ON WEAK CONVERGENCE OF PETTIS INTEGRABLE MULTIFUNCTIONS

TANUSREE CHOUDHURY

ABSTRACT. Here we study weak convergence of sequence of Pettis integrable multifunctions on 2^X.

1. INTRODUCTION

This paper may be considered as the continuation of our paper[5] where we studied some properties of Pettis Integrable Multifunctions. Here we discuss weak convergence of a sequence of Pettis integrable multifunctions and then we use this to characterize sequential weakly compact subset of $P_1(\mu, X)$.
The organization of the paper is as follows. In section 2, we give necessary notations, definitions and preliminaries. In section 3, we discuss our main results.

In [16] , Papageorgiou studied weak convergence of a sequence of integrably bounded multifunctions on 2^X . Here we study weak convergence of a sequence of Pettis integrable multifunctions on 2^X. We actually generalise [16, Theorem 4.1, p.250] to $\mathbb{P}_1(\mu, X)$, the set of all Pettis integrable multifunctions.

2. NOTATIONS, DEFINITIONS AND PRELIMINARIES

Throughout this paper, unless otherwise stated, (Ω, Σ, μ) is a complete finite positive measure space and X is a separable Banach space with $WRNP$ [13]. we shall also assume that the dual X^* of X is norm separable.The closed unit ball of X (respectively X^*) is denoted by B_X (resp. B_{X^*}).$CL(X)$, $C(X)$ and $CWK(X)$ denote the non-empty closed, closed convex and weakly compact convex subsets of X respectively. The symbol $L_p(\mu, X)$, $1 \leq p < \infty$, denotes the Banach space of all equivalence classes of p-th power Bochner integrable functions $f : \Omega \to X$ with respect to the measure μ equipped with the norm

$$\|f\|_p = \left(\int_\Omega \|f\|^p d\mu\right)^{1/p}$$

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A measurable function $f : \Omega \to X$ is said to be scalarly (or weakly) integrable if for each $x^* \in X^*, \langle x^*, f\rangle$ is a member of $L_1(\mu)$, the set of all of μ-integrable real valued functions. A scalarly integrable function is also called Dunford integrable. It is well known that given a scalarly integrable function f and a member $A \in \Sigma$, there exists $x_A^{**} \in X^{**}$, the bidual of X, such that $\langle x^*, x_A^{**}\rangle = \int_A \langle x^*, f\rangle d\mu$, for all $x^* \in X^*$. x_A^{**} is called the Dunford integral of f for all $A \in \Sigma$ and is denoted by $D - \int_A f d\mu$. The scalarly integrable function f is said to be Pettis integrable if for every $A \in \Sigma$, there exists $x_A \in X$ such that $\langle x^*, x_A\rangle = \int_A \langle x^*, f\rangle d\mu$, for all $x^* \in X^*.x_A$ is called the Pettis integral of f over A and is denoted by $P - \int_A f d\mu$ (or simply by $\int_A f d\mu$, if no confusion arises).

We denote by $P_1(\mu, X)$, the space of all scalarly equivalence classes of X-valued Pettis integrable functions $f : \Omega \to X$, equipped with the semivariation norm

$$\|f\|_P = \sup\left\{\int_\Omega |\langle x^*, f\rangle| d\mu; x^* \in B_{X^*}\right\}$$

It is well known that $P_1(\mu, X)$ is a normed linear space which, in general, is not a Banach space.

We can define another topology on $P_1(\mu, X)$ induced by the duality $(P_1(\mu, X), L_\infty(\mu) \otimes X^*)$, since the operation $\langle v \otimes x^*, f\rangle = \int_\Omega v(\omega)\langle x^*, f(\omega)\rangle d\mu, v \in L_\infty(\mu), x^* \in X^*$ is a bilinear form. This topology is known as weak topology of $P_1(\mu, X)$ [11, p. 3].

For every $C \in CL(X)$, the support function of C is denoted by $\sigma(., C)$ and defined on X^* by

$$\sigma(x^*, C) = \sup\{\langle x^*, x\rangle; x \in C\}, \text{ for all } x^* \in X^*.$$

A multifunction $F : \Omega \to CL(X)$ is said to have a measurable graph if the set $G_F = \{(\omega, x) \in \Omega \times X,\ x \in F(\omega)\}$ belongs to $\Sigma \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-algebra. The multifunction $F : \Omega \to CL(X)$ is said to be weakly measurable (or simply measurable) if for every open subset V of X, the set $\{\omega \in \Omega; F(\omega) \cap V \neq \emptyset\}$ belongs to Σ.

