Hydromagnetic Instability of Density Stratified Rotating Layer of Rivlin-Ericksen Visco-Elastic Fluid In Porous Medium

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Abstract:- *Hydromagnetic instability of density stratified rotating layer of rivlin-ericksen visco-elastic fluid in porous medium. A sufficeint condition for instability is found and variational principle for the present problem is verifeid. result obtained in this paper shown graphically also.*

I. INTRODUCTION.

Mohapatra and Mishra [1] discussed the thermal stability of heterogeneous rotating layer between free boundaries and observed that PES is not valid for this problem. Gupta, Bhardwaj et al. [2] described Rayleigh-Taylor instability with rotating and magnetic field. Gupta, Sood et al. [3] studied the characterization of non-oscillatory motions in rotating hydromagnetic thermohaline convection. Pradhan and Samal [4] have shown thermal instability of compressible shear flow in porous medium. Pradhan, Tripathi et al. [6] described thermal instability of fluid layers in a variable gravitational field. Sharma and Singh [7] studied stability of stratified fluid in the presence of suspended particles and variable magnetic field. Sharma and Sunil [8] studied the thermosolutal instability of partially ionized hall plasma in porous medium. Sharma and Kumar [9] discussed the Rayleigh-Taylor instability of visco-elastic fluid through porous media. Chen and Crighton [10] illustrated the instability of large Reynolds number flow of Newtonian fluid over a visco-elastic fluid. Deepak and Evas [11] have described the stability of an interface between viscous fluids subjected to a high frequency magnetic field and consequences for electromagnetic casting. Sharma and Kumar [12] investigated the thermal instability of a layer of Rivlin-Ericksen viscous fluid acted on by a uniform rotation and found that rotation has stabilizing effect and introduces oscillatory modes in the system. Sharma and Sunil [13] discussed thermal instability of compressible finite Larmour-radius hall plasma in a porous medium. Sharma and Kumar [14] discussed the thermal instability of Rivlin-Ericksen elastico-viscous fluid in hydromagnetics. Sharma & Chand et al [15]. investigated the thermosolutal convection of Rivlin-Ericksen fluid in porous medium in hydromagnetics. Sharma and Kumar [16] discussed the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid in the presence of suspended particles.

II. BASIC STATIC CONFIGURATION

We consider the stability of an incompressible, perfectly conducting density stratified horizontal layer of a Maxwell fluid in porous medium confined in the region $0 \le z \le d, -\infty < x, y < 0,$ z-axis being vertical in the presence of a horizontal magnetic field $\bar{H} = (H_0, 0, 0)$. We refer the system with respect to coordinate system rotating with a uniform angular velocity $\mathbf{\Omega}$ about z-axis, so that with respect to this coordinate system, fluid velocity is at rest in the equilibrium. It is assumed that the porous medium obeys Darcy's law.

We consider the density stratified fluid in which the density ρ is an exponential function of z, the vertical coordinate; that is $\rho = \rho_0\, e^{\delta z}$, where δ is a constant. The equation of motion of rotating coordinate system and in the presence of magnetic field are

$$\rho\left[\frac{\partial q}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q}\right] = -\nabla P + \mathbf{g}\rho\lambda + 2\rho(\nu \times \Omega)$$

$$+\frac{\mu}{4\pi}[(\nabla\times\mathbf{H})\times\mathbf{H}]-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q} \quad (1)$$

The equation of continuity is

$$\nabla.\mathbf{q} = 0 \quad (2)$$

The equation of incompressibility of the fluid is as

$$\frac{\partial\rho}{\partial t}+(\mathbf{q}.\nabla)\,\rho=0 \quad (3)$$

The fluid is perfectly electrically conducting, then

$$\frac{\partial\mathbf{H}}{\partial t}=\nabla\times(\mathbf{q}\times\mathbf{H}) \quad (4)$$

The magnetic field satisfies the equation

$$\nabla.\mathbf{H} = 0 \quad (5)$$

Since in basic or the equilibrium state, we have taken the fluid to be at rest with respect to rotating coordinate system, the equilibrium state of the fluid is characterized by

$$\bar{V} = (0,0,0), \quad (6)$$

$$\rho=\rho_0(z), \quad (7)$$

$$P=P_0(z) \quad (8)$$

and $\mathbf{H}=(H_0,0,0)$ (9)

where the pressure $P_0(z)$ satisfies the equation

$$P_0+\frac{1}{8\pi}H_0^2+g\int\rho(z)dz = \text{constant} \quad (10)$$

Since the velocity is zero in the basic state and all the variables are time independent.

