On α−Irresolute Topological Rings

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Abstract
We introduce the theory of α-irresolute topological rings. We present some examples of α-irresolute topological rings and discuss their relationship with topological rings. Some basic properties of α-irresolute topological rings are investigated. It is proved that every α-irresolute topological ring is α-regular.

Keywords - *α-open sets, α-irresolute mappings, α-irresolute topological rings.*

I. INTRODUCTION

If a set is endowed with algebraic and topological structures, it is always interesting to probe the relation between these two structures on the set. The aim of this paper is to display the notion of rings with topology. The resulting theory is called the α−irresolute topological ring. The theory of α−irresolute topological rings is structurally similar to the well-known theory of topological rings but we will exhibit in the sequel that α−irresolute topological rings are independent of topological rings. In fact, the theory of α−irresolute topological rings is stronger than the theory of topological rings.

A topological ring is a ring R with a topology τ on R such that the following mappings:

(1) $\phi : R \times R \rightarrow R$ defined by $\phi(x, y) = x - y$, and
(2) $\psi: R \times R \rightarrow R$ defined by $\psi(x, y) = xy$ for all $x, y \in R$

are continuous, where $R \times R$ carries the product topology.

Equivalently, (R, τ) is a topological ring if the following conditions are satisfied:

(1) For each $x, y \in R$ and each open neighbourhood W of $x - y$, there exist open neighbourhoods U and V of x and y, respectively, in R such that $U - V \subseteq W$, and
(2) For each $x, y \in R$ and each open neighbourhood W of xy in R, there exist open neighbourhoods U and V of x and y, respectively, in R such that $U.V \subseteq W$.

The study of topological rings is quite active since 1930s. Different researchers and mathematicians have been given great innovation in the field of topological rings. Kaplansky [3, 4, 5] and Warner [10] have greatly contributed in the field of topological rings and laid the foundation for the study of topological rings. [1] has done a classical work on topological rings. In 2018, Salih [11] introduced the irresolute topological rings which are independent of topological rings.

Topological rings have wide range of applications in the literature of mathematics like group theory, ring theory, number theory, etc, but they are also interesting in their own due to some unique and nice properties. The theory of topological rings has been extensively developed.

II. PRELIMINARIES

Throughout the present paper, for a subset S of a topological space X, the closure of A and the interior of A are denoted by Cl(A) and Int(A) respectively. By R, we mean the ring (R, +, .) without unity unless stated explicitly.

Definition 2.1. A subset A of a topological space X is called

(1) α−open [9] if $A \subseteq Int(Cl(Int(A)))$.
(2) semi-open [7] if $A \subseteq Cl(Int(A))$.

Clearly, every open set is α−open but the converse is not true, in general.

Example 2.1. Let X = R with its usual topology. Consider the set $A = F^c$ in X where F = {1/ n : n ∈ N} and N denotes the set of positive integers. Then A is α−open but not open. The complement of an α−open set is said to be α−closed set; or equivalently, a set A in a topological space X is α−closed if Cl(Int(Cl(A))) ⊆ A. The intersection of all α −closed sets in a topological space X containing a subset A ⊆ X is called the α−closure of A [8] and is denoted by αCl(A). It is shown in [8] that a subset A of X is α−closed if and only if A = αCl(A). Furthermore, it is proved in [8] that a point x ∈ αCl(A) if and only if A∩U/ = ∅ for each α−open set U in X containing x. The α−interior of a subset A ⊆ X is the union of all α−open sets in X that are contained in A and is denoted by αInt(A). A point x of X is called an α−interior point of a subset A ⊆ X if there exists an α−open set U in X containing x such that U ⊆ A. The set of all α−interior points of A is equal to αInt(A). The family of all α−open (resp. α−closed) sets in X is denoted by τ^α (resp. C^α). Njastand [9] showed that τ^α forms a topology on X. The collection of all α−open sets in X containing a point x of X is denoted by $N_x(X)$.

We also recall some definitions that will be helpful for the better understanding of our future study:

Definition 2.2. A mapping f : X → Y from a topological space X to a topological space Y is called:

(1) irresolute if for every semi-open set V in Y , $f^{-1}(V)$ is semi-open set in X.
(2) α-irresolute [8] if for every α−open set V in Y , $f^{-1}(V)$ is α−open set in X.