The reader is referred to Theorem 1.0 of [3] and [12] for different notions of measurability of a multifunction and their equivalences. A function $f : \Omega \to X$ is said to be a selector of $F : \Omega \to CL(X)$ if $f(\omega) \in F(\omega)$, μ-a.e. The collection of all measurable selectors of F is denoted by S_F. S_F^1 (respectively S_F^P) denotes the family of all Bochner (resp. Pettis) integrable selectors of the multifunction F.

A measurable multifunction $F : \Omega \to CL(X)$ is said to be scalarly integrable if the scalar function $\sigma(x^*, F(\cdot))$ is integrable with respect to μ, for each $x^* \in X^*$.

A measurable multifunction $F : \Omega \to CL(X)$ is said to be Aumann-Pettis integrable (respectively Aumann integrable or simply integrable) if S_F^P (resp. S_F^1) is non-empty. In this case we denote the Aumann-Pettis integral of F over $A \in \Sigma$

by $I_A(F)$ and is defined by $I_A(F) = \left\{\int_A f d\mu; f \in S_F^P\right\}.I_\Omega(F)$ is simply denoted by $I(F)$ [1, p. 341].

A measurable multifunction $F : \Omega \to C(X)$ is said to be Pettis integrable if F is scalarly integrable and for each $A \in \Sigma$, there exists $C_A(F) \in C(X)$ such that

$$\sigma(x^*, C_A(F)) = \int_A \sigma(x^*, F) d\mu, \text{ for each } x^* \in X^*.$$

$C_A(F)$ is called the Pettis integral of F over $A \in \Sigma$ and is denoted by $\int_A F d\mu$.

If $F : \Omega \to CWK(X)$ is a scalarly integrable multifunction then it follows from [1, Theorem 5.4, p. 352] or [17, Theorem 3.2, p. 126] or [18, Theorem 1, p. 228] that F is Aumann-Pettis integrable iff it is Pettis integrable and in this case $I_A(F) = C_A(F) \in CWK(X)$, for each $A \in \Sigma$.

The set of all Pettis integrable multifunctions is denoted by $\mathbb{P}_1(\mu, X)$.
A sequence of Pettis integrable multifunctions $F_n : \Omega \to CWK(X)$ is said to converge weakly to a Pettis integrable multifunction $F : \Omega \to CWK(X)$ if for $x^* \in X^*$,

$$\lim_{n\to\infty} \sigma(x^*, \int_A F_n(\omega) d\mu) = \sigma(x^*, \int_A F(\omega) d\mu), \text{ for all } A \in \Sigma.$$

A multifunction $F : \Omega \to CL(X)$ is said to be integrably bounded if the real valued function

$\omega \to \|F(\omega)\| = \sup\{|\sigma(y^*, F(\omega))|; y^* \in B_{X^*}\}$ is integrable.

A multimeasure is a map $M : \Sigma \to 2^X \setminus \{\emptyset\}$ such that $M(\emptyset) = \{0\}$ and for $\{A_n\} \subseteq \Sigma$ pair-wise disjoint, we have

$$M\left(\underset{n\geq 1}{\cup} A_n\right) = \sum_{n\geq 1} M(A_n).$$

Depending on how we interpret the sum in the right hand side, we have different types of multimeasures. Here we present the two basic ones that we will use in this work.

a) $M(\cdot)$ is a multimeasure (or strong multimeasure) if and only if

$$\sum_{n\geq 1} M(A_n) = \{x : x = \sum x_n \text{ (unconditionally convergent)}, x_n \in M(A_n)\}.$$

b) If the values of $M(\cdot)$ are closed, we define $M(\cdot)$ a weak multimeasure if and only if for every $x^* \in X^*, A \to \sigma(x^*, M(A))$ is a signed measure.

