The Classification of Semisimple Lie Algebra With Some Applications

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ABSTRACT: *This paper presents the classification of semisimple Lie algebras and its application. Starting on the level of Lie groups, we concisely introduce the connection between Lie groups and Lie algebras. We then further explore the structure of Lie algebras, which we introduced semisimple Lie algebras and their root decomposition. We then turn our study to root systems as separate structures, and finally simple root systems, which can be classified by Dynkin diagrams. Then also considered quantum mechanics and its rotation invariance as its physical application.*

Keywords: *Lie algebra, Lie group, root decomposition, root system, Dynkin diagram*

A. BACKGROUND OF THE STUDY

Lie theory is a rich area of mathematics named after Marious Sophus Lie (1873-1874). While considering the solution to partial differential equations,Marious discoveredsomething called an infinitesimal group, which was not a group by the modern definition of group but rather a Lie algebra (considering there axioms). The classification of simple Lie algebra began with William Killing and Elie Cartan[1]. Killingdiscovered Lie algebra independent of Lie's work. The main tools in classification of semisimple Lie algebras, as well as the idea of a root, were first introduced by Killing. In part of this work, similar to the work of killing, we will shall lay out the classification by considering roots as geometric objects independent of Lie algebras.

Lie theory has many applications in physical sciences among them is quantum mechanics. Michael Weiss explain one of such application [2]. Which he said "One would like to visualize the electron as a little spinning ball. A spinning ball which spins about an axis, you can imagine changing the axis by rotating the space containing the ball. Analogously, the quantum spin state of an electron has an associated axis, which can be changed by rotating the ambient space". These ideas of a ball spinning on an axis and that axis turning in space are related to the classical Lie groups. The Lie group $SO(3)$ (the rotation group in three dimensions) describes motions in three-dimensional space. The idea of spin is also related to the compact real form of $SO(3)$, that is the Lie group $SU(2)$ (the special unitary group in two dimensions). Where both the $SO(3)$ and $SU(2)$ are example of group of transformations. Hence, these groups are said to be locally isomorphic. We can say that $SU(2)$ is the double cover of $SO(3)$ which means geometrically a rotation of 2π which gives an identity transformation in $SO(3)$, while in $SU(2)$ a rotation of 4π is required to return to the identity.

However, the Lie algebras related to $SO(3)$ and $SU(2)$ are isomorphic. This is the reason that it is almost incorrect to think of the spin of an electron as a spinning ball. The fact that $SO(3)$ and $SU(2)$ as Lie groups are not quite isomorphic gives subtle differences in the behavior of electrons (thought of as fermions) and photons (a type of boson).

In science, groups play the role of describing symmetries of a system. Where both finite groups and Lie groups can be used for such a purpose. A classification of all the possible groups one can construct is thus not just a purely mathematical problem, but also has physical applications and other science related, since it states all the possible symmetries a physical system might have. In the case of Lie groups, such a classification for example could prove useful for finding a suitable Gauge group for a grand unified theory. In light of the title of this work, and with some other works more application oriented, this text will be mainly mathematical in nature, and should be regarded as a reference for the Lie groups/ Lie algebras in one's repertoire. Moreover, the classification result might be guarantee that we do not overlook any symmetry, which could conceivably arise. To put the results for the classification of (simple) Lie groups/ (simple) Lie algebras into perspective, it is interesting to consider a still ongoing episode in the history of mathematics and the classification of finite groups.

B. PRELIMINARIES

Definition 1.2.1: Aset V of elements is called a vector space over a field F if it satisfies the following properties:

i) The set V is an abelian group under addition

ii) For any vector v in V and for any c in F , is defined cv in V (Field elements are called scalars and elements of V are called vectors)

iii) If V is a vector, c and d are scalars then $(c + d)v = cv + dv$ (distributive law)

iv) If u and v are vectors in V and c is a scalars then $c(u + v) = cu + cv$ (distributive law).

v) If V is a vector in V $\wedge c$ and d are scalars, then $(cd)v = c(dv)$

vi) $1v = v$ for any v in V [3].

Definition 1.2.2: An algebra consist of a vector space V over a field F together with a binary operation of multiplication on the set V of vectors such that for all $\alpha \in F$, $u, v, w \in V;$ the following conditions are satisfied.

i. $(\alpha u)v = \alpha(uv) = u(\alpha v)$,

ii. $(u + v)w = uw + vw$,

iii. $u(v + w) = uv + uw$,

iv. $(uv)w = u(vw)$ [4].

Definition 1.2.3:A real (or complex) vector space G is a real (or complex) Lie algebra, if it is equipped with an additional mapping $[a,a]=0$ $[\cdot,\cdot]$:G × G → G, which is called the Lie bracket and satisfies the following properties:

i. Bilinearity: $[\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c]$ and $[c, \alpha a + \beta b] = \alpha[c,a] + \beta[c,b]$ for all $a,b,c \in$ G and $\alpha, \beta \in F$,

ii. $[a,a] = 0$ for all $a \in$ G,

iii. Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [b,[c,a]] + [c,[a,b]] = 0$ .for all $a,b,c \in$ G[5].

Definition 1.2.4: Let G be a group,

• If G is also a smooth (real) manifold, and the mappings $(a,b) \mapsto ab$ and $a \mapsto a^{-1}$ are smooth, G is a real Lie group,

• If G is also a complex analytic manifold, and the mappings $(a,b) \mapsto ab$ and $a \mapsto a^{-1}$ are analytic, G is a complex Lie group [6].

Definition 1.2.5: Let G be a Lie algebra.

• G is semisimple, if there are no nonzero solvable ideals in G,

• G is simple, if it is non-Abelian, and contains 0 and G as the only ideals [5].

Definition 1.2.6: Let V and W be vector spaces over a field F . A map $T : V \to W$ is said to be linear if it satisfies

$T(Au + fv) = AT(u) + 3T(v)$ for all u, v in V and A , f in F [3].

Definition 1.2.7: Let $S \subseteq R$ be a simple root system in n dimensional space, and let us choose an order of labeling for the elements $\alpha_i \in S$ where $i \in \{1,...,n\}$. The Cartan matrix a is then a $n \times n$ matrix, which has the following entries component wise: $a_{ij} = n_{\alpha_i \alpha_j}$ [3].

Definition 1.2.8: Suppose $S \subseteq R$ is a simple root system. The Dynkin diagram of S is a graph constructed by the following prescription:

i. For each $\alpha_i \in S$ we construct a vertex (visually, we draw a circle),

ii. For each pair of roots α_i, α_j, we draw a connection depending on the angle φ between them.

- If $\varphi = 90^\circ$, the vertices are not connected (we draw no line).
- If $\varphi = 120^\circ$, the vertices have a single edge (we draw a single line).
- If $\varphi = 135^\circ$, the vertices have a double edge (we draw two connecting lines).
- If $\varphi = 150^\circ$, the vertices have a triple edge (we draw three connecting lines).

iii. For double and triple edges connecting two roots, we direct them towards the shorter root (we draw an arrow pointing to the shorter root) [3].

Definition 1.2.9: A representation of a group G on a vector space V is define as a homomorphism $\phi : G \to GL(V)$. To each $g \in G$, the representation map assigns a linear map, $\rho_g : V \to V$. Although V is actually the representation space, one may for short refer to V as the representation of G [3].

Definition 1.2.10: The general linear group over the real numbers, denoted by $GL(n, R)$ is the group of all $n \times n$ invertible matrices with real number entries. Which can be similarly be define over the complex numbers, C denoted by $GL(n, C)$ [8].

II. REVIEW OF SOME LITERATURES

A. THE ALGEBRAIC HISTORY

It was Sopus Lie (1842-1899) who started investigating all possible (local) group actions on manifolds. Lie's seminal idea was to look at the action *infinitesimally*. If the local action is by $\mathbf{R}$, it gives rise to a vector field on the manifold which integrates to capture the action of the local group. In general case we get a Lie algebra of vector fields, which enables us to reconstruct the local group action. The simplest example is the one where the local Lie group act on itself by left (or right) translations and we get the Lie algebra of the Lie group. The Lie algebra, being a linear object, is more immediately accessible than the group. It was Wilhelm Killing who insisted that before one could classify all group actions one should begin by classifying all (finite dimensional real) Lie algebras. The gradual evolution of the ideas of Lie, Friedrich Engel and Killing, made it clear that determining all Simple Lie algebras was fundamental.