Now, we will take the equations (1) to (5) in Cartesian form as,

$$\rho\left[\frac{\partial u}{\partial t}+\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]=-\frac{\partial P}{\partial x}-2\rho(v\times\Omega)$$

$$+\frac{\mu}{4\pi}\left[H_z\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)-H_y\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\right]-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)u, \quad (11)$$

$$\rho\left[\frac{\partial v}{\partial t}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\right]=-\frac{\partial P}{\partial y}-2\rho(u\times\Omega)$$

$$+\frac{\mu}{4\pi}\left[H_x\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)-H_z\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\right]-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)v, \quad (12)$$

$$\rho\left[\frac{\partial w}{\partial t}+\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\right]=-\frac{\partial P}{\partial z}-g\rho$$

$$+\frac{\mu}{4\pi}\left[H_y\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)-H_x\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\right]-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)w, \quad (13)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad (14)$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}=0, \quad (15)$$

$$\frac{\partial H_x}{\partial t}=\frac{\partial}{\partial y}\left(uH_z-vH_x\right)-\frac{\partial}{\partial z}(wH_x-uH_z), \quad (16)$$

$$\frac{\partial H_y}{\partial t}=\frac{\partial}{\partial z}\left(vH_z-wH_y\right)-\frac{\partial}{\partial x}(uH_y-vH_x), \quad (17)$$

$$\frac{\partial H_z}{\partial t}=\frac{\partial}{\partial x}\left(wH_x-uH_z\right)-\frac{\partial}{\partial y}(vH_z-wH_y) \quad (18)$$

and $$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0. \quad (19)$$

III. Perturbation Equations

To discuss the stability of the system, we perturb the basic state and take the perturbed velocity, density, pressure, magnetic field respectively, as $(u,v,w),\ \rho+\delta\rho,\ P+\delta P,\ (H_0+h_x, h_y, h_z)$, where $\delta\rho,\ \delta P$ and (h_x, h_y, h_z) denote the perturbations. Substituting these variables in the equations (11) to (19) and linearizing the equations, we get the perturbation equations as

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}\delta P-2\rho(v\times\Omega)-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)u, \quad (20)$$

$$\rho\frac{\partial v}{\partial t}=-\frac{\partial}{\partial y}\delta P+2\rho(u\times\Omega)+\frac{\mu H}{4\pi}\left[\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right]-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)v, \quad (21)$$

$$\rho\frac{\partial w}{\partial t}=-\frac{\partial}{\partial z}\delta P-g\delta\rho+\frac{\mu H}{4\pi}\left[\frac{\partial H_z}{\partial x}-\frac{\partial H_x}{\partial z}\right]-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)w, \quad (22)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{23}$$

$$\frac{\partial}{\partial t}\delta\rho + w\frac{\partial \rho}{\partial z}=0 \;\Rightarrow\; \frac{\partial}{\partial t}\delta\rho + wD\rho = 0, \tag{24}$$

$$\frac{\partial H_x}{\partial t}=H_0\frac{\partial u}{\partial x}, \tag{25}$$

$$\frac{\partial H_y}{\partial t}=H_0\frac{\partial v}{\partial x}, \tag{26}$$

$$\frac{\partial H_z}{\partial t}=H_0\frac{\partial w}{\partial x} \tag{27}$$

and
$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0. \tag{28}$$

The perturbation variables are the functions of space coordinate x, y, z and time t. We have taken the fluid layer to be horizontal extending to infinity in both x and y directions. The applied magnetic field is along x-axis and the gravitational force along negative z -axis. In this case, it would be sufficient to consider the perturbations to be two dimensional. Therefore, to discuss the stability of the fluid layer, we take the perturbations to be two-dimensional only and so express the perturbation variables in the form

$$f(x,y,z,t)=f(z)\exp.[ik_x+nt] \tag{29}$$

where $f(z)$ is some regular function of z, representing the perturbation variables $f(x, y, z, t)$. In this expression, k is the wave number and n the complex growth rate of the perturbation modes. We substitute this form of the perturbation variables in the perturbation equations (20) to (28) and then solve the resultant equations simultaneously. We observe from equation (29) that in this form of perturbation variables the dependence on t is exponential. Now using (29) form of the perturbations in equations (20) to (28), we have the equations governing the perturbations as

$$\rho nu = -ik\delta P + 2\rho(v\times\Omega) - \frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)u$$

$$\rho nu - 2\rho(v\times\Omega) = -ik\delta P - \frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)u, \tag{30}$$

$$\rho nv = \frac{\mu H_0}{4\pi}\left[ikH_y - ikH_x\right] - 2\rho(u\times\Omega) - \frac{1}{k_1}(\mu+\mu' n)v$$

$$\rho nv + 2\rho(u\times\Omega) = \frac{\mu H_0}{4\pi}\left[ikH_y - ikH_x\right] - \frac{1}{k_1}(\mu+\mu' n)v, \tag{31}$$

$$\rho nw = -D\delta P - g\delta\rho + \frac{\mu H_0}{4\pi}\left[ikH_z - DH_x\right] - \frac{1}{k_1}(\mu+\mu' n)w, \tag{32}$$

$$iku + Dw = 0, \tag{33}$$

$$n\delta\rho + w(D\rho) = 0 \quad \Rightarrow \quad \delta\rho = \frac{-w(D\rho)}{n}, \tag{34}$$

$$nH_x = H_0 iku, \tag{35}$$

$$nH_y = H_0 ikv, \tag{36}$$

$$nH_z = H_0 ikw \tag{37}$$

and $$ikH_x + DH_z = 0 \tag{38}$$

where D is the differential operator.

