

Remarks and Specific Examples on Symmetric and Skew-Symmetric Operators in Hilbert Spaces

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Abstract: In this paper, classes of symmetric and skew-symmetric operators on a Hilbert Space are characterised. It is demonstrated that skew-symmetric operators admit skew-symmetric matrix representation with respect to some orthonormal basis. It will also be shown that the characteristic polynomial of a self adjoint operator on an n -dimensional Hilbert Space, H has n real zeros (counted with multiplicity). Further, a specific example of a normal form of a skew-adjoint operator shall be given and then be shown that the rank of a skew-symmetric operator is always even. By considering a forward shift operator on a Hilbert space, it is demonstrated that not every skew-symmetric operator is biquasitriangular. Finally, the relationship between complex symmetric and skew-symmetric operators is established.

Keywords and Phrases: Self-adjoint operators, Hilbert space, Complex-symmetric operators, Skew-symmetric operators, Bitriangular and Conjugation operators.

I. INTRODUCTION

In this paper, Hilbert spaces or subspaces will be denoted by capital letters, H, H_1, H_2, K, K_1, K_2 etc and T, T_1, T_2, A, B , denote bounded linear operators where an operator means a bounded linear transformation. $B(H)$ will denote the bounded linear operators on a complex separable Hilbert space H . $B(H, K)$ denotes the set of bounded linear transformations from H to K , which is equipped with the (induced uniform) norm. The following definitions are of essence:

Definition 1.1: Let H be a linear (vector) space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$. Suppose we have the function $\| \cdot \|: H \rightarrow [0, \infty)$ such that

1. $\|x\| = 0$ if and only if $x = 0$
2. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in H$ and
3. $\|\alpha x\| = |\alpha| \|x\|$ for all scalars α and vectors x .

We call $(H, \| \cdot \|)$ a *normed linear space*. The second property in the above definition is called the triangle inequality and the third is the homogeneity property.

The *Euclidean space* \mathbb{C}^n , defined by $H = \mathbb{C}^n = \{(z_1, z_2, \dots, z_n) : z_j \in \mathbb{C}\}$ with

$\|(z_1, z_2, \dots, z_n)\| = (\sum_{j=1}^n |z_j|^2)^{1/2}$ is a normed linear space. The *Euclidean space* \mathbb{R}^n is defined similarly where we restrict the set H to real values.

Definition 1.2: Let H be a linear (vector) space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$.

An *inner product* is a bilinear function $\langle \cdot, \cdot \rangle: H \times H \rightarrow \mathbb{C}$ with the following properties:

1. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \quad \forall x, y, z \in H$ and $a, b \in K$, that is, linearity to the first argument is satisfied;
2. $\langle z, ax + by \rangle = \bar{a}\langle z, x \rangle + \bar{b}\langle z, y \rangle \quad \forall x, y, z \in H$ and $a, b \in K$, that is, semi-linearity to the second argument is satisfied;
3. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in H$. This property is called the complex conjugation;
4. $\langle x, x \rangle \geq 0 \quad \forall x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$. This is the non-negative (or positive definite) property.

A linear space equipped with an inner product is called an inner product space. This will be denoted by the set $(H, \langle \cdot, \cdot \rangle)$. A *Hilbert space* is a complete inner product space. The norm $\|x\|$ of a vector $x \in H$ is defined as the positive square-root $\|x\| = \langle x, x \rangle^{1/2}$.

We note that the restriction of the bilinear function $\langle \dots \rangle$ to a subspace $K \subset H$ satisfies the properties of an inner product and by this fact, every subspace of an inner product space is itself an inner product space.

Definition 1.3: A function T which maps H_1 into H_2 is called a linear operator if for all $x, y \in H_1$ and α a complex number, then the following two properties hold: $T(x + y) = T(x) + T(y)$ and $T(\alpha x) = \alpha T(x)$. If H_1 and H_2 are normed linear spaces, we say that the linear transformation is a bounded linear operator from H_1 to H_2 if there exists a finite constant C such that $\|Tx\|_{H_2} \leq C \|x\|_{H_1} \forall x \in H_1$.

Definition 1.4: If $T \in B(H)$ then its adjoint T^* is the unique operator in $B(H)$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle \forall x, y \in H$.

Definition 1.5: An operator $T \in B(H)$ which is self adjoint is said to be positive if $\langle Tx, x \rangle \geq 0 \forall x \in H$.

Definition 1.6: An operator $T \in B(H_1, H_2)$ is said to be invertible if there exists an operator $T^{-1} \in B(H_2, H_1)$ such that $T^{-1}Tx = x$ for every $x \in H_1$ and $TT^{-1}y = y$ for every $y \in H_2$. The operator T^{-1} is called the inverse of T .

We also need the following terminologies in this research:

An operator $T \in B(H)$ is said to be:

self – adjoint or *Hermitian* if $T^* = T$ or equivalently, if $\langle Tx, y \rangle = \langle x, T^*y \rangle \forall x, y \in H$,

unitary if $T^*T = TT^* = I$,

a *symmetry* if $T^* = T = T^{-1}$, that is T is a self-adjoint unitary,

normal if $T^*T = TT^*$ (equivalently, if $\|Tx\| = \|T^*x\| \forall x \in H$),

binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$,

If H and K are Hilbert spaces, then their (orthogonal) direct sum will be denoted by $H \oplus K$, which itself is a Hilbert space. By a subspace of a Hilbert space H we mean a closed linear manifold of H , which is also a Hilbert space. If M and N are orthogonal (denoted by $M \perp N$) subspaces of a Hilbert space H , then their (orthogonal) direct sum $M \oplus N$ is a given subspace of H . For any set $M \subseteq H$, M^\perp will denote the orthogonal complement of M in H which is a subspace of H . If M is a subspace of H , then H can be decomposed as $H = M \oplus M^\perp$.