If $M(\cdot)$ is a multimeasure and $A \in \Sigma$, we define

$|M|(A) = \sup_\pi \sum_{i=1}^n \|M(A_i)\|$, where the supremum is taken over all finite partitions $\pi = \{A_1, A_2, \ldots, A_n\}$ of A.

If $|M|(\Omega) < \infty$ then $M(\cdot)$ is said to be of bounded variation. The multimeasure M is said to be of σ-finite variation if there exists a countable partition $\{A_1, A_2, \ldots, A_n, \ldots, \}$ of Ω such that $|M|(A_n) < \infty$, for all $n = 1, 2, \ldots$.

It is easy to see that $|M|(\cdot)$ is a positive measure.

Finally, $M(\cdot)$ is said to be $\mu(\cdot)$ continuous if $\mu(A) = 0$ implies $M(A) = \{0\}$. Again $M(\cdot)$ is $\mu(\cdot)$ continuous if and only if $|M|(\cdot)$ is so.

A vector measure $m : \Sigma \to X$ such that $m(A) \in M(A)$ for all $A \in \Sigma$ is said to be a measure selector of $M(\cdot)$. The set of all measure selectors of M is denoted by S_M.

3. Main Results

3.1. **Weak Convergence of Pettis Integrable Multifunction.**

Theorem 3.1.1. *If $F_n : \Omega \to CWK(X)$ are Pettis integrable multifunctions and $F_n(\omega) \subset F(\omega)$, μ a.e. for all $n \geq 1$, where $F : \Omega \to CWK(X)$ is a Pettis integrable multifuonction, then there exists a subsequence $\{F_{n_k}\} = \{F_k\}$ of F_n and a Pettis integrable multifunction $G : \Omega \to CWK(X)$ such that $\{F_k\}$ converges to G weakly.*

Proof. Put $M_n(A) = \int_A F_n(\omega)d\mu$, $n \geq 1$, for each $A \in \Sigma$. It follows from [1, Theorem 5.4, p.352] that $M_n(\cdot)$ is $CWK(X)-$ valued for all $n \geq 1$.
Now $\sigma(x^*, M_n(A)) = \int_A \sigma(x^*, F_n(\omega))d\mu$, for all $n \geq 1$, for each $x^* \in X^*$ and $A \in \Sigma$.
Since $F_n(\cdot)$ are scalarly integrable for all $n \geq 1$,
for all $n \geq 1$, $\sigma(x^*, M_n(\cdot))$ are signed measure for each $x^* \in X^*$.
Therefore,$M_n : \Sigma \to CWK(X)$ are weak multimeasures for all $n \geq 1$ and hence multimeasure by [10, Proposition 3, p.113].
Now by the hypothesis, $F_n(\omega) \subset F(\omega), \mu$ a.e., for all $n \geq 1$.
So we have,
$\sigma(x^*, F_n(\omega)) \leq \sigma(x^*, F(\omega))$, for all $n \geq 1$ and $x^* \in X^*$.
Hence $\sup_n \int_A |\sigma(x^*, F_n(\omega))|\, d\mu \leq \int_A |\sigma(x^*, F(\omega))|\, d\mu$, for each $x^* \in X^*$ and $A \in \Sigma$.
As F is Pettis integrable multifunction, $\{\sigma(x^*, F(\cdot))\}$ is uniformly integrable for each $x^* \in X^*$ [1,Theorem 5.4, p.352].
Hence,

$$\lim_{\mu(A)\to 0} \sup_n \int_A |\sigma(x^*, F_n(\omega))| d\mu = 0, \text{ for each } x^* \in X^*.$$

Fixed $x^* \in X^*$.
The set $\{\sigma(x^*, F_n(\cdot)), n \geq 1\}$ is uniformly integrable subset of $L_1(\mu)$.
Hence By [8,Theorem 15,p.76] ,the set $\{\sigma(x^*, F_n(\cdot)), n \geq 1\}$ is relatively weakly

compact subset of $L_1(\mu)$.
So by Eberlien-Smulian Theorem [9,Theorem1,p.430], the set $\{\sigma(x^*, F_n(\cdot)), n \geq 1\}$ is weakly sequentially compact in $L_1(\mu)$.
So there exists a subsequence $\{\sigma(x^*, F_{n_k}(\cdot))\} = \{\sigma(x^*, F_k(\cdot))\} \subset \{\sigma(x^*, F_n(\cdot)), n \geq 1\}$ and $h_{x^*} \in L_1(\mu)$, such that, $\{\sigma(x^*, F_k(\cdot))\}$ converges weakly to $h_{x^*} \in L_1(\mu)$.
So,

$$\lim_{k\to\infty} \int_A \sigma(x^*, F_k(\omega))d\mu = \int_A h_{x^*} d\mu, \text{ for each } A \in \Sigma. \tag{3.1.1}$$