We substitute the value of $\delta\rho, h_x, h_y$ and h_z from the equations (34) to (37) into the equations (31) and (32), we get

$$\rho nv - 2\rho u\Omega = -\frac{\mu H_0^2 k^2}{4\pi n} v - \frac{1}{k_1}(\mu + \mu' n)v \tag{39}$$

$$\rho nw = -D\delta P + g\left(\frac{D\rho}{n}\right)w + \frac{\mu H_0^2}{4\pi n}[-k^2 + D^2 Dw] - \frac{1}{k_1}(\mu + \mu' n)w$$

$$= -D\delta P + g\left(\frac{D\rho}{n}\right)w + \frac{\mu H_0^2}{4\pi n}[D^2 - k^2]w - \frac{1}{k_1}(\mu + \mu' n)w. \tag{40}$$

We eliminate δP from the equation (30) and (40) by multiplying equation (40) by ik and differentiating the equation (30) and subtracting. We get

$$n\rho(Du - ikw) - 2\rho\Omega Dv = \frac{-ikg(D\rho)}{n} w - \frac{ik\mu H_0^2}{4\pi n}[D^2 - k^2]w$$

$$-\frac{1}{k_1}(\mu + \mu' n)\,(Du - ikw). \tag{41}$$

Now, multiplying equation (41) by ik and eliminating u by equation (33), we get

$$n\rho\left[D \times -\frac{Dw}{ik} - ikw\right]ik - 2\rho\Omega ikDv = -\frac{k^2 g(D\rho)}{n} w$$

$$-\frac{k^2 \mu H_0^2}{4\pi n}(D^2 - k^2)w + \frac{1}{k_1}(\mu + \mu' n)\left(D \times -\frac{Dw}{ik} - ikw\right)ik$$

or $$n\rho\left[D^2-k^2\right]-2\rho\Omega ikDv=-\frac{k^2g(D\rho)}{n}w$$

$$-\frac{k^2\mu H_0^2}{4\pi n}(D^2-k^2)w-\frac{1}{k_1}(\mu+\mu' n)\left(D^2-k^2\right)w$$

or $$(D^2-k^2)w\left[-n\rho-\frac{k^2\mu H_0^2}{4\pi n}-\frac{1}{k_1}(\mu+\mu' n)\right]-\frac{gk^2(D\rho)}{n}=2\rho i\Omega kDv \qquad (42)$$

Further eliminating u between equations (31) and (33), we get

$$\left[\rho n+\frac{\mu H_0^2k^2}{4\pi n}+\frac{1}{k_1}(\mu+\mu' n)\right]v=-\frac{2\rho\Omega}{k}Dw$$

or $$\left[-n\rho-\frac{\mu H_0^2k^2}{4\pi n}-\frac{1}{k_1}(\mu+\mu' n)\right]v=\frac{2i\rho\Omega}{k}Dw. \qquad (43)$$

We thus have to solve the equations (42) and (43) to determine the nature of the perturbations.

We now non-dimensionalize the perturbation variables in the equations (42) and (43) by taking the following transformations :

$$D^*=dD,\ a=Kd,\ P=\frac{nd^2}{\nu},\ \ u^*=Uv,\ w^*=Uw,\ \ \upsilon=\frac{\mu}{\rho},\ Q=\frac{\mu H_0^2d^2}{4\pi\rho\upsilon^2},$$

$$R_d=\frac{d^2}{k_1},\ \ R_a=\frac{4\Omega^2d^4}{\upsilon^2},\ \ R_1=\frac{\delta gd^4}{\upsilon^2},\ \ \lambda^*=\frac{\lambda\upsilon}{d^2}$$

where Q is the magnetic force number or Chandrasekhar number, R_d the porosity number, R_a the Taylor's number, R_1 the Froude's number and λ^* the non-dimensional relaxation time. Using the above transformation in equations (42) and (43) and removing the stars over u,v,w and λ, these equations become

$$(D^2-a^2)w\left[-P-\frac{a^2Q}{P}-R_d\left(1+\frac{P\upsilon'}{d^2}\right)\right]-\frac{a^2}{P}R_1=\frac{2\Omega iad^2}{\upsilon}dv$$

Since, $$-R_aDv=\frac{-4\pi^2a^4}{\upsilon^2}Dv$$

$$=\frac{-4\Omega^2a^4}{\upsilon^2}\cdot\frac{iaDv\times\upsilon}{2\Omega d^2}=\frac{-2ia\Omega d^2}{\upsilon}dv$$

then, $$(D^2-a^2)w\left[-P-\frac{a^2Q}{P}-R_d\left(1+\frac{P\upsilon'}{d^2}\right)\right]-\frac{a^2}{P}R_1=-R_aDv \qquad (44)$$

$$\frac{\rho\upsilon}{d^2}\left[-P-\frac{a^2Q}{P}-R_d\left(1+\frac{P\upsilon'}{d^2}\right)\right]v=\frac{2i}{k}\,\rho\Omega Dw$$

$$\left[-P-\frac{a^2Q}{P}-R_d\left(1+\frac{P\upsilon'}{d^2}\right)\right]v=Dw\,. \tag{45}$$

IV. Marginal State of the System

We shall first examine the nature of perturbation modes, whether stable, unstable or neutral modes. We can get some conditions by taking the integral approach. Now multiplying the equation (42) by w^*, the complex conjugate of w and integrating over the interval (0, 1), we have

$$-\left[P+R_d\left(\frac{\upsilon'P}{d^2}+1\right)+\frac{a^2Q}{P}\right]\int_0^1 w^*(D^2-a^2)wdz$$

$$-\frac{a^2R_1}{P}\int_0^1 w^*wdz=-R_a\int_0^1 w^*Dv\,dz\,. \tag{46}$$

We express $G=(D^2-a^2)w$ and evaluate the above integrals. We take these integrals one by one.