A set M in H is invariant for T if $T(M) \subseteq M$. M is an invariant subspace for T if it is a subspace of H which, as a subset of H , is invariant for T . A subspace M of H is invariant for T if and only if M^\perp is invariant for T .

A subspace M reduces T (or M is a reducing subspace for T) if both M and M^\perp are invariant under T (equivalently, if M is invariant for both T and T^*).

If M is an invariant subspace for T then, relative to the decomposition $H = M \oplus M^\perp$, the operator T can be written as

$$T = \begin{bmatrix} T|_M & X \\ 0 & Y \end{bmatrix} \text{ for operators } X: M^\perp \rightarrow M \text{ and } Y: M^\perp \rightarrow M^\perp,$$

where $T|_M: M \rightarrow M$ is the restriction of T on M .

An operator $T \in B(H)$ is said to be a *right shift* operator if $Tx = y$ where $x = (x_1, x_2, \dots)$ and $y = (0, x_1, x_2, \dots) \in l^2$, also called the *unilateral shift* or *forward weighted shift* operator with weights 1. The kernel (or null space) of T , $Ker(T)$ is the set $N(T) = \{x \in H: Tx = 0\}$. The range of $T \in B(H, K)$ is the set $Ran(T) = R(T) = T(H) = \{y \in K: y = Tx \text{ for some } x \in H\}$. $N(T)$ is a subspace of H i.e, it is a linear manifold that is closed in H for every $T \in B(H, K)$ and $R(T)$ is a linear manifold that is not necessarily closed in K . An operator T is finite-dimensional if $R(T)$ is finite-dimensional.

A scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of an operator $T \in B(H)$ if there exists a nonzero vector $x \in H$ such that $Tx = \lambda x$; equivalently, if $Ker(\lambda I - T) \neq \{0\}$.

Let H be a Hilbert space and $T \in B(H)$. The set $\rho(T)$ of all complex number λ for which $(\lambda I - T)$ is invertible is called the *resolvent set* of T . Equivalently,

$$\rho(T) = \{ \lambda \in \mathbb{C}: Ker(\lambda I - T) = \{0\} \text{ and } Ran(\lambda I - T) = H \}.$$

The complement of the resolvent set $\rho(T)$ denoted by $\sigma(T)$ is called the *spectrum* of T i.e,

$\sigma(T) = \mathbb{C} \setminus \rho(T) = \{ \lambda \in \mathbb{C}: Ker(\lambda I - T) \neq \{0\} \text{ or } Ran(\lambda I - T) \neq H \}$, which is the set of all λ such that $(\lambda I - T)$ fails to be invertible (i.e. fails to have a bounded inverse on $Ran(\lambda I - T) = H$). On the basis of this failure, the spectrum can be split into many disjoint parts. A classical disjoint partition comprises of three parts: the set of those $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ has no inverse, denoted by $\sigma_p(T)$ is called the point spectrum of, T i.e, $\sigma_p(T) = \{ \lambda \in \mathbb{C}: Ker(\lambda I - T) \neq \{0\} \}$, which is exactly the set of all eigenvalues of T . The set of all

those $\lambda \in \mathbb{C}$ for which $(\lambda I - T)$ has a densely defined but unbounded inverse on its range, denoted by $\sigma_c(T)$ is called the continuous spectrum of T , i.e.,

$$\sigma_c(T) = \{ \lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}, \overline{Ran(\lambda I - T)} = H \text{ and } Ran(\lambda I - T) \neq H \}.$$

If $(\lambda I - T)$ has an inverse that is not densely defined, then λ belongs to the residual spectrum of T , denoted $\sigma_r(T)$. That is, $\sigma_r(T) = \{ \lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}, \overline{Ran(\lambda I - T)} \neq H \}$.

The parts $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are pairwise disjoint and $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$.

In the study of Hilbert space theory, self-adjoint, skew-adjoint and normal operators find a wide range of applications especially in quantum mechanics. Such classes of operators have been extensively studied in [11], [12] and [13]. Garcia and Putinar ([2], [3]) and Garcia and Wogen [4] also analyzed an important class of operators on a Hilbert space, namely the class of complex symmetric operators. Under this, a conjugation mapping was considered such that $CTC^{-1} = T^*$, where $C = C^{-1}$ for a given bounded linear operator T . The same authors demonstrated also that normal operators, binormal operators, Hankel operators and Volterra integration operators are complex symmetric.

Closely related to complex symmetric operators is the class of skew symmetric operators, which have been studied by Li and Zhu [10] and also Zhu S [16]. This class was also defined in terms of a conjugation map such that $CTC^{-1} = -T^*$, where $C = C^{-1}$, i.e. an operator T is considered skew symmetric if it is represented as a skew symmetric matrix relative to some orthonormal basis $\{e_n\}$ of a Hilbert space H .