Definition 2.3. Let R be a ring and let τ be a topology on R such that the following mappings:

(1) $\phi_1 : R\times R \to R$ defined by $\phi_1(x,y) = x + y$
(2) $\phi_2 : R \to R$ defined by $\phi_2(x) = -x$ and
(3) $\phi_3 : R\times R \to R$ defined by $\phi_3(x,y) = xy$, for all x,y ∈ R

are continuous with respect to τ. Then the pair (R, τ) is called topological ring.

Definition 2.4. [11] Let R be a ring and let τ be a topology on R such that the following mappings:

(1) $\phi_1 : R\times R \to R$ defined by $\phi_1(x, y) = x + y$
(2) $\phi_2 : R \to R$ defined by $\phi_2(x) = -x$ and
(3) $\phi_3 : R\times R \to R$ defined by $\phi_3(x, y) = xy$, for all x,y ∈ R

are irresolute with respect to τ. Then the pair (R , τ) is called irresolute topological ring.

III. α−IRRESOLUTE TOPOLOGICAL RINGS

In this section, we define and present some examples of α−irresolute topological rings. We discuss about the relationship of α−irresolute topological rings with the topological rings.

Definition 3.1. Let R be a ring endowed with a topology τ such that the following conditions are satisfied:

(1) For each x, y ∈ R and each α−open set W in R containing x+y, there exist α−open sets U and V in R containing x and y respectively, such that U + V ⊆ W;
(2) For each x ∈ R and each α−open set V in R containing −x, there exists an α−open set U in R containing x such that −U ⊆ V and
(3) For each x, y ∈ R and each α−open set W in R containing xy, there exist α−open sets U and V in R containing x and y respectively, such that U.V ⊆ W

Then the pair (R, τ) is called an α−irresolute topological ring.

If we remind the theory of topological rings, we observe that the structure of α−irresolute topological rings and the structure of topological rings are ideologically same but conceptually the theory of α−irresolute topological rings is different from the theory of topological rings. In fact, the theory of α−irresolute topological rings is stronger than the theory of topological rings.

Example 3.1. Consider the ring R of real numbers with the discrete topology D. Then (R, D) is an α−irresolute topological ring as well as topological ring.

In fact, every ring with its discrete topology is an $\alpha-$irresolute topological ring as well as topological ring. We next give an example of a topological ring which is not an $\alpha-$irresolute topological ring.

Example 3.2. Consider the ring R of real numbers with the standard topology U on R. Then (R, U) is topological ring but in the sequel we will show that (R, U) is not an $\alpha-$irresolute topological ring.

It can be readily seen that every ring R with its trivial topology is an $\alpha-$irresolute topological ring. So, in further study we assume that the topology of an $\alpha-$irresolute topological ring is non-trivial. Presenting some basic properties of $\alpha-$irresolute topological rings.

Theorem 3.1. Let (R, τ) be an $\alpha-$irresolute topological ring. Then for $A \in \tau^{\alpha}$, the following are valid:

(1) $-A \in \tau^{\alpha}$,
(2) $x + A \in \tau^{\alpha}$, for each $x \in R$.

Proof. (1) Let y be any element of $-A$. Then $y = -x$ for some $x \in A$. By definition 3.1, there exists an $\alpha-$open set U in R containing y such that $-U \subseteq A$. This means that $U \subseteq -A$ and hence y is an $\alpha-$interior point of $-A$; that is $y \in \alpha Int(-A)$. Consequently, $-A \in \tau^{\alpha}$.

(2) Choose an element y in x+A. Then there exist $U, V \in \tau^{\alpha}$ such that $-x \in U$, $y \in V$ and $U + V \subseteq A$. In particular, $-x + V \subseteq A$. This means that $V \subseteq x + A \Rightarrow y \in \alpha Int\,(x + A) \Rightarrow \alpha Int(x + A) = x + A$. Therefore, $x + A \in \tau^{\alpha}$.

Corollary 3.1.1. Let A be any $\alpha-$open set in an $\alpha-$irresolute topological ring (R, τ). Then $B + A \in \tau^{\alpha}$, for any $B \subseteq R$.

Theorem 3.2. Let F be any $\alpha-$closed set in an $\alpha-$irresolute topological ring (R, τ). The following are valid:

(1) $-F \in C^{\alpha}$ and
(2) $x + F \in C^{\alpha}$ for each $x \in R$.