Again, Killing came with the idea of simple Lie algebras (of finite dimension) over $\mathbf{C}$. Although his proofs were incomplete at crucial places and the overall structure of the theory was confusing, Killing arrived at the astounding conclusion that the only simple Lie algebras were those associated to the linear, orthogonal, and symplectic groups, apart from a small number of isolated ones. The problem was completely solved by Elie Cartan (1869-1951), who through the reviewing the ideas and results of Killing but adding crucial innovations of his own (*Cartan-Killing form*), obtained the rigorous classification of simple Lie algebras in his work which is one of the greatest work of the nineteen century. In 1914s he classified the simple real lie algebras by determining the real form (the compact form) on which the Cartan- Killing form is negative definite,[1].

a) THE CLASSIFICATION

The simple Lie algebras over C fall into four infinite families $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3)$ and respectively corresponding to the groups $SL(n+1, C), SO(2n+1, C), SP(2n, C), SO(2n, C)$ and five isolated ones (the exceptional Lie algebras) denoted by G_2, F_4, E_6, E_7, E_8, with dimensions 14, 52, 78,133 and 248 respectively. The key concept for the classification is that of a Cartan sub algebras $\mathfrak{h}$ which is a special maximal nilpotent subalgebra unique up to conjugacy. A classification of the real four-dimensional connected Lie groups is also obtained by Rory Biggs and Claudiu C. Remsing[8]. Those groups which are linearizable are identified accompanying matrix Lie groups are exhibited. In the spectral decomposition of $ad\,\mathfrak{h}$, the eigenvalues α are certain linear forms on $\mathfrak{h}$ called roots and the corresponding (generalized) eigenvectors X_α are root vectors, the eigenspaces g_α are root spaces and the structure of the set of the roots captures a great deal of the structure of the Lie algebra itself. For instance, if α and β are root but $\alpha+\beta$ is non zero but a root then $[X_\alpha, X_\beta]=0$.

b) REPRESENTATIONS

Cartandetermined the irreducible finite dimensional representations of the simple Lie algebras. Among the weights of an irreducible representation there is a distinguished one λ, the highest weight which has multiplicity 1, determines the irreducible representation, and is dominant i.e. $\lambda_{\alpha i} \geq 0$ for $1 \leq i \leq n$. The obvious question is whether every dominant integral element of $\mathfrak{h}_R$ is the highest weight of an irreducible representation. For an irreducible affine variety X over an algebraically closed field of characteristic zero we define two new classes of modules over the Lie algebra of vector fields on $X -$ gauge modules and Rudakov modules, which admit a compatible action of the algebra of functions [1]. Gauge modules are generalizations of modules of tensor densities whose construction was inspired by non-abelian gauge theory while Rudakov modules are generalizations of a family of induced modules over the Lie algebra of derivations of a polynomial ring studiedby some authors includingNikitin,Tsilevich andVershik[9].

c) GENERAL ALGEBRAIC METHODS

In the late 1940s Claude Chevalley and Harish-Chandra (independently) discovered the way to answer, without using classification, the two key questions here:

i. Whether every Dynkin diagram comes from a semi simple Lie algebra,[10], and also
ii. If every dominant integral weight is the highest weight of an irreducible representation.

In the mid 1920's, Hermann Weyl had settled (ii) as well as the complete reducibility of all representations by global methods without classification.

For (ii) one works with the universal enveloping algebra of g, say $\mathcal{U}$. For any linear function $\lambda \in \mathfrak{h}^*$there is a unique irreducible module I_λ with highest weight λ, and one has to show that I_λ is finite dimensional if and only if λ is dominant and integral. For (i) one notes that in a semi simple Lie algebra g with a Cartan matrix $A=(a_{ij})$, if $0 \neq X_{\pm i}$ are in the root spaces $g_{\pm \alpha i}$, then we have the commutation rules

$$[H_i, H_J]=0, [H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, [X_i, X_{-j}] = \delta_{ij} H_i \qquad \text{(I)}$$

However a deeper study of the adjoint representation yields the higher order commutation rules

$$[X_{\pm i}, [X_{\pm i}, [\ldots X_{\pm i}, X_{\pm j},]]..] = ad(X_{\pm i,})^{-aij+1}(X_{\pm i}) = 0 \qquad \text{(II)}$$

The universal associative algebra $\mathcal{U}_A$ defined by the relations (I) and (II) bears a close resemblance to the algebra $\mathcal{U}$ mentioned earlier and one can construct a theory of its highest weight representations. One obtains the same criterion for the finite dimensionality of the irreducible representations, [11]. Let $\mathfrak{l}$ be the Lie algebra inside $\mathcal{U}_A$generated by the $H_i, X_{\pm j}$. If the highest weight has a value strictly > 0 at each node of the diagram this representation will be faithful on $\mathfrak{h}$, and the image of $\mathfrak{l}$under this representation will be the semi simple Lie algebra corresponding to the diagram. Much later Serre discovered the beautiful result that $\mathfrak{l}$is already finite dimensional and hence is the required semi simple Lie algebra with the given Cartanmatrix A, thus defining a presentation of the semi simple Lie algebra associated to any given diagram, [3].

d) INFINITE DIMENSIONAL LIE ALGEBRAS

Cartan also studied what he called the infinite simple continuous groups. Roughly speaking they are the infinite dimensional analogues of the simple Lie groups, the general theory of infinite dimensional Lie groups is still very much of a mystery.

The concept of a versal deformation of a Lie algebra is investigated and obstructions to extending an infinitesimal deformation to a higher-order one are described [3].

In the late 1960's, Victor Kac and Robert Moody independently initiated the study of certain infinite dimensional Lie algebras somewhat different from Cartan's. If we relax the properties of a Cartan matrix, especially the one requiring the Weyl group to be finite (I) and (II) will lead, by the methods of Chevalley and Harish-Chandra, to new Lie algebras that will no longer be finite dimensional. If we extend the scalars from C to the ring of finite Laurent series in an indeterminate, the simple Lie algebras give rise to certain Lie algebras, which have universal central extensions with one-dimensional center. The latter are the affine Lie algebras which are special Kac-Moody algebras, which along with the Virasoro algebras are important in conformal field theory. Their structure and representation theory resemble closely those of the finite dimensional simple Lie algebras,and their root systems are very beautiful infinite combinatorial objects related to many famous classical formulae.

e) CLASSIfiCATION OF RESTRICTED SIMPLE LIE ALGEBRAS WITH CHARACTERISTIC $p > 0$

It is natural to ask what the classification of simple Lie algebras looks like in characteristic $p \succ 0$ Here one has the concept of a restricted Lie algebra which is a Lie algebra together with an automorphism $X \mapsto X^{[p]}$ that is an infinitesimal version of the Frobenius morphism for algebraic groups. Interestingly there are additional simple Lie algebras, namely those that are finite dimensional analogues of Cartan's infinite simple Lie algebras, the so-called Cartan-type Lie algebras, [11]. That the class of restricted simple Lie algebras is exhausted by the classical and Cartan-type Lie algebras (Kostrikin-Shafarevich conjecture).

f) INVARIANT THEORY

For the adjoint action of a Lie group G (or a subgroup of G) on the Lie algebra Lie we suggest a method for constructing its invariants. The method is easy to implement and may shed light on the algebraical independence of invariants [4].

The work of Paul Gordan, had led to the result that the subalgebra $I_{n,d}$ is finitely generated and to an algorithmic construction of a set of generators for it, [13]. when David Hilbert came into the picture and took the entire subject to a new level. In a celebrated paper Hilbert proved the finite generation of $I_{n,d}$ by very general abstract arguments, but under prodding from Gordan, later examined the question of the finite determination of the invariants.

g) MODERN DEVELOPMENT

Nowadays groups with additional structures are viewed as group objects in categories. One starts with a Lie group G of whatever category one wants to be in, and associates its Lie algebra Lie(G) to get a functor $G \mapsto Lie(G)$, the fundamental theorems of Lie amount to studying how close this functor comes to being an equivalence of categories. It was only after the appearance of Chevalley's great 1946 book "The Theory of Lie Group", that the global view became accessible to the general mathematical public.