$$I_1=-\int_0^1 w^*(D^2-a^2)wdz$$

$$=\int_0^1 |Dw|^2+a^2|w|^2\,dz\,, \tag{47}$$

$$I_2=\int_0^1 w^*wdz=\int_0^1 |w|^2\,dz \tag{48}$$

and $$I_3=\int_0^1 w^*Dvdz=|w^*v|_0^1-\int_0^1 vDw^*dz$$

$$=-\int_0^1 v.Dw^*dz \tag{49}$$

Now to evaluate above integral, we take the complex conjugate of the equation (45), we thus have

$$-\left[P^*+R_d\left(\frac{\upsilon'P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right]v^*=Dw^*$$

Substituting the value of Dw^* in the integral (49), we have

$$\int_0^1 w^* Dv dz = \int_0^1 v\left[\left\{P^* + R_d\left(\frac{\upsilon'\rho^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}v^*\right]dz$$

$$=\left[\left\{P^* + R_d\left(\frac{\upsilon'\rho^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}\right]\int_0^1 |v|^2\, dz$$

$$=\left[P^* + R_d\left(\frac{\upsilon'\rho^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right]I_4 \qquad (50)$$

where, $I_4 = \int_0^1 |v|^2\, dz$

Substituting the value of integrals $\int_0^1 w^* Dv dz,\ I_1,\ I_2$ and I_3 from the equations (47) to (50) into the equation (46), we thus have

$$\left[P + R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right]I_1 - \frac{a^2R_1}{P}I_2$$

$$= -R_a\left[P^* + R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right]I_4$$

$$\left[\left(P+\frac{a^2Q}{P}\right)I_1 + \left(P^* + \frac{a^2Q}{P^*}\right)R_aI_4\right] + R_d\left(\frac{\upsilon' P}{d^2}+1\right)I_1$$

$$+ R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)R_aI_4 - \frac{a^2R_1P^*}{|P|^2}I_2 = 0$$

$$\left[\left(P+\frac{a^2QP^*}{|P|^2}\right)I_1 + \left(P^* + \frac{a^2QP}{|P^*|^2}\right)R_aI_4\right] + R_d\left(\frac{\upsilon' P}{d^2}+1\right)I_1$$

$$+ R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)R_aI_4 - \frac{a^2R_1P^*}{|P|^2}I_2 = 0. \qquad (51)$$

Now, we consider the marginal state of the system. We therefore take $P = P_r + iP_i$. Substituting this in the equation (51), we have

$$\left\{P_r + iP_i + \frac{a^2Q(P_r - iP_i)}{|P|^2}\right\}I_1 + \left\{P_r - iP_i + \frac{a^2Q(P_r + iP_i)}{|P|^2}\right\}R_aI_4 + R_d\left(\frac{\upsilon'(P_r + iP_i)}{d^2}+1\right)I_1$$

$$+ R_d\left(\frac{\upsilon'(P_r - iP_i)}{d^2} + 1\right)R_a I_4 - \frac{a^2 R_1 (P_r - iP_i)}{|P|^2} I_2 = 0 .$$

We have the real part

$$\left(P_r + \frac{a^2 Q}{|P|^2} P_r\right) I_1 + \left(P_r + \frac{a^2 Q}{|P|^2} P_r\right) R_a I_4 + R_d\left(\frac{\upsilon' P_r}{d^2} + 1\right) I_1$$

$$+ R_d\left(\frac{\upsilon' P_r}{d^2} + 1\right) R_a I_4 - \frac{a^2 R_1 P_r I_2}{|P|^2} = 0$$

$$\left\{P_r\left(1 + \frac{a^2 Q}{|P|^2}\right)(I_1 + R_a I_4)\right\} + \left\{R_d\left(\frac{\upsilon' P_r}{d^2} + 1\right)(I_1 + R_a I_4)\right\}$$

$$- \frac{a^2 R_1 P_r}{|P|^2} I_2 = 0 . \qquad (52)$$

In this equation, P_r must be negative, that is system is stable.

and

$$\left(P_i - \frac{a^2 Q}{|P|^2} P_i\right) I_1 + \left(-P_i + \frac{a^2 Q}{|P|^2} P_i\right) R_a I_4 + R_d\left(\frac{\upsilon' P}{d^2}\right) I_1$$

$$+ R_d\left(\frac{-\upsilon' P_i}{d^2}\right) R_a I_4 + \frac{a^2 R_1 P_i}{|P|^2} I_2 = 0$$

$$\left[P_i\left\{\left(1 - \frac{a^2 Q}{|P|^2}\right) I_1 + \left(1 - \frac{a^2 Q}{|P|^2}\right) R_a I_4 + \frac{a^2 R_1}{|P|^2} I_2\right\}\right] = 0 . \qquad (53)$$

In this equation, the expression within the curly bracket of the equation would be positive if

$$1 - \frac{a^2 Q}{|P|^2} > 0$$

$$\frac{a^2 Q}{|P|^2} < 1 \Rightarrow Q < \frac{|P|^2}{a^2} \qquad (54)$$

Therefore, for statically unstable density configuration, the modes would be non-oscillatory in character.