Proof. Petty.

Theorem 3.3. Let A be any subset of an $\alpha-$irresolute topological ring (R, τ). Then

(1) $\alpha Cl(-A) = -\alpha Cl(A)$, and
(2) $\alpha Cl(x + A) = x + \alpha Cl(A)$ for each $x \in R$.

Proof. (1) Let $x \in \alpha Cl(-A)$ be an arbitrary. Let $y = -x$ and suppose V is an $\alpha-$open set in R containing y. Then there exists $U \in \tau^{\alpha}$ such that $x \in U$ and $-U \subseteq V$. By assumption, $(-A)\cap U \neq \emptyset$. There is some $a \in (-A)\cap U$. Consequently, $-a \in A\cap V \Rightarrow A\cap V \neq \emptyset$. Therefore, $y \in \alpha Cl(A)$. That is, $x \in -\alpha Cl(A)$. For the converse, suppose that $y \in -\alpha Cl(A)$. Then $y = -x$ for some $x \in \alpha Cl(A)$. Let $V \in N_y(R)$. Then by definition 3.1, there exists $U \in N_x(R)$ with $-U \subseteq V$. Since $x \in \alpha Cl(A)$, $A\cap U \neq \emptyset$. So, there is $a \in A\cap U$. This gives that $-a \in (-A)\cap V$. Thus, $y \in \alpha Cl(-A)$. Thereby the assertion follows.

(2) Let $y \in \alpha Cl(x+A)$. Suppose $z = -x+y$ and W be an $\alpha-$open set in R containing z. From the definition of an $\alpha-$irresolute topological ring, we obtain $\alpha-$open sets U and V in R such that $-x \in U$, $y \in V$ and $U + V \subseteq W$. By assumption, $(x + A)\cap V \neq \emptyset$. So, there is $a \in (x + A)\cap V$. Now $-x+a \in A\cap(U+V\,) \subseteq A\cap W \Rightarrow A\cap W \neq \emptyset$ and hence $y \in x+\alpha Cl(A)$. Conversely, suppose that $z \in x+\alpha Cl(A)$. Then $z = x+y$ for some $y \in \alpha Cl(A)$. Choose any $W \in \tau^{\alpha}$ such that $z \in W$. Then there exist $U \in N_x(R)$ and $V \in N_y(R)$ such that $U + V \subseteq W$. Since $y \in \alpha Cl(A)$, $A\cap V \neq \emptyset$. Consequently, $(x + A)\cap W \neq \emptyset$. That is, $z \in \alpha Cl(x + A)$. Combining the facts from above, we conclude $\alpha Cl(x + A) = x + \alpha Cl(A)$. □

Theorem 3.4. For any subset A of an $\alpha-$irresolute topological ring (R, τ), the following are valid:

(1) $\alpha Int(-A) = -\alpha Int(A)$, and
(2) $\alpha Int(x + A) = x + \alpha Int(A)$ for each $x \in R$.

Proof. (1) Choose an arbitrary element y from $\alpha Int(-A)$. Then $y = -x$ for some $x \in A$. By definition 3.1, there exists $U \in N_x(R)$ such that $-U \subseteq \alpha Int(-A)$. This gives $-U \subseteq -A \Rightarrow U \subseteq A \Rightarrow x \in \alpha Int(A)$. Therefore, $y = -x \in$

$-\alpha Int(A)$. Thus, $\alpha Int(-A) \subseteq -\alpha Int(A)$. For the reverse inclusion, let $y \in -\alpha Int(A)$. Then $y = -x$ for some $x \in A$ because $-\alpha Int(A) \subseteq -A$. There exists $U \in N_x(R)$ such that $-U \subseteq -A$. By Theorem 3.1, $-U \in \tau^\alpha$. Consequently, $y = -x \in -U \subseteq \alpha Int(-A)$. Therefore, $-\alpha Int(A) \subseteq \alpha Int(-A)$. Hence the assertion follows.

