Certain Transformation Formulae for q-Hypergeometric Series

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Abstract
In present paper we have taken certain known summation formulae due to Verma and Jain and by making use of Bailey's transformation an attempt has made to establish certain beautiful and interesting transformation formulae for q-Hypergeometric series.

Keywords: *Basic Hypergeometric series; transformation; summation formulae*

Introduction

In 1947 W.N. Bailey [1] established the following result.

If

$$\beta_n = \sum_{r=0}^{n} \alpha_r U_{n-r} V_{n+r} \tag{1}$$

$$\text{and } \gamma_n = \sum_{r=0}^{n} \delta_r U_{r-n} V_{r+n} = \sum_{r=0}^{n} \delta_{r+n} U_r V_{r+2n} \tag{2}$$

Then under suitable convergence conditions

$$\sum_{r=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{3}$$

Where α_r, δ_r, U_r and V_r are any functions of r only, such that the series γ_r exists.

Making use of (3), Bailey developed a technique to obtain various transformation formulae for ordinary and q-series which play an important role in the number theory and transformation theory of hypergeometric series. Recently Singh (2), has obtained many transformation formulae for q-series by using baileys transformation and certain known results due to Verma and Jain (3). In present paper, we have made to establish certain transformation formulae for q-hypergeometric series by using Baileys transformation and some known summation formulae due to Verma and Jain (3) and also by Verma (5).

II. Notation and Definitions

A generalized basic hypergeometric function in defined by L.J. Stater (4); and Exton (6): also by Srivastava and Karlson (7) is as under.

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots\ldots, a_r; q; z \\ b_1, b_2, \ldots\ldots, b_s;\ q^i \end{matrix}\right] = \sum_{n=o}^{\infty} q^{i\binom{m}{2}} \frac{(a_1)_n (a_2)_n \ldots\ldots (a_r)_n Z^n}{(b_1)_n (b_2)_n \ldots\ldots (b_s)_n (q)_n} \tag{4} $$

Valid for |z|<1 provided no zeroes appears in denominator. Here a_1, a_2, a_3,..........a_r and b_1, b_2...........,b_s and Z are assumed to be complex numbers.

The shifted factorial in defined by

$$ (a;q)_n = \begin{cases} 1, & \text{if} \quad n = o \\ (1-a)(1-aq)\ldots\ldots(1-aqn-1) & \text{if} \quad n = 1,2,3\ldots\ldots \end{cases} \tag{5} $$

And for real or complex q |q|<1 we have

$$ (a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n) $$

$$ \text{and } (a;q)_\infty = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{6} $$

$$ \text{and } {}_A\phi_B\left[\begin{matrix} (a); q; z \\ (b); i \end{matrix}\right] = \sum_{n=o}^{\infty} q^{in(n-1)/2} \frac{\prod_{J=1}^{A} (a_J;q)_n Z^n}{\prod_{J=1}^{B} (b_J;q)_n (q;q)_n} \tag{7} $$

in the special case when i=0 the first member of (7) will be written simply as

$$ {}_A\phi_B\left[\begin{matrix} (a); q; z \\ (b) \end{matrix}\right] $$

We shall use the following known results to establish our transformations.

$$ {}_2\phi_1\left[\begin{matrix} a, b; q; c/ab \\ c \end{matrix}\right] = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c/ab;q)_\infty (c;q)_\infty} \tag{8} $$

Slater [4; APP IV IV-2]

$$ {}_4\phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q; -q^{n-1/2} \\ \sqrt{a}, -\sqrt{a}, aq^{1+n} \end{matrix}\right] $$

$$ = \frac{1}{2}\left\{\frac{(aq;q)_n(-q^{1/2};q)_n}{(\sqrt{a}q;q)_n(-q\sqrt{a};q)_{n-1}}\right\} + \left\{\frac{(aq;q)_n(-q^{1/2};q)_n}{(-\sqrt{a}q;q)_n(q\sqrt{a};q)_{n-1}}\right\} \tag{9} $$

[Verma and Jain 3; (4.1)]

$$ {}_2\phi_1\left[\begin{matrix} a,q^{-n};q;-q^{n+1/2} \\ aq^{n+1} \end{matrix}\right] = \frac{1}{2}\left\{\frac{(1+\sqrt{a})(aq;q)_n(-\sqrt{q};q)_n}{(-\sqrt{a}q;q)_n(q\sqrt{a};q)_n}\right\}+\left\{\frac{(1-\sqrt{a})(aq;q)_n(-\sqrt{q};q)_n}{(\sqrt{a}q;q)_n(-q\sqrt{a};q)_n}\right\} \qquad (10) $$

[Verma and Iain 3; (4.3)]

III. Main Results

In this section we shall establish our main results.