V. Sufficient Condition for Instability of the System

Next, we consider the non-oscillatory modes, whether stable or unstable. That is, we take P to be real. Then the dispersion relation (46) can be expressed as

$$\left\{P + R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right\}I_1 - \frac{a^2R_1}{P}I_2 = -R_a\left\{P^* + R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}I_4$$

$$\frac{a^2R_1}{P}I_2 = \left\{P + R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}(I_1+R_aI_4)\right\}$$

$$a^2R_1I_2 = \left[P\left\{P^2 + R_dP\left(\frac{\upsilon' P}{d^2}+1\right)+a^2Q(I_1+R_aI_4\right\}\right]$$

$$= \left[\left\{P^3 + R_d\frac{\upsilon' P^3}{d^2}+R_dP^2+Pa^2Q\right\}(I_1+R_aI_4)\right]$$

$$= P^3\left[(I_1+R_aI_4)+R_d\frac{\upsilon'}{d^2}(I_1+R_aI_4)\right]+P^2\left[R_d(I_1+R_aI_4)\right]$$

$$+P\,[a^2Q(I_1+R_aI_4)] - a^2R_1I_2 = 0$$

If $\quad a^2Q(I_1+R_aI_4) > a^2R_1I_2$

or $\quad Q > \dfrac{R_1I_2}{I_1+R_aI_4}$

then, if $R_1 < 0$, the given equation does not allow any positive value of P. Then, non-oscillatory modes, if exist are stable in nature.

If $R_1 > 0$, then product of roots is positive. In both cases, at least one root is positive. Therefore, non-oscillatory modes if exist are unstable under the condition $R_1 > 0$.

VI. Variational Principle

We shall now proceed to establish the variational principle in R_1 for the solution of this problem. For this multiplying the equation (49) by w^* and integrating the resulting over the interval $z=0$ and $z=1$. The equation (41) can then be expressed as

$$-\left[P + R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right]\int_0^1 w^*(D^2-a^2)w\,dz - \frac{a^2R_1}{P}\int_0^1 w^*w\,dz$$

$$= -R_a\int_0^1 w^*Dv\,dz$$

$$\left[P+R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2 Q}{P}\right]I_1-\frac{a^2 R_1}{P}I_2=-R_a\left[P^*+R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2 Q}{P^*}\right]I_4 \tag{52}$$

Taking the complex conjugate of the above equation (52) and adding into it, we get

$$\left[\left(P+P^*+R_d\left(\frac{\upsilon'}{d^2}(P+P^*)+1\right)+a^2 Q\left(\frac{1}{P}+\frac{1}{P^*}\right)\right)\right]I_1-a^2 R_1\left(\frac{1}{P}+\frac{1}{P^*}\right)I_2$$

$$=-R_a\left[(P+P^*)+R_d\left(\frac{\upsilon'}{d^2}(P+P^*)+1\right)+a^2 Q\left(\frac{1}{P}+\frac{1}{P^*}\right)\right]I_4$$

or $$\left[P_r+R_d\left(\frac{\upsilon' P_r}{d^2}+1\right)+\frac{a^2 Q P_r}{|P|^2}\right]I_1-\frac{a^2 R_1 P_r}{|P|^2}I_2$$

$$=-R_a\left[P_r+R_d\left(\frac{\upsilon'}{d^2}P_r+1\right)+\frac{a^2 Q P_r}{|P|^2}\right]I_4$$

or $$\left[P_r+R_d\left(\frac{\upsilon' P_r}{d^2}+1\right)+\frac{a^2 Q P_r}{|P|^2}\right](I_1+R_a I_4)-\frac{a^2 R_1 P_r}{|P|^2}I_2=0$$

$$\frac{a^2 R_1 P_r}{|P|^2}I_2=I$$

or $$R_1=\frac{|P|^2\, I}{a^2 P_r I_2}$$

or $$R_1=\frac{|P|^2}{a^2 P_r}\,\frac{I}{J} \tag{53}$$

where, $$I=\left[\left\{\left(P_r+R_d\left(\frac{\upsilon' P_r}{d^2}+1\right)+\frac{a^2 Q P_r}{|P|^2}\right)(I_1+R_a I_4)\right\}\right]$$

and $$J=I_2$$

Small variation in the solution of the problem gives variation in R_1, i.e., $\delta R_1=0$. Now, prove this principle

$$\delta R_1=\frac{|P|^2}{a^2 P_r J}\left(\delta I_1-\frac{a^2 P_r}{|P|^2}R_1\delta J\right)$$

$$\delta I = \left[P_r + R_d \left(\frac{\upsilon'}{d^2} P_r + 1 \right) + \frac{a^2 Q P_r}{|P|^2} \right] (\delta I_1 + R_a \delta I_4)$$

$$2\delta I = \left[\left\{ \left(P + R_d \left(\frac{\upsilon' P}{d^2} + 1 \right) + \frac{a^2 Q}{P} \right) + \left(P^* + R_d \left(\frac{\upsilon' P^*}{d^2} + 1 \right) + \frac{a^2 Q}{P^*} \right) \right\} \right.$$