Li and Zhu [10] further characterized normal operators that were skew symmetric and gave two structure results for skew symmetric normal operators. In their paper, they showed that for a given bounded linear operator T on a Hilbert space H , it can be decomposed as a sum of a complex symmetric operator A and a skew symmetric operator B such that $T = A + B$. This was achieved by arbitrarily making a choice of a conjugation C on H and setting $A = \frac{1}{2}(T + CT^*C^{-1})$ and $B = \frac{1}{2}(T - CT^*C^{-1})$. In a certain sense, this reflected some universality of complex symmetric and skew symmetric operators.

A generalized result of constructing a skew symmetric operator for a given complex symmetric operator was proved, (see [10], Proposition 1.9). Examples of skew symmetric operators on finite dimensional spaces and in particular the Jordan blocks with even ranks were also outlined. Zhu [16] further gave an analysis of skew symmetric operators with non-zero eigenvalues by providing an upper triangular operator matrix representation. In addition, a description of triangular operators based on the geometric relationship between eigenvalues was also given.

II. SYMMETRIC OPERATORS

Remark 2.1: We demonstrate that a self-adjoint unitary (or a symmetric) operator of a finite dimensional Hilbert space H (of dimension, say n) has n eigenvectors which are an orthogonal basis. This is described in what we call the *eigen-value problem*. In other terms, we aim to show that the characteristic polynomial of a self-adjoint operator has n zeros if every zero is counted with multiplicity (see more results on the eigenvalue problem which have extensively been discussed in Greub [5]).

Now consider a system of n -orthogonal vectors can be found once an eigenvector of an operator T has been constructed. The system of these e_v n -eigenvectors $e_v (v = 1, 2, \dots, n)$ is such that

$$\langle e_v, e_\mu \rangle = \delta_{v\mu}. \tag{2.1}$$

An orthonormal basis of the eigenvectors e_v takes the form

$$Te_v = \lambda_v e_v \tag{2.2}$$

where λ_v denotes the eigenvalues of e_v . These equations show that the matrix of a self adjoint operator has diagonal form if the eigenvectors are used as a basis. The following definition follows immediately as a consequence of these results:

Definition 2.2: If λ is an eigenvalue of T , the corresponding *eigen-space* $H(\lambda)$ is the set of all vectors x satisfying the equation $Tx = \lambda x$. It is also clear that two eigen-spaces $H(\lambda)$ and $H(\lambda')$ corresponding to different values are orthogonal, that is if λ and λ' are eigenvalues, then

$Te = \lambda e$ and $Te' = \lambda e'$. It follows then that

$$\langle e', Te \rangle = \lambda \langle e, e' \rangle \tag{2.3}$$

and

$$\langle e', Te' \rangle = \lambda' \langle e, e' \rangle \tag{2.4}$$

Subtracting these two equations we obtain $(\lambda' - \lambda)\langle e, e' \rangle = 0$ in which case $\langle e, e' \rangle = 0$ if $\lambda' \neq \lambda$.

If we denote by $\lambda_v (v = 1, 2, \dots, r)$ the different values of T , then every two eigenspaces $H(\lambda_i)$ and $H(\lambda_j)$ $i \neq j$ are orthogonal. Since every $x \in H$ can be written as a linear combination of eigenvectors it follows that the different sum $H(\lambda_i)$ is H and is obtained as

$$H = H(\lambda_1) \oplus \dots \oplus H(\lambda_r). \tag{2.5}$$

Letting T_i be the transformation induced by T in $H(\lambda_i)$, we have

$$T_i x = \lambda_i x, x \in H(\lambda_i).$$

This implies that the characteristic polynomial of T_i is given by

$$\det(T_i - \lambda_i) = (\lambda - \lambda_i)^{k_i} \quad (i = 1, 2, \dots, r) \tag{2.6}$$

where k_i is the dimension of $H(\lambda_i)$. Thus by (5) and (6), the characteristic polynomial of T is equal to the product

$$\det(T - \lambda) = (\lambda_1 - \lambda)^{k_1} \dots (\lambda_r - \lambda)^{k_r} \tag{2.7}$$

that is (2.7) clearly shows that the characteristic polynomial of a self-adjoint operator has n real zeros, if every zero is counted with multiplicity.

Remark 2.3: In view of the illustration of an adjoint, we see that a self-adjoint operator (in this case an $n \times n$ matrix) $A = (a_{\mu\nu}^H)$ has n real eigenvalues. For instance we can consider the transformation given by $T x_\nu = \sum_{\mu} a_{\mu\nu}^H x_\mu$ ($\nu = 1, 2, \dots, r$) where x_ν ($\nu = 1, 2, \dots, r$) is an orthonormal basis of H . Then T is self-adjoint and hence the characteristic polynomial of T has the form (2.7). It is also known that

$$\det(A - \lambda_i) = \det(A - \lambda_j) \tag{2.8}$$

These two equations (2.7) and (2.8) give us

$$\det(A - \lambda_j) = (\lambda_1 - \lambda)^{k_1} \dots (\lambda_r - \lambda)^{k_r}.$$

III. SKEW-ADJOINT AND SKEW-SYMMETRIC OPERATORS

To study these classes of operators, we shall consider the following definitions.

Definition 3.1: If $x \neq 0$ is a vector such that for some λ , $Tx = \lambda x \Leftrightarrow (T - \lambda I)x = 0$ then x is the eigenvector of the operator T associated with the eigenvalue λ . The *characteristic polynomial* of an operator (matrix) is the polynomial $\det(\lambda I - T)$.