In his book $(Theory\ of\ Lie\ gruops)$ Chevalley developed all the major results the construction of the Lie algebra of a Lie group, the exponential map, the subgroup–subalgebra correspondence, Von Neumann's theorem that a closed subgroup of a real Lie group is a Lie group, and the fact that every C^{∞} (in fact, every C^{2}) Lie group is a real analytic Lie group, the analytic structure underlying the topology is unique because any continuous homomorphism between Lie groups is analytic. In addition he treated compact Lie groups in depth complete reducibility of all representations, Peter-Weyl completeness theorem, Tannaka-Krein duality, existence of a faithful finite dimensional representation σ and the theorem that every irreducible representation is contained in the tensor product of a number of copies of σ and its contragredient. This list does not indicate the originality of his treatment of these topics. For instance he had to extend the notion of Lie subgroups to include the cases when the subgroup is not closed and its topology and smooth structures are not induced by the ambient group. He constructed the subgroup and its cosets as the maximal global integral manifolds of the involutive distribution on the group defined by the subalgebra, giving in the process the first globaltreatment of the Frobenius theorem of integrability of involutive distributions. In the Tannaka duality he proved that there is a unique complex Lie group of which the given compact Lie group is a real form, thereby giving an entirely new perspective on the Weyl correspondence between compact and complex groups. Chevalley's theorem is the beginning of the Tannakian point of view that reconstructs an algebraic group from the tensor category of its finite dimensional modules. For Chevalley, the ring of matrix elements of a compact Lie group is a reduced finitely generated algebra with a Hopf algebra structure, and its spectrum is the complex semi simple group enveloped by the compact group, thus foreshadowing the point of view of quantum groups which arose almost forty years later.

Perhaps some remarks on the fifth problem of Hilbert are in order here. Hilbert, motivated by his insights into foundations of geometry, felt that the condition of differentiability in the definition of a Lie group was a deficiency, and proposed the problem of proving that any topological group which is locally homeomorphic to a manifold, must be a Lie group. The problem was eventually solved in the affirmative by the efforts of Gleason, Iwasawa, Montgomery-Zippin, Yamabe, and Lazard (in the p-adic case) after partial solutions by Von Neumann (compact groups), and Chevalley (solvable).

B. LINEAR ALGEBRAIC GROUPS AND THE CLASSIfiCATION OF SIMPLE GROUPS OVER AN ALGEBRAICALLY CLOSED FIELD OF ARBITRARY CHARACTERISTIC

Chevalley himself, along with Armand Borel, they were central player in the next great development of Lie theory, the theory of linear algebraic groups in arbitrary characteristic. Chevalley's initial attempts did not go very far because they were tied to the exponential map. But the work of Borel, which used only global methods based on algebraic geometry, changed the picture dramatically. Starting from Borel's work Chevalley went forward (by "analytic continuation"in his own words) to the classification of semi simple algebraic groups and their representations.

He discovered the remarkable fact that complex semi simple groups form group schemes over $\mathbf{Z}$, so that one can tensor them with any field to produce algebraic semi simple groups over that field. If the field is algebraically closed this procedure will yield essentially all semi simple algebraic groups. If the field is finite one will get new finite simple groups beyond those first studied by.Finally, the notion of a quantum group arose from the idea that quantum mechanics is a deformation of classical mechanics, namely, there is an essentially unique deformation of the Lie algebra of smooth functions on phase space with the Poisson Bracket. Given this point of view it is natural to ask whether the symmetry groups of classical geometry can also be deformed into interesting objects. In the 1980's such a theory of deformations emerged, under the impulses of several groups of people. Since classical semi simple Lie algebras are classified by discrete data, they are rigid. So, in order to deform them one must enlarge the category, [14].

III. METHODOLOGY

In this section, we state some methods which will eventually culminate in the fundamental theorems of Lie theory, and then lead us to see the connection between Lie groups and Lie algebras and also introduce the concept of semisimple Lie algebras and root systems.

A. LIE GROUPS

a) GROUPS OF TRANSFORMATIONS

Groups of transformations can be divided into discrete (finite and infinite) and continuous (finite and infinite). Both discrete and continuous groups are of importance in natural sciences, here we shall briefly describe some continuous groups (Lie groups) and their association with Lie algebras.

b) GROUPS OF MATRICES

Among the groups of transformations, particularly important are groups of square matrices $A = \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \end{pmatrix} (n \times n)$

And these matrices satisfy all the axioms of a group:

- The identity I of any given matrix is the unit matrix.
- Matrix multiplication gives closure.
- If $dei|A| \neq 0$ an inverse A^{-1} exists.
- Matrix multiplication gives associativity.

Groups of matrices can be written in terms of all number fields R, C, Q, O. The matrix elements of the matrix A will be denoted by A_{ik}, with $i =$ row index and $k =$ column index. We shall also introduce real and complex vectors in n dimensions. The components of vectors will be denoted by x_i and z_i.

c) EXAMPLES OF GROUPS OF TRANSFORMATIONS

1) The rotation group in two dimensions($SO(2)$)

As a first example we consider the rotation group in two dimensions $SO(2) \equiv SO(2, R),$ under a general linear real transformation the two coordinates x, ytransform as

$$x^{'} = a_{11}x + a_{12}y,$$
$$y^{'} = a_{21}x + a_{22}y.$$

The corresponding group $GL(2, R)$ is afour parameter group. The invariance of $x^2 + y^2$ is

$$a_{11}^2 x^2 + a_{12}^2 y^2 + 2a_{11}a_{12}xy + a_{21}^2 x^2 + a_{22}^2 y^2 + 2a_{21}a_{22}xy = x^2 + y^2$$

gives three conditions

$$a_{11}^2 + a_{21}^2 = 1,$$
$$2a_{11}a_{12} + 2a_{21}a_{22} = 0,$$
$$a_{22}^2 + a_{12}^2 = 1.$$

this leaves only one parameter.

Example3.1

The group $SO(2)$ is a one parameter group, the parameter can be chosen as the angle of rotation φ.

$$x^{'} = (\cos\varphi)x - (\sin\varphi)y$$
$$y^{'} = (\sin\varphi)x - (\cos\varphi)y$$

2) The rotation group in three dimensions ($SO(3)$)

As another example consider the rotation group in three dimensions $SO(3) \equiv SO(3, R)$, under a general linear transformation $GL(3, R)$ the coordinates x, y, z transform as

$$x' = a_{11}x + a_{12}y + a_{13}z$$
$$y' = a_{21}x + a_{22}y + a_{23}z$$
$$z' = a_{31}x + a_{32}y + a_{33}.$$

this is a nine parameter groups. Orthogonality

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

gives six conditions. We thus have a three parameter group.

d) OTHER IMPORTANT GROUPS OF TRANSFORMATIONS

An important class of transformations is formed by the combination of the translation group with the general linear group and its subgroups. These groups are still Lie groups but the associated Lie algebras are non-semisimple.

1) Translation Group($T(n)$)

Translations in n-dimentions form a group. Under a translation $\mathbf{a}$, the new coordinates are

$$\mathbf{x}' = x + a;\ x_i' = x_i + a_i\ (i = 1,...,n)$$

The translation group is a n parameter group.

2) Affine group ($A(n)$)

General linear transformations with $\det|A| \neq 0$ $det\ |A| \neq 0$ plus translations form a group, called the affine group $A(n)$ with

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{a};\ x_i' = \sum_k A_{ik}x_k + a_i\ \ (i = 1,...,n).$$

This group is the semi direct product of the general linear group and the translation group,the number of parameters of $A(n)$ for real transformations is $n^2 + n$ .Matrix representations of the affine group can be constructed in terms of $(n+1)\times(n+1)$ matrices.