(2) Let $z \in \alpha Int(x+A)$. Then $z = x+y$ for some $y \in A$. By definition 3.1, there exist α −open sets U and V in R containing x and y respectively, such that $U+V \subseteq \alpha Int(x+A)$. In particular, $x+V \subseteq \alpha Int(x+A) \subseteq x+A \Rightarrow x+V \subseteq x+\alpha Int(A) \Rightarrow z \in x+\alpha Int(A)$. This shows that $\alpha Int(x+A) \subseteq x+\alpha Int(A)$. For the converse, let $y \in x+\alpha Int(A)$. Then $-x + y \in \alpha Int(A) \Rightarrow$ there exist $U,V \in \tau^\alpha$ such that $-x \in U$, $y \in V$ and $U+V \subseteq \alpha Int(A)$. This gives $U + V \subseteq A$. In particular, $-x + V \subseteq A \Rightarrow V \subseteq x + A$. Since $V \in N_y(R)$, $y \in \alpha Int(x + A)$. Thus, $x + \alpha Int(A) \subseteq \alpha Int(x + A)$. Hence $\alpha Int(x + A) = x + \alpha Int(A)$. □

We say that (R, τ) is an α−irresolute topological ring with unity if (R, τ) is an α−irresolute topological ring and R is a ring with unity. In this case, we denote the set of all invertible elements in R by R^*.

Theorem 3.5. Let (R, τ) be an α−irresolute topological ring with unity. If $A \in \tau^\alpha$, then $rA, Ar, rAr \in \tau^\alpha$ for all $r \in R^*$.

Proof. Let x be an element of rA. We show that $x \in \alpha Int(rA)$. Since $x \in rA$ and $r \in R^*$, $r^{-1}x \in A$. Therefore, there exist $U \in N_r{-1}(R)$ and $V \in N_x(R)$ such that $U.V \subseteq A$. This results in $r^{-1}V \subseteq A \Rightarrow V \subseteq rA \Rightarrow x \in \alpha Int(rA)$. This implies $rA \subseteq \alpha Int(rA)$ and hence $\alpha Int(rA) = rA$. That is, $rA \in \tau^\alpha$.

By a mirroring style, we can show that $Ar, rAr \in \tau^\alpha$.

Theorem 3.6. Let (R, τ) be an α−irresolute topological ring with unity. Then for $A \subseteq R$, the following hold:

(1) $\alpha Cl(rA) = r\alpha Cl(A)$ for each $r \in R^*$.
(2) $\alpha Int(rA) = r\alpha Int(A)$ for each $r \in R^*$.

Proof. (1) Suppose $x \in \alpha Cl(rA)$ and let $y = r^{-1}x$. Let W be an α−open set in R containing y. Then we find α−open sets U containing r^{-1} and V containing x in R such that $U.V \subseteq W$. By assumption, $(rA) \cap V \neq \emptyset \Rightarrow$ there is $y \in (rA) \cap V$. This gives $r^{-1}y \in A\cap(U.V) \subseteq A\cap W \Rightarrow A\cap W \neq \emptyset \Rightarrow y \in \alpha Cl(A) \Rightarrow x \in r\alpha Cl(A)$. Therefore $\alpha Cl(rA) \subseteq r\alpha Cl(A)$. Conversely, let $y \in r\alpha Cl(A)$. Then $y = rx$ for some $x \in \alpha Cl(A)$. For any $W \in N_y(R)$, there exist $U \in N_r(R)$ and $V \in N_x(R)$ satisfying $U.V \subseteq W$. Also, there is $a \in A\cap V$. This yields $ra \in (rA) \cap(U.V) \subseteq (rA) \cap W \Rightarrow (rA)\cap W \neq \emptyset \Rightarrow y \in \alpha Cl(rA)$. Thus $r\alpha Cl(A) \subseteq \alpha Cl(rA)$. By above calculation, we get the assertion.

(2) Pick up an element y from $\alpha Int(rA)$. Then $y = rx$ for some $x \in A$. Also, there exist $U \in N_r(R)$ and $V \in N_x(R)$ such that $U.V \subseteq \alpha Int(rA)$. This means $U.V \subseteq rA$. In particular, $rV \subseteq rA$. Since V is α−open, $rV \subseteq r\alpha Int(A) \Rightarrow y \in r\alpha Int(A)$. This implies that $\alpha Int(rA) \subseteq r\alpha Int(A)$. For the converse, let x be any element of $r\alpha Int(A)$. Since $r \in R^*$, $r^{-1}x \in \alpha Int(A)$. Therefore, there exist $U, V \in \tau^\alpha$ with $r^{-1} \in U$, $x \in V$ and $U.V \subseteq \alpha Int(A)$. This yields $r^{-1}V \subseteq \alpha Int(A) \subseteq A \Rightarrow V \subseteq rA \Rightarrow x \in \alpha Int(rA)$. Hence $\alpha Int(rA) = r\alpha Int(A)$. □

On applying the same argument of Theorem 3.6, the following two results are immediate.