$$\left. (\delta I_1 + R_a \delta I_4) \right]. \tag{54}$$

Taking the variation δI_1 and integrating this by parts using boundary condition

$$\delta I_1 = \int_0^1 [|Dw|^2 + a^2 |w| \ dz$$

$$= \int_0^1 \left[Dw D\delta w^* + a^2 w \delta w^* \right] dz + \int_0^1 \left[Dw^* D\delta w + a^2 w^* \delta w \right] dz$$

$$= -\left[\left(\int_0^1 (D^2 - a^2) w \right) \delta w^* dz - \int_0^1 (D^2 - a^2) \delta w \right] w^* dz \,, \tag{55}$$

$$\delta I_2 = \int_0^1 |w|^2 \ dz = \int_0^1 w \delta w^* dz + \int_0^1 w^* \delta w \, dz \tag{56}$$

and $$\delta I_4 = \int_0^1 |v|^2 dz = \int_0^1 v \delta v^* \ dz - \int_0^1 v^* \delta v \ dz \tag{57}$$

Substituting the values of δI_1, δI_2 and δI_4 into the equation (54), we get

$$2\delta I = \left\{ \left(P + R_d \left(\frac{\upsilon' P}{d^2} + 1 \right) + \frac{a^2 Q}{P} \right) + \left(P^* + R_d \left(\frac{\upsilon' P^*}{d^2} + 1 \right) + \frac{a^2 Q}{P^*} \right) \right\}$$

or $$\left\{ -\int_0^1 (D^2 - a^2) w \right\} \delta w^* dz - \left\{ \int_0^1 (D^2 - a^2) \delta w \right\} w^* dz$$

$$+ \left[R_a \int_0^1 v \delta v^* dz - \int_0^1 v^* \delta v \ dz \right]$$

or $$\left[\int_0^1 \left\{ P + R_d \left(\frac{\upsilon' P}{d^2} + 1 \right) + \frac{a^2 Q}{P} \right\} (D^2 - a^2) w \right] \delta w^* dz$$

$$-R_a \left[\int_0^1 - \left\{ P^* + R_d \left(\frac{\upsilon' P}{d^2} + 1 \right) + \frac{a^2 Q}{P^*} \right\} \delta v^* \right] v \, dz$$

$$+ \int_0^1 - \left[\left\{ P^* + R_d \left(\frac{\upsilon' P}{d^2} + 1 \right) + \frac{a^2 Q}{P^*} \right\} (D^2 - a^2) w^* \right] \delta w \, dz$$

$$-R_a\left[\int_0^1 -\left\{P+R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right\}\delta v\right]v^* dz$$

$$+\int_0^1 -\left[\left\{P^*+R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right\}(D^2-a^2)\delta w\right]w^* dz$$

$$-R_a\left[\int_0^1 -\left\{P^*+R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P}\right\}v^*\right]\delta v\, dz$$

$$+\int_0^1 -\left[\left\{P^*+R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}(D^2-a^2)\delta w^*\right]wdz$$

$$-R_a\int_0^1 -\left[\left\{P+R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right\}v\right]\delta v^*\, dz\,, \tag{58}$$

$$\delta J=\int_0^1 w\delta w^* dz+\int_0^1 w^*\delta w\, dz\,, \tag{59}$$

$$\delta R_1=\frac{|P|^2}{a^2P_rJ}\left(\delta I_1-\frac{a^2P_rR_1}{|P|^2}\delta J\right)$$

$$=\frac{|P|^2}{2a^2P_rJ}\left(2\delta I_1-\frac{2P_ra^2R_1}{|P|^2}\right)$$

$$=\frac{|P|^2}{2a^2\ P_r\ J}(\delta A_1+\delta A_1^*+\delta A_2+\delta A_2^*) \tag{60}$$

where

$$\delta A_1=\int_0^1\left[-\left\{P+R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right\}(D^2-a^2)w-\frac{a^2R_1}{w}w\right]\delta w^* dz$$

$$-R_a\int_0^1 -\left\{\left(P^*+R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right)\delta v^*\right\}v\, dz \tag{61}$$

$$\delta A_2=\left[-\left\{\int_0^1\left(P+R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right)(D^2-a^2)\delta w-\frac{a^2R_1}{P}\delta w\right\}w^* dz\right]$$

$$-R_a\int_0^1\left\{\left(P^*+R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right)v^*\right\}\delta v\, dz\,. \tag{62}$$

Let us first take up δA_1. For this taking the variation in δv and δw consistent equation (45), and taking the complex conjugate, we have

$$\left[-(P^* + R_d\left(\frac{P\upsilon'}{d^2}+1\right)+\frac{a^2Q}{P^*}\right]\delta v^* = D\delta w^*$$

$$\int_0^1 -\left[\left\{P^* + R_d\left(\frac{P\upsilon'}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}dv^*\right]vdz = \int_0^1 D\delta w^* vdz$$

or $$\int_0^1 D.\delta w^* dz = -\int_0^1 Dv.\delta w^* dz. \tag{63}$$

Using the above integral (63) in the equation (61), we get

$$\delta A_1 = \int_0^1\left[-\left\{P + R_d\left(\frac{\upsilon' P}{d^2}+1\right)+\frac{a^2Q}{P}\right\}(D^2-a^2)w - \frac{a^2R_1}{P}w\right.$$

$$\left. -R_a\int_0^1 Dv\right]\delta w^* dz. \tag{64}$$

Using the above variation (64) in δA_2 in the equation (62), we get

$$\delta A_2 = \int_0^1 -\left\{P + R_d\left(\frac{P\upsilon'}{d^2}+1\right)+\frac{a^2Q}{P}\right\}(D^2-a^2)dw - \frac{a^2R_1}{P}\delta w = -R_a D\delta v$$

$$\delta A_2 = -\int_0^1 R_a Ddv.w^* dz - R_a\int_0^1 -\left[\left\{P^* + R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}v^*\right]\delta v\, dz$$

$$= -R_a\left[\left\{\int_0^1\left(P^* + R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right)\right\}v^* - Dw^*\right]\delta v\, dz$$