It has to be noted that for any n by n matrix, then the matrix $(T - \lambda I)$ is obtained by multiplying each of the n rows by -1 and hence $\det(\lambda I - T) = (-1)^n \det(T - \lambda I)$.

Definition 3.2: Let α be an eigenvalue of an operator T in a Hilbert space H . The *algebraic multiplicity* (or multiplicity) of the eigenvalue α is the multiplicity of α as a root of the characteristic polynomial $\det(\lambda I - T)$. (See details on algebraic multiplicities of the Jordan canonical form of n by n matrices (Weintraub, [15])).

Definition 3.3: A linear transformation (operator) T in a Hilbert Space H is said to be skew-adjoint if $T^* = -T$. Equivalently, we can write this equation as

$$\langle Tx, y \rangle + \langle x, Ty \rangle = 0 \quad \text{for all } x, y \text{ in } H. \tag{3.1}$$

From this equation, we see that the matrix of a skew-adjoint relative to an orthonormal basis is *skew symmetric*. Substituting y for x in (3.1) yields

$$\langle x, Tx \rangle = 0, x \in H \tag{3.2}$$

This equation shows that every vector is orthogonal to its range-vector.

On the other hand, an operator T having this property is skew adjoint. Replacing x by $x + y$ in (3.2) gives $\langle x + y, Tx + Ty \rangle = 0$, that is $\langle x, Tx \rangle + \langle x, Ty \rangle + \langle y, Ty \rangle = \langle x, Tx \rangle + \langle y, Ty \rangle$ for all x, y in H . By Nzimbi et al [12], the spectrum of a skew adjoint operator is contained in the imaginary axis.

Remark 3.4: If T is a bounded linear operator on a finite dimensional Hilbert space, then the equation $T^* = -T$ implies that $\text{tr } T = 0$ and $\det T = (-1)^n \det T^*$. Hence from the last equation we see that $\det T = 0$ if the dimension of H is odd. More generally the rank of a skew-adjoint transformation is always even.

It is also well known that every skew operator is normal, and its range is the orthogonal complement of the kernel. As a result, the induced transformation $T_1: \text{Ran } T \rightarrow \text{Ran } T$ is regular and T_1 is the restriction of T to the T -invariant subspace $M = \text{Ran}(T)$. Note also that an operator is T in a Hilbert space H regular (or non-singular) exactly when it has a unique inverse mapping T^{-1} such that for all x, y in H $Hy = Tx$ implies that $x = T^{-1}y$. Since T_1 is again skew adjoint mapping, it follows that the dimension of $\text{Ran } T$ must be even and so the rank of a skew-symmetric matrix is always even.

Example 3.5: The normal form of a skew-adjoint operator.

Consider a skew adjoint mapping T having an orthonormal basis a_ν ($\nu = 1, \dots, p$) and define T as

$$T = \begin{pmatrix} 0 & k_1 & & & \\ k_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & k_p \\ & & & k_p & 0 \end{pmatrix} \tag{3.3}$$

and consider the mapping $S = T^2$. Then $S^* = S$. Thus there exists an orthonormal basis e_v ($v = 1, \dots, n$) in which S has the form $Se_v = \lambda_v e_v$ ($v = 1, \dots, n$). All the eigenvalues λ_v are negative or zero. That is $Se = \lambda e \Rightarrow \lambda = \langle e, Se \rangle = \langle e, T^2 e \rangle = -\langle Te, Te \rangle \leq 0$. Since the rank of T is even, and T^2 has the same rank as T the rank of S must be even. As a result, the number of negative eigenvalues is even and we can enumerate the vectors e_v ($v = 1, \dots, n$) such that $\lambda_v < 0$ ($v = 1, \dots, 2p$) and $\lambda_v = 0$ where $v = 2p + 1, \dots, n$. Define the orthonormal basis a_v ($v = 1, \dots, n$) by $a_{2v-1} = e_v, a_{2v} = \frac{1}{k_v} Te_v, k_v = \sqrt{-\lambda_v}, (v = 1, \dots, 2p)$ and $a_v = e_v (v = 2p + 1, \dots, n)$. In this basis, the matrix of T has the form (3.3).

To describe *skew – symmetric* operators on a complex Hilbert space H , we consider an operator T which can be represented, with respect to some orthonormal basis as skew symmetric matrix. The following basic definitions are required for this description.

Definition 3.6: An operator $T \in B(H)$ is said to be *complex symmetric* if there exists a conjugation C on H such that $CTC = T^*$. Hankel operators, Volterra integration operators and Toeplitz operators as discussed in Garcia and Putinar([2], [3]) and Garcia and Wogen, [4] are examples of complex symmetric operators.

Definition 3.7: An operator $T \in B(H)$ of bounded linear transformations on a complex separable Hilbert space is called *triangular* if H has an upper triangular matrix representation with respect to some orthonormal basis $\{e_n\}$ of H with the property that $Te_n \in \text{span}\{e_1, e_2, \dots, e_n\}$ for each n .

T is *bitriangular* if both T and T^* are triangular, with respect to different orthonormal bases.