(i) Let us suppose

$$ U_n=\frac{1}{(q;q)_n},V_n=\frac{1}{(aq;q)_n},\alpha_n=\frac{(a;q\sqrt{a},-q\sqrt{a};q)_n}{(\sqrt{a},-\sqrt{a},q;q)_n}q^{n\frac{(n-2)}{2}} $$

and $\delta_n=(b;c;q)_n\left(\frac{aq}{bc}\right)^n$

In Baiely's transformation (1) and (2) respectively, weget

$$ \beta_n=\left[\frac{1+q^n\sqrt{a}}{2}\frac{(-q^{-1/2};q)_n}{(\sqrt{aq},-q\sqrt{a},q;q)_n}+\frac{1-q^n\sqrt{a}}{2}\frac{(-q^{-1/2};q)_n}{(-\sqrt{aq},q\sqrt{a},q;q)_n}\right] \text{ by using (9)} $$

$$ \text{and} \quad \gamma_n=\frac{\left(\frac{aq}{b},\frac{aq}{c};q\right)_\infty(b,c;q)_n}{\left(aq,\frac{aq}{bc};q\right)_\infty\left(\frac{aq}{b},\frac{aq}{c};q\right)_n}\left(\frac{aq}{bc}\right)^n \quad \text{By using (8)} $$

Now, putting these value of α_n,β_n, γ_n and δ_n in (3), we get the transformation formulae.

$$ {}_5\phi_4\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,c;q;\frac{aq^{1/2}}{bc} \\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{c};q \end{matrix}\right] $$

$$ =\frac{\left(aq,\frac{aq}{bc};q\right)_\infty}{2\left(\frac{aq}{b},\frac{aq}{c};q\right)_\infty} \quad \text{x} \quad \left\{{}_3\phi_2\left[\begin{matrix} b,c,-q^{-1/2};q;\frac{aq}{bc} \\ \sqrt{aq},-q\sqrt{a} \end{matrix}\right]\right. $$

$$ +\sqrt{a}\,{}_3\phi_2\left[\begin{matrix} b,c,-q^{-1/2};q;\frac{aq^2}{bc} \\ \sqrt{aq},-q\sqrt{a} \end{matrix}\right]+{}_3\phi_2\left[\begin{matrix} b,c,-q^{-1/2};q;\frac{aq}{bc} \\ -\sqrt{aq},q\sqrt{a} \end{matrix}\right] $$

$$ \left.-\sqrt{a}\,{}_3\phi_2\left[\begin{matrix} b,c,-q^{-1/2};q;\frac{aq^2}{bc} \\ -\sqrt{aq},q\sqrt{a} \end{matrix}\right]\right\} $$

(ii) Choosing

$$U_n = \frac{1}{(q;q)_n}, V_n = \frac{1}{(aq;q)_n}, \alpha_n = \frac{(a;q)_n}{(q;q)_n} q^{\frac{n^2}{2}}$$

and $\delta_n = (e_1, e_2; q)_n \dfrac{aq}{e_1 e_2}$ in (1) and (2)

We get

$$\left[\frac{1+\sqrt{a}}{2}\frac{(-\sqrt{q};q)_n}{(-\sqrt{aq}, q\sqrt{a}, q;q)_n} + \frac{1-\sqrt{a}}{2}\frac{(-\sqrt{q};q)_n}{(\sqrt{aq}, -q\sqrt{a}, q;q)_n}\right] \text{by making use of (10)}$$

$$\text{and } \gamma_n \frac{\left(\frac{aq}{e_1}, \frac{aq}{e_2}; q\right)_\infty}{\left(aq, \frac{aq}{e_1 e_2}; q\right)_\infty} \frac{(e_1, e_2; q)_n}{\left(\frac{aq}{e_1}, \frac{aq}{e_2}; q\right)_n} \left(\frac{aq}{e_1 e_2}\right)^n \qquad \text{By using (8)}$$

Now putting these values of α_n, β_n, γ_n and δ_n in (3), we get the following transformation.

$${}_3\phi_2\left[\begin{matrix} a, e_1, e_2; q; \frac{aq^{3/2}}{e_1 e_2} \\ \frac{aq}{e_1}, \frac{aq}{e_2}; q \end{matrix}\right]$$

$$= \frac{\left(aq, \frac{aq}{e_1 e_2}; q\right)_\infty}{\left(\frac{aq}{e_1}, \frac{aq}{e_2}; q\right)_\infty} \text{x} \left\{ \frac{1+\sqrt{a}}{2} {}_3\phi_2\left[\begin{matrix} e_1, e_2 - \sqrt{q}; q; \frac{aq}{e_1 e_2} \\ \sqrt{aq}, -q\sqrt{a} \end{matrix}\right] \right.$$

$$\left. + \frac{1-\sqrt{a}}{2} {}_3\phi_2\left[\begin{matrix} e_1, e_2 - \sqrt{q}; q; \frac{aq}{e_1 e_2} \\ \sqrt{aq}, -q\sqrt{a} \end{matrix}\right] \right\}$$

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