ 0.0764 & -0.2670 & -0.1013 & 0.9553 \end{bmatrix}$$

Which is used for the next step iteration

We obtain $X_2 = Q_2\Lambda_2Q_2^t = \begin{bmatrix} 2.4743 & 0.8684 & 2.1906 & -0.2672 \\ 0.8684 & 1.6319 & 2.1182 & 0.7336 \\ 2.1906 & 2.1182 & 4.6272 & 0.5604 \\ -0.2672 & 0.7336 & 0.5604 & -1.0334 \end{bmatrix}$ with eigenvalues -1.3, 0.5, 1.5, and 7

A skew-symmetric matrix is obtained as follows

$$K_2 = \frac{1}{2}(Q_2 - Q_2^t) = \begin{bmatrix} 0 & 0.2111 & -0.5511 & 0.0417 \\ -0.2111 & 0 & -0.2712 & 0.0132 \\ 0.5511 & 0.2712 & 0 & 0.0190 \\ -0.0417 & -0.0132 & -0.0190 & 0 \end{bmatrix}$$

$$A_2 = X_2 + X_2K_2 - K_2X_2 = \begin{bmatrix} 4.5444 & 2.7806 & 2.6760 & 0.0860 \\ 2.7806 & 3.1280 & 2.8930 & 0.9408 \\ 2.6760 & 2.8930 & 1.0426 & 0.7354 \\ 0.0860 & 0.9408 & 0.7354 & -1.0150 \end{bmatrix}$$

Therefore

$$Q_3 = \begin{bmatrix} 0.6693 & 0.7109 & -0.2025 & -0.0756 \\ 0.5784 & -0.6066 & -0.0144 & -0.5452 \\ 0.4565 & -0.2141 & 0.4925 & 0.7094 \\ 0.0957 & -0.2843 & -0.8463 & 0.4402 \end{bmatrix}$$

$$X_3 = Q_3\Lambda_2 Q_3^t = \begin{bmatrix} 3.9066 & 2.0106 & 1.9305 & 0.2740 \\ 2.0106 & 2.5072 & 2.5423 & 0.9641 \\ 1.9305 & 2.5423 & 0.9947 & -0.2174 \\ 0.2740 & 0.9641 & -0.2174 & 0.2915 \end{bmatrix} \text{ with eigenvalues 7, -1.3, 1.5, and 0.5}$$

Again the iterates X_1, X_2, X_3 obtained using an initial 4×4 singular symmetric matrix in each of the two given cases are non-singular symmetric matrices whose eigenvalues are respectively the same as their assumed diagonal matrices Λ_1 andΛ_2.

IV. CONCLUSION

Singular symmetric and non-singular symmetric matrices have useful applications in many scientific and engineering fields. In the paper, a Newton-type numerical iteration method is employed to deriving sequences of non-singular symmetric matrices from a given singular symmetric matrix. The research was limited to three and four dimensional singular symmetric matrices. We began by deriving a special orthogonal matrix Q , and a skew-symmetric matrix K from some given singular symmetric matrix. Q and K were used as initial guess matrices for the iterative process. We again assumed a diagonal matrix Λ in the neighbourhood of the eigenvalues of the singular matrix. The outcome of these iterations are sequences of non-singular symmetric matrix X_i for $i = 1, 2, 3$. It was realized that the eigenvalues of the non-singular symmetric matrices X_i had the same eigenvalues as the diagonal entries of Λ .

REFERENCES

[1] Emmanuel Akweittey, Kwasi Baah Gyamfi, and Gabriel Obed Fosu. Solubility existence of inverse eigenvalue problem for a class of singular hermitian matrices. *Journal of Mathematics and System Science*, 9:119–123, November 2019.

[2] Li Luoluo. Sufficient conditions for the solvability of an algebraic inverse eigenvalue problem. *Linear Algebra and Its Applications*, 221:117–129, 1995.

[3] Daniel Boley and Gene H Golub. A survey of matrix inverse eigenvalue problems. *Inverse problems*, 3(4):595, 1987.

[4] Chu Moody, Golub Gene, and Golub H Gene. *Inverse eigenvalue problems: theory, algorithms, and applications*, volume 13. Oxford University Press, 2005.

[5] Xu Ying-Hong and Jiang Er-Xiong. An inverse eigenvalue problem for periodic jacobi matrices. *Inverse Problems*, 23(1):165, 2006.

[6] K. Baah Gyamfi, Francis T. Oduro, and Anthony Y. Aidoo. Solution of an inverse eigenvalue problem by newton's method on a fibre bundle with structure group so(n). *Journal of Mathematics and System Science*, 3:124–128, 2013.

[7] Jean Gallier. Notes on differential geometry and lie groups. Gallier Dept. of Computer and Information Science, University of Pennsy lvania, 2011.