Remark 3.8: In view definitions 3.6 and 3.7, we have that $T \in B(H)$ is skew symmetric exactly when there exists an orthonormal basis $\{e_n\}$ such that $\langle Te_n, e_m \rangle = -\langle Te_m, e_n \rangle \forall m, n$. In other words, T admits a skew-symmetric matrix representation with respect to $\{e_n\}$. These operators can be visualized as infinite dimensional skew-symmetric matrices.

In the sequel, we give some known results on skew-symmetric operators and related examples.

Lemma 3.9 (Li and Zhu [10], Lemma 1.4 pp. 2756): Let C be a conjugation on a complex Hilbert space H . Denote $S_C(H) = \{X \in B(H) : CXC = -X^*\}$. Then

- a) If $A, B \in B(H), CAC = A^*, CBC = B^*$ then $[A, B] = AB - BA \in S_C(H)$.
- b) If $T \in S_C(H)$, then $CT^{2n}C = (T^{2n})^*$ for all $n \in \mathbb{N}$.
- c) The class $S_C(H)$ is norm-closed and forms a Lie algebra under the commutator bracket $[.,.]$.
- d) If $T \in S_C(H)$, then $\sigma(T) = -\sigma(T)$.

Example 3.10: Consider an operator $T \in B(\mathbb{C}^3)$ given by the matrix representation

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ with respect to a basis } \{e_1, e_2, e_3\} \text{ basis of } \mathbb{C}^3.$$

Since the trace of a skew symmetric matrix is zero and the trace is invariant under unitary transformation, it is evident that T is not a skew symmetric operator.

Remark 3.11: From Example 3.10, we see that the spectral condition $\sigma(T) = -\sigma(T)$ as provided for in part (d) of Lemma 3.9 above is not a sufficient condition for a normal operator T to be skew -symmetric.

If $T \in B(H)$ and C is a conjugation on H satisfying $CTC = -T^*$, then $C(T - \lambda)C = -(T + \lambda)^*$ and $C(T + \lambda)^*C = -(T - \lambda)$ for $\lambda \in \mathbb{C}$. We then have that $\dim(\ker(T - \lambda)) = \dim(\ker(T - \lambda)^*)$. This shows that multiplicity is an essential invariant in the structure of skew symmetric operators.

On a finite dimensional setting, we provide an example of skew symmetric operators by corresponding Jordan blocks with even ranks.

Example 3.12: Let an operator $T \in B(\mathbb{C}^3)$ be represented in a matrix form as

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ basis of \mathbb{C}^3 . Then T is skew symmetric matrix. To see this, re-write $x \in \mathbb{C}^3$ as a linear combination $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ and define $Cx = -\bar{\alpha}_3 e_1 + \bar{\alpha}_2 e_2 - \bar{\alpha}_1 e_3$. Then C in matrix representation is given by

$$C = \begin{bmatrix} 0 & 0 & -\bar{\alpha}_3 \\ 0 & \bar{\alpha}_2 & 0 \\ -\bar{\alpha}_1 & 0 & 0 \end{bmatrix}$$

and is a conjugation on \mathbb{C}^3 . A quick computation shows that $CTC = -T^*$. Hence, T is skew symmetric.

Now set $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{bmatrix}$. Then U is unitary, i.e. $U^*U = UU^* = I$ and

$$U^*TU = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 \end{bmatrix}.$$

Definition 3.13: An operator $T \in B(H)$ is called quasidiagonal (quasitriangular), denoted by (QT) if there exists an increasing sequence $\{P_n\}_{n=1}^\infty$ of finite rank (orthogonal) projections such that $P_n \rightarrow 0$ (strongly $n \rightarrow \infty$) and $\|TP_n - P_n T\| \rightarrow 0$ ($\|TP_n - P_n TP_n\| \rightarrow 0$, respectively) as $n \rightarrow \infty$ (see Herrero[7]). We can illustrate as follows.

An operator $Q = (q_{ij})$ is a quasitriangular (Hessenberg) matrix if $q_{ij} = 0$ whenever $i > j + 1$. That is, Q is a Hessenberg matrix if all the entries below the sub-diagonal are zero (see Jonson, et al[13], pp.433). For example, by computation, it is seen that the sum $S_+^* + S_+$ is a quasidiagonal operator Q , where S_+ is the unilateral shift of finite multiplicity. The class of biquasitriangular operators, denoted by (BQT) is defined as $(BQT) = \{T \in B(H) : T \text{ and its adjoint } T^* \text{ are quasitriangular}\}$.

Remark 3.14: Recall that an operator T is said to be a *Fredholm operator* if both $\ker(T)$ and $\ker(T^*)$ are finite dimensional and the range of T is closed. If T is Fredholm, then its *index* is denoted by $ind(T)$ and is defined as $ind(T) = \dim(\ker(T)) - \dim(\ker(T^*))$ (see Lee [18] for a detailed discussion on Fredholm operators). It has been shown in Garcia[2] that every complex symmetric operator is bitriangular in nature. We can study the structure of skew symmetric operators analogously from the symmetric operators. However, not every skew- symmetric operator is biquasitriangular as illustrated in the following example.