That is,

$$\lim_{k\to\infty} \sigma(x^*, M_k(A)) = \int_A h_{x^*} d\mu, \text{ for each } A \in \Sigma. \tag{3.1.2}$$

Put

$$\lim_{k\to\infty} \sigma(x^*, M_k(A)) = \lambda_{x^*}(A), \text{for each} A \in \Sigma. \tag{3.1.3}$$

Then by Nikodym Convergence Theorem,
$A \to \lambda_{x^*}(A)$ is a signed measure, for $x^* \in X^*$.
Now $M_k(A) = \int_A F_k(\omega)d\mu \subseteq \int_A F(\omega)d\mu = I_A(F)$, for all $A \in \Sigma$ and $k \in \mathbb{N}$.

So, $|\sigma(x^*, M_k(A)| \leq |\sigma(x^*, I_A(F)|$, for all $A \in \Sigma$ and $k \in N$.

And hence, $|\lambda_{x^*}(A))| \leq |\sigma(x^*, I_A(F))|$, for all $A \in \Sigma$ and $k \in N$.
Since $I_A(F)$ is $CWK(X)$-valued [1, Theorem 5.4,p.352] ,$\sigma(x^*, I_A(F))$ is continuous for Mackey Topology $\tau(X^*, X)$.[1, Proposition 1.5c,p.333]
So by the same Proposition, there exists an $M(A) \in CWK(X)$ such that

$$\lambda_{x^*}(A) = \sigma(x^*, M(A)), \text{ for each } A \in \Sigma. \tag{3.1.4}$$

So the map $M : \Sigma \to CWK(X)$ is a weak multimeasure and hence multimeasure.[10, Proposition 3, p.113].
It is easy to prove that M is μ-continuous. Also M is of σ-finite variation by [7, Proposition 1,p.1516]
Also any measure selection $m \in S_M$ of M possesses a Pettis integrable density. Hence by [7, Theorem 3, p. 1517] , M posseses a Pettis integrable density i.e. there exists a Pettis integrable multifunction $G : \Sigma \to CWK(X)$ such that

$$M(A) = \int_A G(\omega)d\mu, \text{ for all } A \in \Sigma$$

and so,

$$\lambda_{x^*}(A) = \sigma(x^*, M(A)) = \int_A \sigma(x^*, G(\omega))d\mu, \text{ for all } A \in \Sigma. \tag{3.1.5}$$

So by (3.1.1), (3.1.2), (3.1.3,(3.1.6) and (3.1.7) we have,

$$\lim_{K\to\infty}\int_A \sigma(x^*, F_k(\omega))d\mu = \int_A h_{x^*}(\omega)d\mu$$

Or,

$$\lim_{k\to\infty}\int_A \sigma(x^*, F_k(\omega))d\mu = \lambda_{x^*}(A) = \sigma(x^*, M(A)) = \int_A \sigma(x^*, G(\omega))d\mu, \text{ for each } A \in \Sigma.$$

As F_n ,for all $n \in \mathbb{N}$, and G are Pettis integrable , We have,

$$\lim_{k\to\infty}\sigma(x^*, \int_A F_k(\omega))d\mu = \sigma(x^*, \int_A G(\omega))d\mu, \text{ for each } A \in \Sigma.$$

Let $\{x^*_m\}$ be a sequence in X^* which is dense in strong topology of X^* i.e. the topology of uniform convergence on bounded subset of X.
Now using a diagonal process , a subsequence $\{F_{n_k}\} = \{F_k\}$ of $\{F_n\}$ exists such that

$$\lim_{k\to\infty}\sigma(x^*_m, \int_A F_k(\omega))d\mu = \sigma(x^*_m, M(A)) = \sigma(x^*_m, \int_A G(\omega))d\mu, \text{ for each } A \in \Sigma \text{ and for each } m \in \mathbb{N}.$$