3) Euclidean Group($E(n)$)

Rotations plus translations in an n-dimensional space form a groupcalled the Euclidean group $E(n)$. A vector $\mathbf{x}$ transforms under $E(n)$ as

$$\mathbf{x}' = \mathbf{Rx} + \mathbf{a}\ \ x_i' = \sum_k R_{ik}x_k + a_i$$

Where R_{ik} is the rotation matrix and a_i are the components of the translation vector. The Euclidean group is the semi-direct product of $SO(n)$ and $T(n)$

$$E(n) = T(n) \otimes_s SO(n)$$

a case of particular interest is

$$E(3) = T(3) \otimes_s SO(3)$$

the Lie algebra $e(n)$ associated with $E(n)$ are the semidirect sums

$$e(n) = t(n) \otimes_s so(n).$$

4) Poincare Group $(\boldsymbol{P}(\boldsymbol{n}))$

Lorentz transformations plus translations form a group called the Poincare' group, $P(n)$.A vector $\mathbf{x}'$ transforms under $P(n)$ as

$$\mathbf{x}' = L\mathbf{x} + \mathbf{a};\ x_\mu = \sum L_\mu^\nu x_\nu + a_\mu$$

Where L_μ^ν are Lorentz transformations and a_μ are the components of the translation. This group is the semidirect product of $SO(p,q)$ and $T(p,q)$ with $p+q=n$

$$P(n) = T(p,q) \otimes_s SO(p,q),\ p+q=n$$

a case of particular interest is

$$P(4) = T(3,1) \otimes_s SO(3,1),$$

this group is also denoted by $ISO(3,1) \equiv P(4)$ or the inhomogeneous Lorentz group.

5) Dilatation Group $(\boldsymbol{D}(\mathbf{1}))$

Scale transformations form a one parameter group called the dilatation group.

$$D(1): x^{i\mu} = \rho x^\mu.$$

6) Special Conformal Group $(\boldsymbol{C}(\boldsymbol{n}))$

The set of non-linear transformations

$$x^{i\mu} = (x^\mu + c^\mu x^2) \big/ \sigma(x)$$

$$\sigma(x) = 1 + 2c^\nu x_\nu + c^2 x^2$$

Form a group called the special conformal group $C(n)$. In four dimensions, the group $C(4)$ has four parameters $c_\mu(\mu = 0,1,2,3)$.

7) General conformal group $(\boldsymbol{GC}(\boldsymbol{n}))$

The set of Lorentz transformations plus translations plus dilatations plus special conformal transformations form a group, the General Conformal Group $GC(n)$ or simply the Conformal Group. In four dimensions, the number of parameters of $GC(4)$ is 10 for the Poincare' group $ISO(3,1) \equiv P(4)$, 1 for the dilatation $D(1)$ and 4 for the special conformal transformations $C(4)$, for a total of 15.

The group $GC(4)$ is isomorphic to $SO(4,2)$. It is possible to introduce a six-dimensional space and realize the conformal group linearly in this space. A differential realization of the elements of the Lie algebra $so(4,2)$ associated with the Lie group $SO(4,2)$ is

$$M_{\mu\nu} = x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\ SO(3,1)$$

$$P_{\mu} = \partial_{\mu}\ T(3,1)$$

$$K_{\mu} = 2x_{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial_{\mu}\ C(4)$$

$$D = x^{\nu}\partial_{\nu}\ D(1)$$

With $\mu, \nu = 0,1,2,3$ Conformal transformations can be written as linear transformations in a six-dimensional space with coordinates $\eta^{\mu} = kx^{\mu}, k, \lambda = kx^{2}$ Dilatations and special conformal transformations acting in this space are

$$D(1): \eta^{'\mu} = \eta^{\mu}, k^{'} = \rho^{-1}k, \lambda^{'}$$

$$C(4): \eta^{'\mu} = \eta^{\mu} + c^{\mu}\lambda, k^{'} = -2c_{\nu}\eta^{\nu} + k + c^{2}\lambda, \lambda^{'} = \lambda.$$

Theorem 3.1 (Connected Lie groups of a given Lie algebra)

Let G be a finite dimensional Lie algebra. Then there exists a unique (up to isomorphism) connected and simply connected Lie group G with g as its Lie algebra. If $G^{'}$ is another connected Lie group with this Lie algebra, it is of the form G/Z where Z is some discrete central subgroup of G.

We now finally have the whole picture between the connection between Lie groups and Lie algebras. Basically, we now know that we can work with Lie algebras, but thus losing (only) the topology of the group. By knowing possible Lie algebras, we know the possible Lie groups by the following line of thought each Lie algebraGgenerates a unique connected and simply connected Lie group G. Then we also have connected groups of the form G/Z where Z is a discrete central subgroup (central means that it lies in the center of G which is the set of all elements a for which $ab = ba$ for all $b \in G$. We also have disconnected groups with algebra G. But they are just the previous groups G/Z overlayed with another (unrelated) discrete group structure. Before moving on to matrix groups, it is best to look at an example of what we've said so far. We assume some familiarity with matrix groups for this example to make sense. It is a well-known fact that groups $SU(2)$ and $SO(3)$ have the same Lie algebra $\mathfrak{su}(3) = \mathfrak{so}(3)$ Since $SU(2)$ is connected and simply connected, it is the unique group constructed from the Lie algebra $\mathfrak{su}(2)$ And since $SO(3)$ is connected, it means it is isomorphic to $SU(2)/Z$ for some central subgroup Z. It turns out that $SO(3) \cong SU(2)/Z_2$ since $SU(2)$ has the topology of a three dimensional sphere S^3, the quotient group has the topology of the sphere with opposite points identified, which is the real projective space RP^3.

We now come to somewhat more familiar territory. We will consider Lie groups and Lie algebras of matrices.

We define the $GL(n, F)$ as the group of all invertible $n \times n$ matrices, which have either real $(F = R)$ of complex $(F = C)$ entries. Multiplication in this group is defined by the usual multiplication of matrices. The manifold structure is automatic since it is an open set of all $n \times n$ matrices (which form a n^2 dimensional vector space which is isomorphic to F^{n^2}).

We also define G$l(n, F)$ as the set of all matrices of dimension $n \times n$ with entries inF. This set is of course a vector space under the usual addition of matrices and scalar multiplication. One can also define the commutator of two such matrices as $[A,B]=AB-BA$ and this operation satisfies the requirements for the Lie bracket. The set $\mathrm{G}l(n,F)$ therefore has the structure of a Lie algebra.

The notation for $\mathrm{G}l(nF)$ was suggestive. The matrices $\mathrm{G}l(n,F)$ are the Lie algebra of the Lie group $GL(n,F)$ including the Lie bracket being the common commutator. The exponential map, which we were unwilling to define in all generality is in this case given as the usual exponential of matrices

$$\exp(A)=e^{A}=I+\sum_{n=1}^{\infty}\frac{A^{n}}{n!}$$

This map can be inverted near the identity matrix I

$$\exp^{-1}(I+A)=\log(I+A)=\sum_{n=1}^{\infty}\frac{(-1)^{k+1}A^{n}}{n}$$

With this definition of the exponential map, we can easily see that for an arbitrary matrix $A, e^{A}\in GL(n,F)$ really is invertible and its inverse is given by e^{-A} Indeed, because $[A-A]=0$ we have $e^{A}e^{-A}=e^{A-A}=e^{0}=1$.

Now by virtue of the first fundamental theorem, we can construct various matrix subgroups of $GL(n,F)$ by taking Lie subalgebras of $\mathrm{G}l(n,F)$ namely subalgebras of $n\times n$ matrices. There are a number of important groups and algebras of this type, and they are called the classical groups. We will for the purposes of future convenience and reference list them in a large table together with the restrictions by which they were obtained, as well as some other properties. These properties will be their null and first homotopic group, π^{0} and π^{1}. We will not go further into these concepts here, let us just mention that a trivial π^{0} means G is connected and a trivial while π^{1} means G is simply connected. Furthermore, for G which are not connected π^{1} is specified for the connected component of the identity. Another property which we will also list is whether a group is compact (as a topological space) and denote this by a C. Finally, dim will be the dimension of the group as a manifold which is equal to the dimension of the Lie algebra as a vector space. It is easy to check the dimensionality in each case by noting that $n\times n$ matrices form a n^{2} dimensional space by themselves, but then the dimensionality is gradually reduced by the constraints on its Lie algebra. However, the constraints on the Lie algebra are derived from the constraints on the group. In the orthogonal case for example we have $e^{A}(e^{A})^{T}=(e^{A})^{T}e^{A}$ which implies $e^{A}e^{(AT)}=e^{(A+A^{T})}=I$ and consequently $A+A^{T}$. The constraint $\det e^{A}=1$ can be reduced to the constraint $TrA=0$.