From the equation (60), we have

$$\delta R_1 = \frac{|P|^2}{2a^2P_rJ}R_e[\delta A_1 + \delta A_2]$$

$$= \frac{|P|^2}{2a^2\, P_r\, J}R_e\left[\int_0^1 -\left\{P + R_d\left(\frac{P\upsilon'}{d^2}+1\right)+\frac{a^2Q}{P}\right\}(D^2-a^2)w\right.$$

$$\left.-\frac{a^2R_1}{P}\delta w + R_a Dv\right]\delta w^* dz - \left[\int_0^1\left\{P^* + R_d\left(\frac{\upsilon' P^*}{d^2}+1\right)+\frac{a^2Q}{P^*}\right\}v^* - Dw^*\right]\delta vdz.$$

Thus, for the functions U and W satisfying the equations (44) and (45), we see that $\delta R_1 = 0$. This proves the variational principle.

Table 1.1 : ***Unstable modes of maximum growth rate for*** $R_d = 5, Q = 5$ ***and*** $R_a = 100$***, 200 and 300 for different values of density stratification*** R_1 ***for free boundaries***

R_1			p	a	F
$R_a = 100$	$R_a = 200$	$R_a = 300$			
52.4873	53.7595	54.3966	0.00192	0.77	10
55.1956	57.3786	58.4731	0.00615	1.05	12
82.9136	93.009	98.129	0.15068	2.31	14
127.636	141.009	147.861	0.41551	2.96	16
208.118	220.894	227.498	0.74551	3.38	18
344.183	354.944	360.566	1.1114	3.6	20
548.768	557.222	561.665	1.49811	3.904	22
840.084	846.631	850.17	1.89702	4.1	24
1230.49	1235.83	1238.68	2.30257	4.26	26
1740.51	1738.57	1741	2.7117	4.39	28

Table 1.2 : ***Unstable modes of maximum growth rate for*** $R_d = 5, Q = 10$ ***and*** $R_a = 100$***, 200 and 300 for different values of density stratification*** R_1 ***for free boundaries***

R_1			p	a	F
$R_a = 100$	$R_a = 200$	$R_a = 300$			
107.767	109.483	110.346	0.00751	0.925	10
109.287	111.71	112.929	0.01219	0.99	12
138.428	133.135	133.477	0.011618	1.83	14
177.855	186.475	191.281	0.30051	2.3	16
241.103	251.22	256.475	0.54221	2.69	18
341.79	351.492	356.584	0.82521	2.95	20
495.194	503.636	508.108	1.13772	3.19	22
715.564	722.648	726.434	1.47161	3.38	24

Table 1.3 : ***Unstable modes of maximum growth rate for*** $R_d = 5, Q = 5$ ***and*** $R_a = 100$***, for different values of density stratification*** R_1 ***for free boundaries***

R_a	p	a	F
75.1348	0.0162	2.23	10
82.8872	0.0633	2.45	12
103.937	0.11063	3.15	14
156.529	0.31551	4.22	16
240.259	0.73114	4.71	18
403.221	1.23025	5.11	20
663.78	1.74551	5.47	22
995.147	2.20257	5.61	24
1517.53	2.74981	5.83	26
1981.22	3.1114	6.15	28
2623.97	3.53025	6.27	30

Table 1.4 : ***Unstable modes of maximum growth rate for*** $Q=15$, $R_a=100$, ***200 and 300 for different values of density stratification*** R_1 ***for free boundaries***

R_1			p	a	F
$R_a=100$	$R_a=200$	$R_a=300$			
160.115	161.69	162.496	0.00921	0.87	10
163.068	164.921	165.855	0.01342	0.97	12
164.462	166.813	167.999	0.01835	1.005	14
194.457	199.516	202.098	0.10681	1.62	16
234.417	241.495	245.137	0.25452	2.03	18
294.882	302.841	306.987	0.44871	2.37	20
384.373	392.77	397.204	0.68162	2.54	22

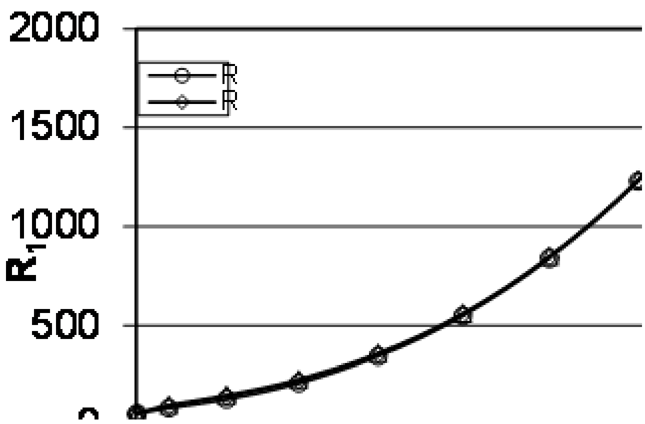

Fig. 1.11 ***Variation of density stratification*** R_1 ***and maximum growth rate*** P ***for different values of Rayleigh number*** R_a ***for free boundaries with magnetic force number*** $Q=5$.