Now for each $x^* \in X^*$, there exists a subsequence $\{x^*_{m_k}\}$ of $\{x^*_m\}$ converging in norm to $x^* \in X^*$.
Since each $F_n(\omega) \subset F(\omega)$,for all $\omega \in \Omega$ and as $F(\omega)$ is $CWK-$ valued, it follows that

$$\lim_{k\to\infty}\sigma(x^*, \int_A F_k(\omega))d\mu = \sigma(x^*, \int_A G(\omega))d\mu, \text{ for each } A \in \Sigma \text{ and for each } x^* \in X^*.$$

Hence $\{F_k\}$ converges to G weakly. □

Theorem 3.1.2. *If $F_n : \Omega \to CWK(X)$ are Pettis integrable multifunctions such that*

(a) *$\{\sigma(x^*, F_n(\cdot)), n \geq 1\}$ is uniformly integrable subset of $L_1(\mu)$ for each $x^* \in X^*$.*
(b) *For all $A \in \Sigma, K(A) = w - cl\{\cup_{n\geq 1}\int_A F_n(\omega)d\mu\}$ is weakly compact in X.*
(c) *For all vector measure $m : \Sigma \to X$ with $m(A) \in \overline{co}K(A)$, for all $A \in \Sigma$, there exists $g \in P_1(\mu, X)$ such that*

$$m(A) = \int_A g(\omega)d\mu.$$

then there exists a subsequence $\{(F_{n_k}\} = \{F_k\}$ *of* F_n *and a Pettis integrable multifunction* $G : \Omega \to CWK(X)$ *such that* $\{F_k\}$ *converges to* G *weakly.*

Proof. Proceeding as in the proof of the Theorem 3.1.1, we produce a signed measure λ_{x^*} such that $|\lambda_{x^*}(A)| \leq |\sigma(x^*, \hat{K}(A))|$, where $\hat{K}(A) = \overline{co}[K(A) \cup (-K(A))] \in CWK(X)$ for all $A \in \Sigma$.

Now as $\hat{K}(A) \in CWK(X)$, $\sigma(x^*, \hat{K}(A))$ is continuous for Mackey Topology $\tau(X^*, X)$.[1, Proposition 1.5c,p.333] and so is $\lambda_{x^*}(A)$ for all $A \in \Sigma$ and for each $x^* \in X^*$.

So by the same Proposition, there exists an $M(A) \in CWK(X)$ such that

$$\lambda_{x^*}(A) = \sigma(x^*, M(A)), \text{ for each } A \in \Sigma. \tag{3.1.6}$$

So the map $M : \Sigma \to CWK(X)$ is a weak multimeasure and hence multimeasure.[10, Proposition 3, p.113].

It is easy to prove that M is μ-continuous. Also M is of σ-finite variation by [7, Proposition 1,p.1516]

Also any measure selection $m \in S_M$ of M possesses a Pettis integrable density. Hence by [7, Theorem 3, p. 1517] , M posseses a Pettis integrable density i.e. there exists a Pettis integrable multifunction $G : \Sigma \to CWK(X)$ such that

$$M(A) = \int_A G(\omega) d\mu, \text{ for all } A \in \Sigma$$

and so,

$$\lambda_{x^*}(A) = \sigma(x^*, M(A)) = \int_A \sigma(x^*, G(\omega)) d\mu, \text{ for all } A \in \Sigma. \tag{3.1.7}$$

Now proceeding as the proof of the Theorem 3.1.1, we find a a subsequence $\{(F_{n_k}\} = \{F_k\}$ of F_n such that

$$\lim_{k\to\infty} \sigma(x^*, \int_A F_k(\omega)) d\mu = \sigma(x^*, \int_A G(\omega)) d\mu, \text{ for each } A \in \Sigma \text{ and for each } x^* \in X^*.$$

Hence

$\{F_k\}$ converges to G weakly. ∎

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Department of Mathematics, Raja Peary Mohan College, Calcutta University, Hooghly-712258, West Bengal, India

E-mail address: choudhurytanusree@yahoo.co.in