Example 3.15: Consider a forward shift operator S defined on a Hilbert Space by $Se_i = e_{i+1}$ for $i \in \mathbb{N}$ where $\{e_i\}$ is an orthonormal basis of H . Set $T = \begin{bmatrix} S+I & 0 \\ 0 & -S^*-I \end{bmatrix}$ on $H \oplus H$ where I is an identity operator. Then T is skew- symmetric. To see this, we define a conjugation C on H by $C(\sum_i \alpha_i e_i) = \sum_i \bar{\alpha}_i e_i$ for $\sum_i \alpha_i e_i \in H$ and set $D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$ on $H \oplus H$. Then D is a conjugation on $H \oplus H$ and by computation, $DTD = -T^*$. In addition, $T - I$ is a Fredholm operator whose index is given by the equation $ind(T - I) = ind S + ind(-S^* - 2I) = -1$. It follows that T is skew-symmetric but not biquasitriangular.

Remark 3.16: By Lemma 3.9 (b), we can establish the relationship between complex symmetric operators and the skew-symmetric operators by writing T as a sum of these classes of operators. If an arbitrary choice of a conjugation C on H and set $A = \frac{1}{2}(T + CT^*C)$, $B = \frac{1}{2}(T - CT^*C)$, then A and B are the respective symmetric and skew-symmetric operators. A quick computation also shows that $T = A + B$. In view of this remark, we have this example in mind.

Example 3.17: Consider a complex symmetric operator $A \in B(H)$ and define $T = A \oplus (-A)$. We claim that T is skew-symmetric. For since A is complex symmetric, there exists, by definition a conjugation C on

H such that $CTC = T^*$. Now set $D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$ on $H \oplus H$. It follows that D is a conjugation on $H \oplus H$ and $DTD = (-CAC) \oplus (CAC) = (-A^*) \oplus A^* = -T^*$, so T is skew-symmetric as earlier claimed. In the sequel, we describe skew-symmetric operators in relation to eigenvalues. This is guaranteed by providing an upper triangular matrix representation for the skew-symmetric operators with non-zero eigenvalues. We also characterize certain skew-symmetric triangular operators based on the geometric relationship between eigenvalues. We begin by outlining the fundamental facts about the relationship between skew-symmetric and complex symmetric operators the following result.

Lemma 3.18 (Zhu [16], Lemma 1.1 pp. 1272). Let $T \in B(H)$ and C be a conjugation on H . Then

- There exists $A, B \in B(H)$ such that $T = A + B$, $CAC = -A^*$ and $CBC = B^*$.
- If $CTC = T^*$, then T^{2n} is complex symmetric with respect to C for $n \geq 1$.
- If T is complex symmetric, then $T \oplus (-T)$ and $T^*T - TT^*$ are both skew-symmetric.

Remark 3.19: By Lemma 3.18, it is evident that complex symmetric operators can be used to construct new symmetric operators. In view of the description of symmetric normal operators (Li and Zhu [10], Theorem 1.10), this provides an alternative approach of describing complex symmetric operators. An example to this is the class of Toeplitz operators as illustrated by Guo and Zhu [6]. More specifically, any commutator of two truncated Toeplitz operators is skew-symmetric. We also require the following definition to describe the upper triangular representation for skew-symmetric operators.

Definition 4.1.20 (Li and Zhu [10], Definition 2.1, pp.1272): Let $T \in B(H)$. An operator $A \in B(H)$ is called a *transpose* of T if $A = CT^*C$ for some conjugation C on H . Since by definition of skew-symmetric operator T , i.e. $-T = CT^*C$ for some conjugation C on H , its transpose is $-T$. It has also to be noted, on the other hand that, any two transposes of an operator T are unitarily equivalent. Further, recall also that a transformation C on a Hilbert space H is an *antiunitary* operator if C is conjugate linear, invertible and $\langle Cx, Cy \rangle = \langle x, y \rangle \forall x, y \in H$. This definition clearly shows that a conjugation is an involutory antiunitary operator.

If $T \in B(H)$ and $N \subset H$, then denote by T_N the compression of T to N , i.e the restriction of $P_N T$ to N . Here, P_N is the orthogonal projection of H onto N . If N is invariant under T , then $T_N = T|_N$.

We now state essential results that can describe some properties of skew-symmetric operators, namely, unitary equivalence of the restriction of the operator T to a subspace M and the compression of the transpose of T to a subspace N of H . These results also provide the description of the eigenvalues of the operator T . First, we need the following known results.

Theorem 3.21 (Zhu [16], Theorem 2.2 pp. 1273). Let $T \in B(H)$ and $\Gamma \subset \mathbb{C}$. Assume that $M = \bigvee_{\lambda \in \Gamma, n \geq 1} \text{Ker}(T - \lambda)^n$, $N = \bigvee_{\lambda \in \Gamma, n \geq 1} \text{Ker}(T + \lambda)^n$, where \bigvee denotes closed linear span. If T is skew symmetric, then $T|_M \cong -T_N^t$ where \cong denotes unitary equivalence.

Lemma 4.1.22 (Zhu [38], Lemma 2.3 pp. 1273-4): Let $T \in B(H)$ and $\{e_i\}_{i=1}^\infty$ be an orthonormal set of H . Assume that $Te_i \in \{e_i : 1 \leq i \leq n\}$ and $\langle Te_n, e_n \rangle = \lambda_n$ for $n \geq 1$ where $\lambda_i \in \mathbb{C}$ and $\lambda_i \neq \lambda_j$ for all $i, j \geq 1$. If C is a conjugation on H and $CTC = -T^*$, then

- $\langle TCe_i, Ce_i \rangle = -\lambda_i$ for all $i \geq 1$ and
- $\langle Ce_i, e_j \rangle = 0$ for all $i, j \geq 1$.