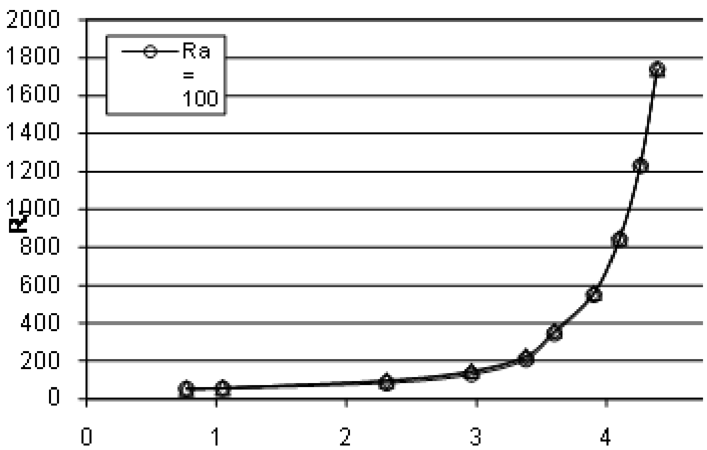

Fig. 1.12 ***Variation of density stratification*** R_1 ***and wave number*** a ***for different values of Rayleigh number*** R_a ***for free boundaries with magnetic force number*** $Q=5$.

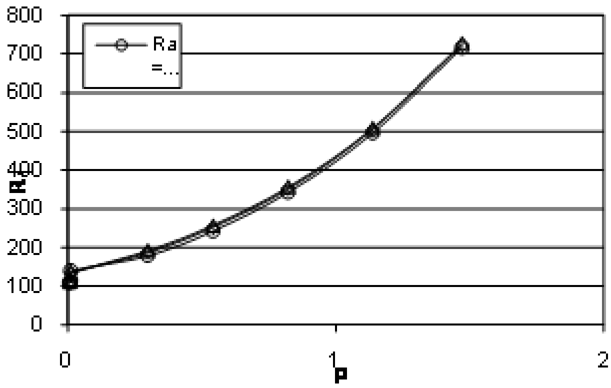

Fig. 1.13 ***Variation of density stratification R_1 and maximum growth rate P for different values of Rayleigh number R_a for free boundaries with magnetic force number $Q=10$.***

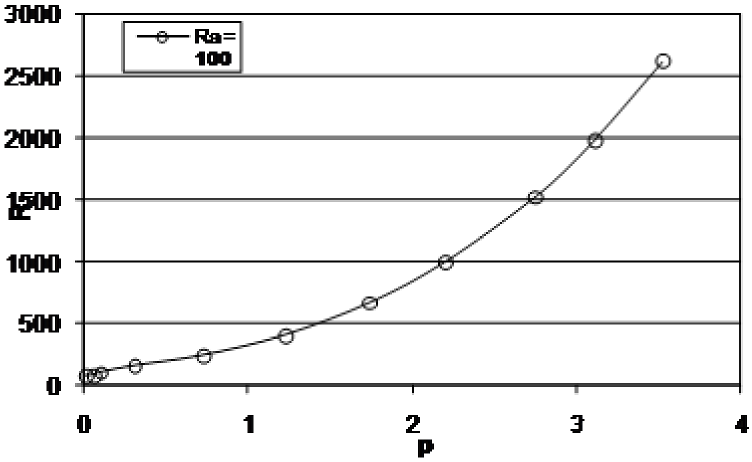

Fig. 1.14 ***Variation of density stratification R_1 and maximum growth rate P for Rayleigh number R_a = 100 for free boundaries with magnetic force number $Q=5$.***

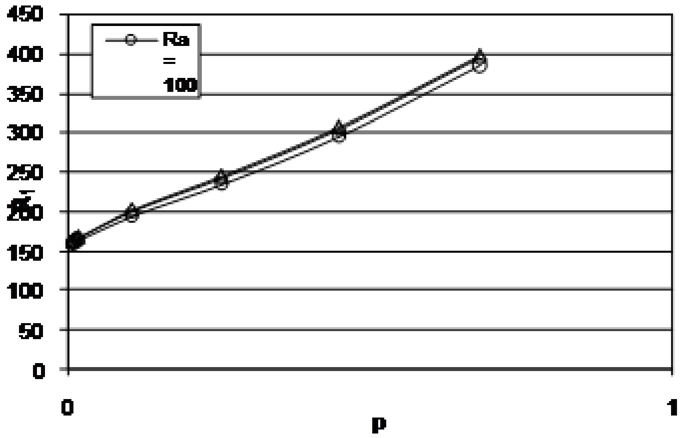

Fig. 1.15 ***Variation of density stratification R_1 and maximum growth rate P for different values of Rayleigh number R_a for free boundaries with magnetic force number $Q=15$.***

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