B. THE ROOT SYSTEM OF A SEMI-SIMPLE LIE ALGEBRA

Semisimple Lie algebras have a very important property called the root decomposition which will be the main concern in this subsection. But first, in order to be able to formulate this decomposition, we shall understand the concept of Cartan subalgebras (not in general but for semisimple Lie algebras).

a) SIMPLE ROOTS

We have defined a root system as a finite collection of vectors in an Euclidean vector space, which satisfy certain properties. Condition 1 (from the definition) stated that the root system must span the whole space. If the space is n-dimensional, we only need n linearly independent vectors. Since R spans V, we know that R contains a basis for V. The only remaining question is whether we can choose these n vectors among the many root vectors in such a way to be able to reconstruct the whole root system out of them. This is our motivation for introducing simple roots.

First we notice that a root system is symmetric with respect to the zero vector namely, we have $-\alpha\in R$ if $\alpha\in R$. Therefore we get the idea that we separate a root system into two parts, which we will call positive and negative roots. We will do this by choosing a polarization vector $t\in V$ which is not located on any of the orthogonal hyperplanes of the roots in R, so that all roots point in one of the two halfspaces divided by the orthogonal hyperplane of t Since α and $-\alpha$ are in different subspaces we have thus separated the root system into two parts. We then look at only "positive" roots and give the concept of a simple root.The simple roots are a very useful concept. Every positive root can be written as a finite sum of simple roots since it is either a simple root or if it is not, it can be written as a sum of two other positive roots since the root system R is finite this has to stop after a finite number of steps.

Also, every negative root can be written as $-\alpha$ for some positive root α. Together that means that for any root $\beta \in R$ we can write it as a linear combination of simple roots with integer coefficients. Let us denote the set of simple roots as S. Therefore, every $\beta \in R$ is a linear combination of vectors in S. Because S spans R and R spans V, simple roots S span the whole space V .Also, it can be proven that simple roots are linearly independent.

IV. RESULTS AND DISCUSSIONS

A. CLASSIFICATION OF SIMPLE ROOT SYSTEMS

The root or root vectors of a Lie algebra are the weight vectors of its adjoint representation. Roots are very important because they can be used both to define Lie algebra and to build their representations. We will see that Dynkin diagrams are in fact really only a way to encode information about roots. The number of roots is equal to the dimension of Lie algebra which is also equal to the dimension of the adjointrepresentation, therefore we can associate a root to every element of the Lie algebra. The most important things about roots is that they allow us to move from one weight to another. Let's see some theorem's to help us get the real picture about the root systems.

Theorem 4.1: Suppose R is a root system with bases Δ and $\Delta^{'}$, then $\exists\ w \in W \ni w(\Delta) = \Delta^{'}$.

Proof: Hence,let R^{+}, R^{-} be the sets of positive and negative roots with respect to Δ and $R^{'+}, R^{'-}$ be the sets of positive and negative roots with respect to $\Delta^{'}$.

Notice that $|R^{+}| = |R^{'+}| = \frac{|R|}{2}$. We write $\Delta = \{\alpha_1, ..., \alpha_n\}$, and $\Delta^{'} = \{\alpha_1^{'}, ..., \alpha_n^{'}\}$. The proof is by induction on $|R^{+} \cap R^{'-}|$.Now if $|R^{+} \cap R^{'-}| = 0$ then $R^{'+} = R^{+}$. which implies that any element from $\Delta^{'}$ is an nonnegative integral combination of elements of Δ i.e, $\alpha_j^{'} = \sum p_{ij}\alpha_i$ and conversely $\alpha_k = \sum_j q_{jk}\alpha_j^{'}$. Hence if we set $P = (p_{ij}), Q = (q_{jk})$ we have $PQ = I_n$. Which shows that P and Q are permutation matrices, hence it implies $\Delta = \Delta^{'}$. Now, if we assume that $|R^{+} \cap R^{'-}| = m \succ 0$. Then we can see that $\Delta \cap R^{'+} \neq 0$, otherwise one would have $\Delta \subset R^{'+}$, hence $R^{+} \subset R^{'+}$ which would imply $R^{'+} \subset R^{+}$, a contradiction with $|R^{+} \cap R^{'-}| \succ 0$.

Let $\alpha \in \Delta \cap R^{'-}$, we have $s_\alpha(R^{+}) = (R^{+} \setminus \{\alpha\}) \cup (-\alpha)$. In particular, $\alpha \notin s_\alpha(R^{+}) \cap R^{'-}$. It implies that $|s_\alpha(R^{+}) \cap R^{'-}| = m - 1$. It is clear that $s_\alpha(\Delta)$ is again a basis of R with corresponding set of positive roots given by $s_\alpha(R^{+})$, hence by induction there exists $y \in W$ such that $y(s_\alpha(\Delta)) = \Delta^{'}$. Setting $w = ys_\alpha$ we get the claim.

Theorem 4.2: Given a root system Φ ,and L to be a complex semisimple Lie algebra. Then L is simple if and only if Φ is irreducible.

Before proving the Theorem, consider the lemma.

Lemma 4.3: Let Φ be a root system with decomposition $\Phi = \Phi_1 \cup \Phi_2$ such that $(\alpha, \beta) = 0$ for any $\alpha \in \Phi_1, \beta \in \Phi_2$. Then

i. $\alpha \in \Phi_1, \beta \in \Phi_2 \Rightarrow \alpha + \beta \notin \Phi_1$,
ii. $\alpha, \beta \in \Phi_1, \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Phi_1$.

Proof (theorem 4.2):

Let's assume that L is not simple, and let $H \subset L$ be a Cartan subalgebra with $I \subset L$ be a nontrivial ideal. Since H consists of semisimple elements and $[H, I] \subset I$, the operation of H on I is semisimple. Hence one has

$$I = H_1 \oplus \left(\oplus_{\alpha \in \Phi} L_\alpha \cap I \right)$$

Where $H_1 = H \cap I$. Since L is semisimple, we have that $\dim_C L_\alpha = 1$ for any $\alpha \in \Phi$, hence $L_\alpha \cap I \in \{0, L_\alpha\}$. Set

$$\Phi_1 = \{\alpha \in \Phi \mid L_\alpha \cap I = L_\alpha\}.$$

Similarly, $I^\perp = H_2 \oplus \left(\oplus_{\alpha \in \Phi_2} L_\alpha \cap I^\perp \right)$, where the symbol $\perp$ denotes the orthogonal with respect to the Killing form $\kappa(-,-)$. Thanks to the non-degeneracy of the Killing form one has that $L = I \oplus I^\perp$. Hence $H = H_1 \oplus H_2$, $\Phi = \Phi_1 \cup \Phi_2$.

If $\Phi_2 = \phi$, then $L_\alpha \subset I$ for any $\alpha \in \Phi$, then $I^\perp = H_2 \subset H$ is an abelian ideal, hence must be trivial by semisimplicity of L. So we can assume that $\Phi_2 \neq \phi$. Let $\alpha_i \in \Phi_i, i = 1,2$ .Then $(\alpha_1, \alpha_2) = \langle \alpha_1, \alpha_2 \rangle = \alpha_1(h_{\alpha 2}) = 0$ since $\alpha_1(h_{\alpha 2}) x_{\alpha 1} = [h_{\alpha 2}, x_{\alpha 1}] \in I \cap I^\perp = 0 \Rightarrow (\alpha_1, \alpha_2) = 0 \Rightarrow \Phi$ is irreducible.

If we let $H \subset L$ be a Cartan subalgebra with root space decomposition $L = H \oplus \oplus_{\alpha \in \Phi} L$. Assume that Φ is reducible, $\Phi = \Phi_1 \cup \Phi_2, (\alpha_1, \alpha_2) = 0$ for $\alpha_1 \in \Phi_1, \alpha_2 \in \Phi_2$ and $\Phi_1, \Phi_2 \neq \phi$. Set

$$I = \langle e_\alpha, f_\alpha, h_\alpha \mid \alpha \in \Phi_1 \rangle$$

Then using (ii) of the Lemma above, one can see that $I \neq L$. Which proofs that I is an ideal. Let $x \in L, \alpha \in \{e_\alpha, f_\alpha, \mid \alpha \in \Phi_1\}$. Since the e_α, f_α generate I, hence, it suffices to show that $[x, \alpha] \in I$. We can without loss of generality assume that either $x \in H$,or $x \in L_\beta$ for some $\beta \in \Phi$.