In view of Theorem 3.21 and Lemma 3.22, it can be remarked that for an operator $T \in B(H)$ and a conjugation D on H such that $DTD = -T^*$, if for a given non-zero e , where $e \in \text{Ker}(T)$ then it is possible that

$\langle De, e \rangle \neq 0$. As an example, consider an operator T defined by $T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & -N \end{bmatrix}$ on $\mathbb{C} \oplus H \oplus H$

where N is an invertible normal operator on H . Then by (Li and Zhu [10], Theorem 1.10), T is skew symmetric. Assume that C is a conjugation on $\mathbb{C} \oplus H \oplus H$ satisfying $CTC = -T^*$. Then $C(\text{Ker}(T)) = \text{Ker}(T^*)$. Since $\text{Ker}(T) = \text{Ker}(T^*) = \mathbb{C}$ is of dimension 1, it follows that $\langle Ce, e \rangle \neq 0$ for all non-zero $e \in \text{Ker}(T)$.

We also need the following:

Remark 3.23: Let H be a separable Hilbert space. Consider $T \in B(H)$ and $S \in B(H)$ where $S \in B(H)$ is a forward shift operator. We define a *Foguel operator*, denoted by R_T on $H \oplus H$ by a matrix of the form

$R_T = \begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix}$. In more general terms, we refer to an operator of the form $R_{T,n} = \begin{bmatrix} S^{*n} & T \\ 0 & S^n \end{bmatrix}$ as a Foguel operator of order n .

For an open connected set $\Omega \in \mathbb{C}$ and $n \in \mathbb{N}$, $T \in B(H)$ is a Cowen – Douglas operator (i.e. $T \in B_n(\Omega)$) if each $\lambda \in \Omega$ is an eigenvalue of T of constant multiplicity n , and the eigenvectors span the Hilbert space H , and the operator $T - \lambda I$ is surjective, for every $\lambda \in \Omega$. (See Cowen and Douglas [1]). By this remark, we can demonstrate that every skew symmetric operator with non-zero eigenvalues admits an upper- triangular operator matrix representation with respect to some chosen orthonormal basis (see Li and Zhu [10], Theorem 2.5). The following is an example.

Example 3.24: Consider a forward shift operator $S \in B(H)$ defined by $Se_i = e_{i+1}$ for $i \geq 1$ where $\{e_i\}$ is an orthonormal basis of H . Assume that $T \in B(H)$ and $i \in \mathbb{N}$. Define $R_T = \begin{bmatrix} S^{*n} & T \\ 0 & S^n \end{bmatrix}$ on $H \oplus H$ where $H = H_1 = H_2$. Then R_T is a Foguel operator of order n . Since by definition S^{*n} is a Cowen-Douglas Operator with index n on

$$\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}, \text{ we have } \bigvee_{k \geq 1} \text{Ker}(S^{*n} - \frac{1}{2})^k = H = \bigvee_{k \geq 1} \text{Ker}(S^{*n} + \frac{1}{2})^k.$$

Then $H_1 = \bigvee_{k \geq 1} \text{Ker}(R_T - \frac{1}{2})^k$ and $H_2 = \bigvee_{k \geq 1} \text{Ker}(R_T + \frac{1}{2})^k$. By (Li and Zhu [10], Theorem 2.5) if R_T is skew symmetric, then there exists a conjugation C on H and $G \in S_C(H)$ where $S_C(H)$ is a set of all skew symmetric operators on H with respect to C , that is

$$S_C(H) = \{X \in B(H): CXC = -X^*\}, \text{ such that } R_T = \begin{bmatrix} S^{*n} & G \\ 0 & -CS^nC \end{bmatrix}.$$

Remark 3.25: As earlier noted in definition 3.7, an operator $T \in B(H)$ is said to be triangular if $\bigvee_{\lambda \in \mathbb{C}, n \geq 1} \text{Ker}(T - \lambda)^n = H$. This also means that T is triangular if and only if it admits an upper triangular

matrix representation of the form $T = \begin{bmatrix} \lambda_1 & * & \dots \\ & \lambda_2 & \dots \\ & & \ddots \end{bmatrix}$ with respect to some orthonormal basis of H , where each

omitted entry is zero. The well-known Cowen – Douglas operators are examples of triangular operators. In some cases, an operator T and its adjoint T^* can be both triangular (with respect to different orthonormal basis). Such operators are said to be bitriangular. Examples include diagonal normal operators, block diagonal operators and all algebraic operators. (Recall that an operator T is algebraic if there is a non-zero polynomial p such that $p(T) = 0$). It is also clear that on a finite dimensional Hilbert space, every operator is bitriangular. However, there are some triangular operators that are not bitriangular. An example of such operators is the adjoint of the forward shift operator.