Theorem 3.7. Let (R, τ) be an α−irresolute topological ring with unity. Then for $A \subseteq R$, the following hold:

(1) $\alpha Cl(Ar) = \alpha Cl(A)r$ for each $r \in R^*$.
(2) $\alpha Int(Ar) = \alpha Int(A)r$ for each $r \in R^*$.

Theorem 3.8. Let (R, τ) be an α−irresolute topological ring with unity. Then for $A \subseteq R$, the following hold:

(1) $\alpha Cl(rAr) = r\alpha Cl(A)r$ for each $r \in R^*$.
(2) $\alpha Int(rAr) = r\alpha Int(A)r$ for each $r \in R^*$.

Our next aim is to show that every α−irresolute topological ring is α−regular. To this end, we start with the following definition:

Definition 3.2. Let (R, τ) be an α−irresolute topological ring. Then a subset $A \subseteq R$ is called symmetric if for each $x \in A$, $-x \in A$, i.e., $A = -A$.

If (R, τ) is an α−irresolute topological ring. Then for any $V \in N_0(R)$, there always exist C, D $\in N_0(R)$ such that $C + D \subseteq V$. Define $U = C \cap D \cap (-C) \cap (-D)$. Using Theorem 3.1 and the fact that finite intersection of α−open sets is α−open, it follows that $U \in N_0(R)$ with $U \subseteq V$. Putting all this together, we conclude:

Theorem 3.9. Let (R, τ) be an α−irresolute topological ring. Then for every $V \in N_0(R)$, there exists a symmetric $U \in N_0(R)$ such that $U \subseteq V$.

Corollary 3.9.1. Let (R, τ) be an α−irresolute topological ring. Then for every $V \in N_x(R)$, there exists a symmetric $U \in N_0(R)$ such that $x + U + U \subseteq V$.

Definition 3.3. A topological space X is called α-regular [6] if for each α-closed set F in X and each element x in X which does not belong to F, there exist disjoint α-open sets U and V in X such that $x \in U$ and $F \subseteq V$.

Theorem 3.10. Every α−irresolute topological ring (R, τ) is α−regular.

Proof. Let F be any α−closed set in R and x be an element in R which does not belong to F. Then $F^c \in N_x(R)$. By corollary 3.9.1, there exists a symmetric $U \in N_0(R)$ such that $x + U + U \subseteq F^c$. This means that $(x+U+U) \cap F = \emptyset$. This yields $(x+U) \cap (F+U) = \emptyset$. For, if $y \in (x + U) \cap (F + U)$. Then $x + u_1 = f + u_2$, for some $u_1, u_2 \in U$ and $f \in F$. This implies $f = x + u_1 - u_2 \in (x + U - U) = (x + U + U) \subseteq F^c$, the impossible. By Theorem 3.1 and corollary 3.1.1, it follows that $x + U, F + U \in \tau^\alpha$. Also, since $x \in x + U$ and $F \subseteq F + U$, (R, τ) is α−regular. □

Remark 3.1. Consider the ring R of real numbers. Let U be the standard topology on R. It is shown in [6] that (R, U) is not α−regular and hence by Theorem 3.10, it follows that (R, U) is not α−irresolute topological ring.

Theorem 3.11. Let (R, τ) be an α−irresolute topological ring with unity. Then the mappings:

(1) $\phi_a : R \to R$ defined by $\phi_a(x) = a + x$
(2) $\psi_b : R \to R$ defined by $\psi_b(x) = bx$ for $x \in R$ ($a \in R$ and $b \in R*$ are fixed)

are α−continuous.

Proof. The proof of part (1) is a direct consequence of Theorem 3.1 whereas part (2) of this theorem follows from Theorem 3.5.

IV .CONCLUSION

The innovation of α−irresolute topological rings is given in this paper. Several basic properties of α−irresolute topological rings are elaborated. Basically, the notion of α−irresolute topological rings is more stronger than the well-known notion of topological rings.

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