If $x \in H$, then $[x, e_\alpha] = \alpha(x) e_\alpha \in I, [x, f_\alpha] = -\alpha(x) f_\alpha \in I, \forall \alpha \in \Phi_1.$

If $x \in L_\beta$, then $[x, e_\alpha] \in L_{\alpha+\beta}$. if $\alpha + \beta \notin \Phi$, then $L_{\alpha+\beta} = 0 \in I.$

Otherwise $\alpha + \beta \in I$ implies that $\beta \in \Phi_1$ i.e. by (i) of the lemma above. Now, again by (ii)of the lemma above, we then have that $\alpha + \beta \in \Phi_1$, hence $L_{\alpha+\beta} \subset I$. The proof in case $\alpha = f_\alpha$ is similar. Hence I is a non-trivial ideal of L, implying that L is not simple.

B. THE CARTAN MATRIX AND DYNKIN DIAGRAMS

Suppose we have a simple root system S. We can ask ourselves, what is the relevant information contained in such a system. Certainly, it is not the absolute position of the roots, or their individual length, since we can take an orthogonal transformation and still obtain an equivalent root system. The important properties are their relative length to each other and the angle between them. Since we have for simple roots $\alpha, \beta \in S$ the inequality $(\alpha, \beta) \leq 0$, the angle between simple roots is $\geq 90^\circ$, and with the help of figure below, we have the four familiar possibilities. Of course, the angle between them also dictates their relative length, so the only relevant information are the angles between the roots (and which root is longer). We can present this information economically as a list of numbers. Instead of angles, we specify the numbers $n_{\alpha\beta} = 2\frac{(\alpha, \beta)}{(\beta, \beta)}$ which are conserved via root system isomorphisms. We call this list the Cartan matrix.

Due to the definition of $n_{\alpha\beta}$, we clearly have $a_{ii} = 2$ for all $i \in \{1, \ldots, n\}$. Also, since the scalar product of simple roots $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$, the non-diagonal entries in the Cartan matrix are not positive i.e. $\alpha_{ij} \leq 0$ for $i \neq j$. It is also possible to present the information in the Cartan matrix in a graphical way via Dynkin diagrams.

C. CLASSIfiCATION OF CONNECTED DYNKIN DIAGRAMS

Dynkin diagrams are a very effective tool for classifying simple root systems S, and consequently the reduced root systems R. Since reducible root systems are a disjoint union of mutually orthogonal subroot systems, the Dynkin diagram is just drawn out of many connected graphs. It is thus sufficient to classify connected Dynkin diagrams which will help to state the result of this classification and as well as to sketch a simplified proof.

Theorem 4.4 (Classification of Dynkin diagrams).

Let R be a reduced irreducible root system. Then its Dynkin diagram is isomorphic to a diagram from the figure below, which is also equipped with labels of the diagrams. The index in the label is always equal to the number of simple roots, and each of the diagrams is realized for some reduced irreducible root system R.

The Families   The 5 exceptional root systems

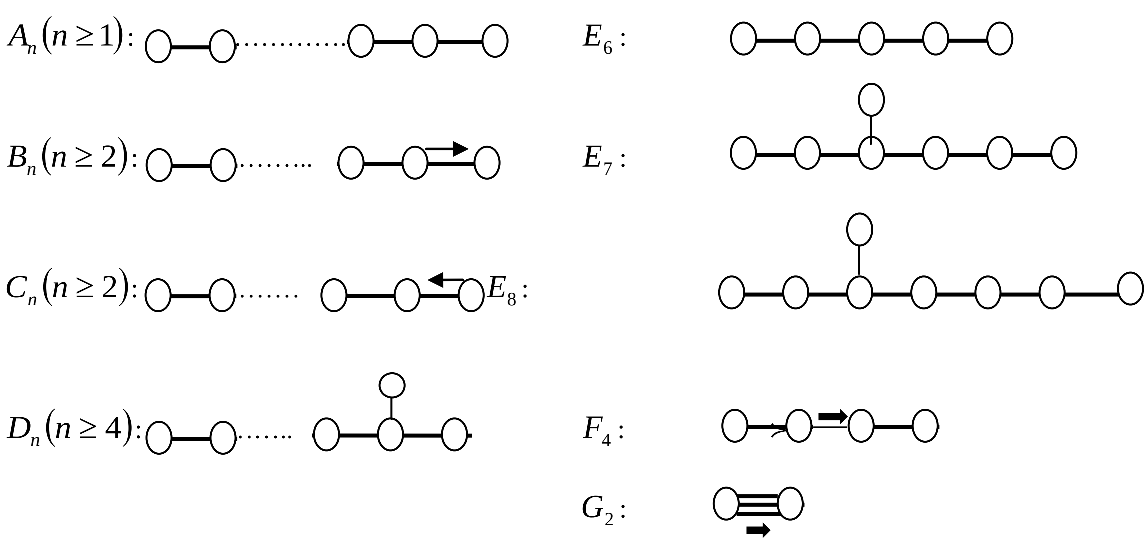


Figure 4.1: Dynkin Diagram of all irreducible root systems R

We now turn to a simplified proof of the theorem. It turns out that all irreducible simple root systems in the figure above can indeed be constructed. In the process of this work we shall see the reason why diagrams with different connections are not valid as Dynkin diagrams of simple root systems. Only connected graphs of vertices with either a null, single, dual or triple connection with another vertex will be considered. The Dynkin diagrams as graphs of an irreducible simple root systems are a subset of all possible graphs under consideration. Before we start, an important notion has to be introduced i.e. the subgraph. If I is the set of vertices of a graph, then a subgraph consists of a subset of vertices $J \subseteq I$, while the types of connections between the vertices in J stay the same as in the original graph I. Also, a special case of a Dynkin diagram is a graph I, which contains no dual or triple connections.

For the purposes of this work, we shall call such a diagram a simple graph. A number of the properties of Dynkin diagrams can be deduced by looking at subgraphs. Suppose we have a true Dynkin diagram I as a realization of an irreducible simple root system. This diagram contains all the necessary information for the construction of the Cartanmatrix a_{ij}. If the root system spans a n-dimensional vector space V, then the n roots constitute a basis of this space, and the Cartan matrix is a linear operator on the vector space V. this operator is written as a matrix in the basis $\{\alpha_i\}_{i\in I}$. Suppose we have a subgraph of this Dynkin diagram, we specify a subset J of simple root vectors $\alpha_i : i \in J$ (for a given labeling of the simple roots). If the chosen number of simple roots is K then the subgraph J has K vertices, and we can construct a $K \times K$ submatrix of the Cartan matrix with entries a_{ij}, where $i, j \in J$. This matrix can be again viewed as a linear operator, this time on the space $\overline{V}$, which is spanned by the roots α_j with $j \in J$. The linear operator $\hat{a}$, constructed by choosing a subset of indices J from the Cartan matrix, is always positive definite which means

$\left(\hat{a}\, x, x\right) \succ 0$ for all $x \in \overline{V} \setminus \{0\}$. Indeed, if $x = \sum_{j\in J} c_j \alpha_j$, then

$$\left(\hat{a}\, x, x\right) = \sum_{i,j,k\in J} \left(c_i a_{ij} \alpha_j, c_k \alpha_k\right) = \sum_{i,j,k\in J} 2 c_i c_k \frac{\left(\alpha_i, \alpha_j\right)}{\left(\alpha_j, \alpha_j\right)} \left(\alpha_j, \alpha_k\right) = \sum_{j\in J} \frac{2}{\left(\alpha_j, \alpha_j\right)} \left(\sum_{i\in J} c_i \alpha_i, \alpha_j\right)\left(\sum_{k\in J} c_k \alpha_k, \alpha_j\right)$$
$$= \sum_{j\in J} 2 \frac{\left(x, \alpha_j\right)^2}{\left(\alpha_j, \alpha_j\right)} \succ 0.$$

The result (which is a sum of nonnegative terms) cannot be zero, because that would imply that $\left(x, \alpha_j\right) = 0$ for all $j \in J$ and therefore x would be orthogonal to all a_j. This is not possible, since x is a nonzero vector in $\overline{V}$ and vectors α_j for $j \in J$ form a basis of the vectorspace $\overline{V}$ that means that given any subgraph J of a given diagram, its Cartan matrix is positive definite. This will allow us to put restrictions, on what kind of subgraphs can be found in the Dynkin diagrams.