Note also that every skew symmetric operator must be triangular. To see this, consider a skew symmetric operator $T \in B(H)$. Then by definition, there exists a conjugation C on H such that $T^*C = -CT$. It follows that

$(-1)^n(T^* + \bar{\lambda})^n C = C(T - \lambda)^n$ and $C(\text{Ker}(T - \lambda)^n) = C(\text{Ker}(T^* + \bar{\lambda})^n)$ for all $\lambda \in \mathbb{C}$ and $\lambda_i \neq \lambda_j$ and $n \geq 1$. But by definition of the triangular operator, $\bigvee_{\lambda \in \mathbb{C}, n \geq 1} \text{Ker}(T - \lambda)^n = H$. Since C is a conjugation it follows that $H = C(H) = \bigvee_{\lambda \in \mathbb{C}, n \geq 1} C(\text{Ker}(T - \lambda)^n) = \bigvee_{\lambda \in \mathbb{C}, n \geq 1} C(\text{Ker}(T^* + \bar{\lambda})^n)$. Thus, T^* is triangular and T is bitriangular. From this argument, it is seen that $\lambda \in \sigma_p(T)$ exactly when $-\bar{\lambda} \in \sigma_p(T^*)$. More particularly, $\dim \text{ker}(T - \lambda) = \dim \text{ker}(T + \lambda)^*$. As a consequence of this remark, we have the following result:

Lemma 3.26 (Zhu [16], Lemma 3.3, pp. 1279): Let $T \in B(H)$. Assume $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and $u \in \text{Ker}(T - \lambda_1), v \in \text{Ker}(T - \lambda_2)^*$. Then $\langle u, v \rangle = 0$.

The proof of the Lemma 3.26 can easily be established when we have the computation of the form $\lambda_1 \langle u, v \rangle = \langle Tu, v \rangle = \langle u, T^*v \rangle = \lambda_2 \langle u, v \rangle$. Since the eigenvalues are distinct, i.e. $\lambda_1 \neq \lambda_2$, it follows that $\langle u, v \rangle = 0$ as required.

Theorem 3.27 (Zhu [16], Theorem 3.4, pp. 1280): Let $T \in B(H)$. Suppose that $\{\lambda_i: i \in \mathbb{N}\}$ are distinct eigenvalues of T and $u_i \in \text{Ker}(T - \lambda_i)$ is a unit vector for $i \in \mathbb{N}$. If $\bigvee\{u_i: i \in \mathbb{N}\} = H$, then T is skew symmetric if and only if there exists unit vectors $\{v_i: i \in \mathbb{N}\}$ with $v_i \in \text{Ker}(T - \lambda_i)^*$ for $i \in \mathbb{N}$ such that $\bigvee\{v_j: j \in \mathbb{N}\} = H$, $\langle u_i, u_j \rangle = \langle v_j, v_i \rangle$ and $\langle u_i, v_j \rangle = \langle u_j, v_i \rangle$ for any $i, j \in \mathbb{N}$.

Remark 3.28: From the sufficiency of Theorem 3.27, we see that if $\{\lambda_i: i \in \mathbb{N}\}$ are the distinct eigenvalues of the operator T and for a given unit vector $u_i \in \text{Ker}(T - \lambda_i)$ for $i \geq 1$, if $\bigvee\{u_i: i \geq 1\} = H$, and T is skew symmetric, then $\{\lambda_i: i \in \mathbb{N}\} = \{-\lambda_i: i \in \mathbb{N}\}$.

It is also important to note that for a general symmetric operator T , $\lambda \in \sigma_p(T)$ does not imply that $-\lambda \in \sigma_p(T)$ as demonstrated in the following example:

Example 3.29: Consider an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of the Hilbert space H and define the forward shift operator $S \in B(H)$ on H as $Se_i = e_{i+1}$ for all $i \geq 1$. For $x \in H$, with $x = \sum_{i=1}^{\infty} \alpha_i e_i$ define $Cx = \sum_{i=1}^{\infty} \bar{\alpha}_i e_i$ where C is a conjugation on H . Then $CSC = S^*$ (i.e. the right shift operator S is complex symmetric). Now set $T = \begin{bmatrix} I & -S \\ 0 & S^* - I \end{bmatrix}$ on $H \oplus H$ where I is an identity operator and $D = \begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix}$ on $H \oplus H$. Then D is a conjugation on $H \oplus H$ and $DTD = -T^*$ so T is skew symmetric. It is also noted that $\sigma_p(T) = \{z \in \mathbb{C}: |z + 1| < 1\}$.

Proposition 3.30 (Nzimbi et al [12]): Every skew-adjoint operator $T \in B(H)$ is binormal.

Example 3.31: We define on the function Hilbert space $L^2[a, b]$ a differential operator by $Tf = \frac{df}{dx}$ and show that it is skew-adjoint. Using integration by parts, we have

$$\begin{aligned} \langle Tf, g \rangle &= \int_a^b \frac{df}{dx} \overline{g(x)} dx \\ &= f(b)\overline{g(a)} - \int_a^b f(x)\overline{g'(x)} dx \\ &= - \int_a^b \frac{dg}{dx} f(x) dx = \langle f, -Tg \rangle. \end{aligned}$$

This clearly shows that $T^* = -T$ is a skew-adjoint operator.

IV. CONCLUSION

In this paper, we have investigated the classes of skew-adjoint and skew-symmetric operators have also been discussed, where we have established that skew-symmetric operators admit skew-symmetric matrix representation with respect to some orthonormal basis $\{e_n\}$. Further, a summary of the basic results for this class of operators has also been outlined. More importantly, we have illustrated that not every skew-symmetric operator is bitriangular. Another example establishing a link between complex symmetric and skew-symmetric operators is also shown, by arbitrarily choosing a conjugation C on H and decomposing the operator T as a direct sum of a complex symmetric and skew-symmetric operator. We have also characterized skew-symmetric operators with non-zero eigenvalues and a description of the same done by providing an upper triangular operator matrix representation

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