Here are some important rules as a motivation for the classification theorem.

i. If I is a connected Dynkin diagram with 3 vertices, then the only two possibilities are shown in Figure 4.2. We shall now derive this result. Consider a Dynkin diagram with 3 vertices, ignoring the relative lengths of the simple roots, the diagram is specified by the 3 angles between the roots. At most one of these angles is equal to 90°, because I is connected. Furthermore, the sum of the angles between 3 vectors is 360° in a plane, and less than 360°, if they are linearly independent. This excludes all possible diagrams with 3 vertices, as shown in Figure 4.2, except the two from the statement. Also, if there is no 90° angle, we have a loop. It suffices to check a loop of 3 vertices with just single connections, since double or triple connections would increase the sum of the angles even further

Allowed Dynkin diagrams: Forbidden diagrams

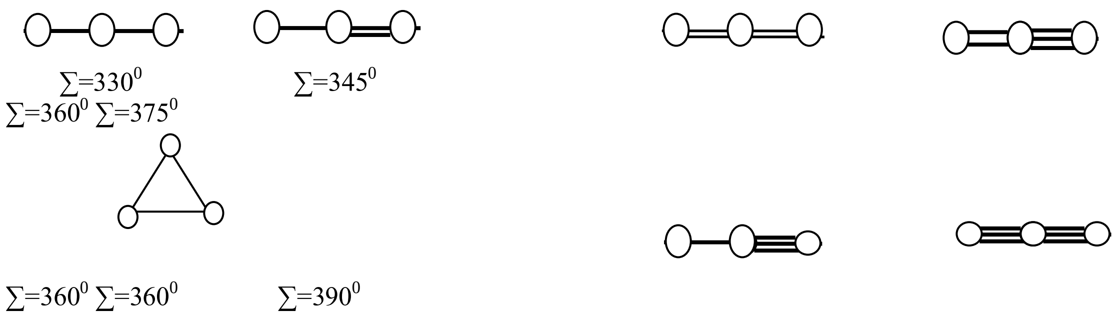


Figure 4.2: Possible diagrams with 3 vertices

Note: If the sum of the angles between the vertices $\sum=360^0$, the diagram is forbidde.

no Dynkin diagram I may contain any of the forbidden 3 vertex diagrams as subgraphs, lest we run into a contradiction. This implies that the only possible diagram with a triple connection is the one with two vertices.

ii. If I is a Dynkin diagram and a simple graph, then I contains no cycles (subgraphs with vertices connected in a loop). If that were not the case, there would exist a subgraph J with $k \geq 3$ vertices, which would be a loop (with only single connections). In this loop, the neighboring vertices would give -1 to the Cartan matrix, the diagonal elements would give 2, while all others would be 0. We relabeled the indices, so that i runs from 1 to k, and we label $\alpha_{k+1} = \alpha_1$ and $\alpha_0 = \alpha_k$. For $x = \sum_{j \in J} \alpha_j$, with the normalization of roots $(\alpha_i, \alpha_i) = 2$ we then have

$$(\hat{a}x, x) = \sum_{i,j,k} \frac{2}{(\alpha_j, \alpha_j)} (\alpha_i, \alpha_j)(\alpha_j, \alpha_k)$$

$$= \sum_{i,j,k} (2\delta_{j,i} - \delta_{j,i+1} - \delta_{j,i-1})(2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1})$$

$$= 0$$

which is a contradiction for the positive definite Cartan matrix of a subgraph of a Dynkin diagram.

Figure 4.3: Cycles and vertices with 4 (or more) connections are forbidden.

Note: This is derived by computing the violation of the positive definiteness of the Cartan matrix.

iii. If I is a Dynkin diagram and a simple graph, then each vertex in I is connected to at most 3 others. If that were not the case, then at least one of the vertices would be connected to at least 4 others, and we would have a subgraph, which is shown in Figure 4.3. For this specific graph, the vector $x = 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ gives $(\hat{a}x, x) = 0$

$$\hat{a}x = \begin{bmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & & & \\ -1 & & 2 & & \\ -1 & & & 2 & \\ -1 & & & & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

iv. If I is a Dynkin diagram, and α, β are two roots connected with a single connection, as shown in Figure 4.4, the two roots can be substituted by a single root, and we obtain a new Dynkin diagram. We will not prove this statement, but it can be shown by constructing a new root system, with the same roots as previously, but taking the root $\alpha + \beta$ instead of roots α and β. One can easily check (via scalar products) that the angles between the new root $\alpha + \beta$ and other roots are consistent with the contraction of the two vertices. As a consequence, it is possible to eliminate some further diagrams by contracting vertices. The reasons why there can be at most one branching point (vertex with 3 connections), and why there cannot be 2 double connections, are illustrated in Figure 4.4.

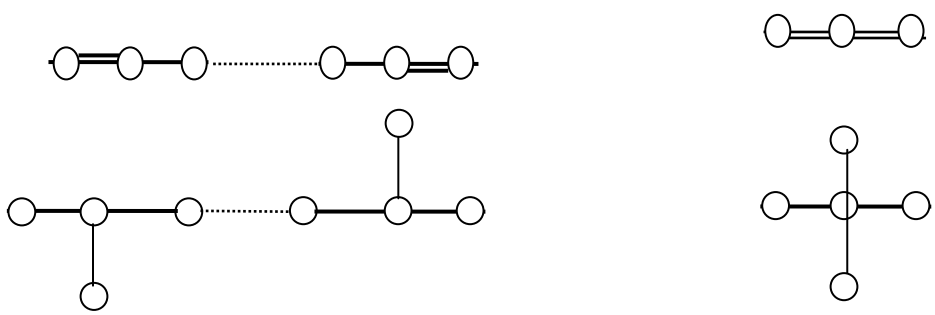


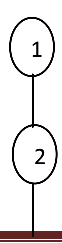
Figure 4.4: Forbidden diagram as a consequence

Note: Contraction of two vertices with a single connection in a valid Dynkin diagrams give a valid Dynkin diagrams.

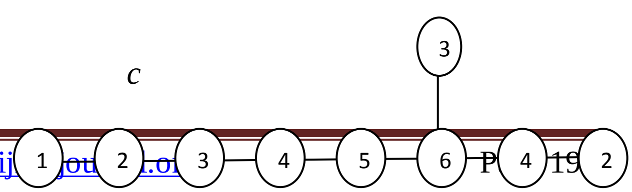
Here are some point to consider about the forbidden diagrams.

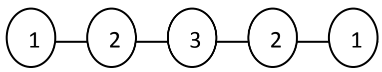
- Branching point is on the third vertex, and the remaining tail is 4 or less vertices long. Otherwise, they contain a sub graph b or c, and therefore are not allowed.
- A double connection between two vertices must be placed at the end of a chain of vertices with single connections, except in the case of the Dynkin diagram $F4$. Otherwise, the chain of single-connection vertices is too long on one side, and the diagram contains d or e as sub graphs, which is not allowed.

a.

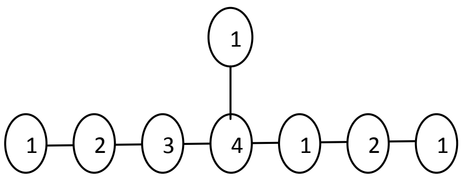


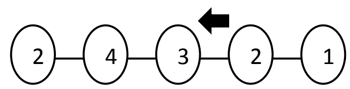
c

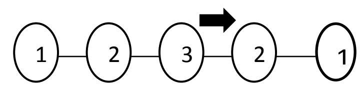




$b\ d\ e$







D. SERRE RELATIONS AND THE CLASSIFICATION OF SEMISIMPLE LIE ALGEBRAS

Now we will turn to the classification of semisimple Lie algebras, and explain how that is related to the classification of irreducible simple root systems. One thing to note is that the decomposition $G = \oplus\ G_i$ of a semisimple Lie algebra into simple Lie algebras is related to the decomposition of the root system, in particular G is simple if and only if its root system R is irreducible.

Theorem 4.5 (Serre relations): Let G be a semisimple Lie algebra with Cartan subalgebra h and its root system $R \subseteq h^*$, and choosing a polarization we have S as its simple root system. Let $(.,.)$ be a scalar product (a non-degenerate symmetric bilinear form) on G.

• We have the decomposition $G = h \oplus n_+ \oplus n_-$, where $n_\pm = \oplus_{\alpha \in R_\pm}\ G_\alpha$.

• Let $H_\alpha \in h$ be the element, which corresponds to $\alpha \in h^*$, and $h_i = h_{\alpha_i} = 2H_{\alpha_i}/(\alpha_i, \alpha_i)$. If we choose $e_i \in g_{\alpha_i}$, $f_i \in G_{-\alpha_i}$ and $h_i = h_{\alpha_i}$, with the constraint $(e_i, f_i) = 2/(\alpha_i, \alpha_i)$, then e_i generate n_+, f_i, generate n_- and h_i form a basis for h (where in all cases $i \in \{1,\dots,r\}$, and thus $\{e_i, f_i, h_i\}_{i\in\{1,\dots,r\}}$ generates G.

• The elements e_i, f_i, g_i satisfy the Serre relations (where a_{ij} are the elements of the Cartan matrix):

$$[h_i, h_j] = 0, [h_i, e_j] = a_{ij} e_j$$

$$[h_i, f_j] = -a_{ij} f_j, [e_i, f_j] = \delta_{ij} h_j$$

$$([e_i,.])^{1-a_{ij}} e_j = 0, \quad ([f_i,.])^{1-a_{ij}} f_j = 0.$$

Theorem 4.6 (Classification of Semisimple Lie Algebras): A simple complex finite dimensional Lie algebra G is isomorphic to a Lie algebra, constructed from one of the Dynkin diagrams in Figure 4.1.Semisimple Lie algebras are all possible finite direct sums of simple Lie algebras. With this, we have classified semisimple Lie algebras.

It is noteworthy that the restrictions on n in the figure above are due to either small diagrams not existing, or they are the same as a previous one. For example, we would have $A_1 = B_1 = C_1$, which would correspond with $sl(2,C) \cong so(3,C) \cong sp(1,C)$ on the Lie algebra level.

E. THE QUANTUM MECHANICS AND ITS ROTATION INVARIANCE

Quantum mechanics tells us that any physical system can be described by a wave function. This wave function is a solution of a diff erential equation (for instance the Schro¨dinger equation, if a non-relativistic limit is applicable) with boundary conditions determined by the physical situation. We will not indulge in the problems of determining this wave function in all sorts of cases, but we are interested in the properties of wave functions that follow from the fact that Nature shows certain symmetries. By making use of these symmetries we can save ourselves a lot of hard work doing calculations. One of the most obvious symmetries that we observe in nature around us, is invariance of the laws of nature under rotations in three-dimensional space. It is be expected that the results of measurements should be independent of the orientation of his or her apparatus in space, assuming that the experimental setup is not interacting with its environment, or with the earth's gravitational field. For instance, one does not expect that the time shown by a watch will depend on its orientation in space, or that the way a calculator works changes if we rotate it. Rotational symmetry can be found in many fundamental equations of physics such as the Newton's laws, Maxwell's laws, and Schr¨odinger's equation for example do not depend on orientation in space. To state things more precisely, Nature's laws are invariant under rotations in three-dimensional space.

We now intend to find out what the consequences are of this invariance under rotation for wave functions. From classical mechanics it is known that rotational invariance of a system with no interaction with its environment, gives rise to conservation of angular momentum in such a system, the total angular momentum is a constant of the motion. This conservation law turns out to be independent of the details of the dynamical laws, it simply follows from more general considerations which can be deduced in quantum mechanics. There turns out to be a connection between the behavior of a wave function under rotations and the conservation of angular momentum. The equations may be hard to solve explicitly, but consider a wave function ψ depending on all sorts of variables, being the solution of some linear diff erential equation

$$D\psi = 0 \qquad (4.1)$$

The essential thing is that the exact form of D does not matter, the only thing that matters is that D be invariant under rotations. An example is Schr¨odinger's equation for a particle moving in a spherically symmetric potential $V(r)$,

$$\left[\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)-V(r)+i\hbar\frac{\partial}{\partial t}\right]\psi(\vec{x},t)=0,\ r=\sqrt{\vec{x}^2} \qquad (4.2)$$

Consider now the behavior of this diff erential equation under rotations. When we rotate, the position vector $\vec{x}$ turns into another vector with coordinates $x_i^{'}$

$$x_i^{'} = \sum_j R_{ij} x_j \qquad (4.3)$$

Here, we should characterize the rotation using a 3×3 matrix R, that is orthogonal and has determinant equal to 1 (orthogonal matrices with determinant -1 correspond to mirror reflections). The orthogonality condition for R implies that

$$\tilde{R}R = R\tilde{R} = 1, \text{ or } \sum_i R_{ij}R_{ik} = \delta_{jk}, \quad \sum_j R_{ij}R_{kj} = \delta_{ik} \quad (4.4)$$

where $\tilde{R}$ is the transpose of R (defined by $\tilde{R}_{ij} = R_{ji}$).

It is not difficult now to check that equation (4.2) is rotationally invariant. To see this, consider the function $\psi'(\vec{x},t)=\psi(\vec{x}',t)$

$$\frac{\partial}{\partial x_i}\psi'(\vec{x},t)=\frac{\partial}{\partial x_i}\psi(\vec{x}',t)=\sum_j\frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x_j^{'}}\psi(\vec{x}',t)=\sum_j R_{ji}\frac{\partial}{\partial x_j^{'}}\psi(\vec{x}',t) \qquad (4.5)$$

Subsequently, we observe that

$$\sum_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\psi(\vec{x}',t) = \sum_{i,j,k} R_{ji}R_{ki}\frac{\partial}{\partial x_j^{'}}\frac{\partial}{\partial x_k^{'}}\psi(\vec{x},t)$$

$$= \sum_i \frac{\partial}{\partial x_i^{'}}\frac{\partial}{\partial x_i^{'}}\psi(\vec{x}',t) \qquad (4.6)$$

where we made use of equation (4.4). Since $\vec{x}'^2 = \vec{x}^2$, the potential $V(r)$ also remains the same after a rotation. From the above, it follows that equation (4.2) is invariant under rotations, if $\psi(\vec{x},t)$ is a solution of equation (4.2), then also $\psi'(\vec{x},t)$ must be a solution of the same equation. In the above, we are meant to believe that rotations can be represented by real 3×3 matrices R. Their determinant must be $+1$, and they must obey the orthogonality condition $R\vec{R} = 1$.

V. CONCLUTION

We have managed to tread the long road from semisimple Lie groups to Dynkin diagrams. For a Lie group G we always have its Lie algebra G, which is the tangent space of the identity, with the commutator arising through the group multiplication law. We know that this Lie algebra can be viewed as a Lie subalgebra of G$l(n.F)$ for some n. We decompose this algebra into a semisimple Lie algebra G and a remainder (the radical). The semisimple part G has a root decomposition, and we thus obtain a reduced root system R of the semisimple Lie algebra G which is a finite set in the dual of the Cartan subalgebra of G. Choosing a polarization, R leads to a simple root system S. We decompose this simple root system into orthogonal parts, whereas each such part can be schematically drawn with a connected Dynkin diagram. There are four families of such diagrams, and an additional five exceptional diagrams. The total Dynkin diagram of S is a disjoint union of the connected Dynkin diagrams for its orthogonal parts. Conversely, we consider all steps in the construction of a Lie algebra from its diagram. We take one of the connected Dynkin diagrams and with this we have a unique (up to isomorphism) simple root system S which enables us to reconstruct a unique reduced root system R. A semisimple Lie algebra is then obtained by taking a direct sum of simple Lie algebras (we get a direct product on the level of groups). A constructed Lie algebra leads to a unique connected and simply connected semisimple Lie group G. The groups, which are not simply connected, are obtained by taking quotients G/Z with discrete central subgroups, and the groups which are not connected have an additional discrete group structure among components. With the understanding of both directions, we have obtained the full picture of possible semisimple Lie algebras, and implications for semisimple Lie groups.

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