An Approach to Van Der Waerden's Theorem using Topological Dynamics

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Abstract

The motivation involved in doing this paper is understanding Ramsey theory in a better way.In this paper a detailed study of topological dynamics in Ramsey theory is made,which is then used to make an appropriate approach to Van der waerdan's theorem.We will start by using some basics in topological dynamics.

Keywords: *Vanderwaeden theorem, Topological space, Ramsey theory*

I.INTRODUCTION

If X is a set then a topology on X is a subset of the power set of X, $\mathcal{T} \subset \mathcal{P}$ (X), that is closed under finite intersections, closed under arbitrary unions, and that contains both X and $\emptyset$. The members of $\mathcal{T}$ are called open sets and their complements are called closed sets. The collection of closed sets is therefore closed under finite unions and arbitrary intersections. The closure E of a set $E \subset X$ is the intersection of all closed sets containing E. the pair (X, $\mathcal{T}$) (or just X, it $\mathcal{T}$ is understood) is called a topological space. If $E \subset X$ then we can create a topology S on E by taking S to consist of all intersections of E with a member of $\mathcal{T}$. We call this the induced topology on E [page 5,[4]]

Axiom(1.1)

The confinite topology

Let X be any set and put $\mathcal{T} = \{U \subset X : \left|U^c\right| < \infty\}$ is a topological space.

A metric on a set X is a function $\rho: X \times X \rightarrow [0, \infty$ with $\rho\,(x,y) = 0$ if and only if $x = y$ and with $\rho\,(x,z) \leq \rho\,(x,y) + \rho\,(y,z) for\ all\ x,y,z \in X$. The pair X, ρ is called a metric space.

Example(1.1).

Let R be the set of real numbers and put $\rho\,(x,y) = |x - y|, x,y \in R$. Than ρ is a metric.

A metric space has a natural topology consisting of those sets U having the property that for every $x \in$ U, there exists $\epsilon > 0$ such that $B\epsilon(x) = \{y \in X : \rho\,(x,y) < \epsilon\} \subset U$. For any topological space (X, $\mathcal{T}$), if there is a metric on X such that the topology induced by ρin this fashion is just $\mathcal{T}$, then the space (X, $\mathcal{T}$) is called metrizable.)

A sequence $(x_n)\,{}^{\infty}_{n}= 1$ in a topological space is said to converge to $x \in$ X, and we write either $lim_{n\to\infty\, x_n \to\infty\, x_n \to x\ as\ n\to\infty,}$ if for every open set U containing x (such sets are called neighborhoods of x), $x_n \in U$ but for finitely many n.

A necessary and sufficient condition for a metric space X to be compact is that it be totally bounded and complete. X is totally bounded if for every $\epsilon > 0$ there exits a finite set $\{x_1 \ldots., x_k\} \subset X$ having the property that for every $y \in$ X there exists $i, 1 \leq i \leq k$, with $\rho\,(x,y) < \epsilon$. (Such a set $\{x_1 \ldots., x_k\}$is called an $\epsilon - net.$) X is complete if every Cauchy sequence in X converges ($(x_n)\,{}^{\infty}_{n}= 1$is Cauchy if for every $\epsilon > 0$ there exists $k \in$ N such that $\rho\,(x_n, x_m) < \epsilon$ whenever $n, m > k$).

If X is a set and $\mathcal{G}$ is a family of its subsets, than the topology $\mathcal{T}$ generated by $\mathcal{G}$ consists of all unions of families of sets each of whose elements are finite intersections of the members of $\mathcal{G}$. We say that $\mathcal{G}$ is a sub basis for $\mathcal{T}$.

Definition:A subset $U \subset X$ is called residual if it contains a countable intersection of dense open sets. Equivalenty, if $U^c$is a countable union of nowhere dense sets (a set is nowhere dense if its closure contains non-empty open sets). In complete metric spaces, residual sets are non-empty. (this is the Baire category theorem.) indeed, in some sense, they are quite large. The complement of a residual set is called a set of first category.

A relation on a set Ƶ is a subset of Ƶ ×Ƶ with the properties:

(a) If $(x, y) \in \mathcal{E}$ *and* $(y, x) \in \mathcal{E}$ *then* $x = y$,
(b) If $(x, y) \in \mathcal{E}$ *and* $(y, z) \in \mathcal{E}$ *then* $(x, z) \in Z$, and
(c) $(x, x) \in \mathcal{E}$ *for all* $x \in$ Ƶ.

Will be called a partial order on Ƶ. If E is a relation then we may write $x \prec y$ if and only if $(x, y) \in \mathcal{E}$, and say simply that "≺" is a partial order on Ƶ.

Remark 1:

An application of Zorn's Lemma.

Let X be a compact topological space. We claim that there exists a minimal non-empty closed set B in X. (this is not obvious. There are topological spaces for which singletons are not closed sets.) Let Ƶ be the family of non-empty closed subsets of X. Ƶ is partially ordered by ⊂. Let $\mathcal{T}$ be a totally ordered subset of Ƶ. Putting $C = \bigcap_{V \in \mathcal{T}}$ V, C is non-empty by the finite intersection property. (The intersection of any finite subset of $\mathcal{T}$ is equal to the least element of that subset and hence non-empty.) Moreover C ⊂ V for every V ∈$\mathcal{T}$, so the hypotheses of Zorn's Lemma are met and there exists a non-empty, minimal closed subset of X relative to ⊂.[2]

II. INTREPETATION OF VAN DER WAERDEN'S THEOREM

A fundamental result of Ramsey theory is van der Waerden's theorem ([7]); see also [6], which has many equivalent formulations. The most basic is:

(I)vdW1. Let $k, r \in$ N. if N = $\bigcup_{i=1}^{r} C_i$ then some C_i contains an arithmetic progression of length k.

An another formulation,

(II)vdW2. Let $k, r \in$ N. There exists $N = N\,(k, r) \in$ N with the property that for any partition {1,2,…,N} = $\bigcup_{i=1}^{r} C_i$ some C_i contains an arithmetic progression of length k.

A purely combinatorial formulation of van der Waerden's theorem is the following form,

if F⊂ N, then an affine image of F is a set of the form $a + bF = \{a + bf : f \in F\}$, where $a, b \in G$ *with* $b \neq 0$.

(III)vdW3. For any finite partition of N, one of the cells contains affine images of every finite set.

Some terminology of partitions are as follows: the sets of a partition (the cells, that is) are often termed colors, and by color of a point we mean the cell of which it belongs. Hence, a partition into two cells is also called a 2-coloring. A configuration (such as an arithmetic progression) which is a contained in a single cell of the partition is called monochromatic. Hence we can say

> Ramsey theory is a collection of results which, given a finite coloring of some structure, guarantee the existence of certain monochromatic configurations or substructures.
>
> Now we will work on monochromatic configurations, avoiding the matter of monochromatic substructures to the next. Throughout, we use the methods of topological dynamics;i.e, we consider topological spaces and their contains self-maps. This approach to Ramsey theory was pioneered by Furstenberg and Weiss [5]. For this, we need to formulate results on this setting which are equivalent to (or at least imply) the corresponding Ramsey-theoretic theorems under consideration.

Let us consider this another example

Example 2.1 :vdW4. ([5]).

Let $k \in$ N and $\epsilon > 0$. For any compact metric space X and continuous map T : X → X there exist $x \in X$ and $n \in$ N such that $\rho(x, T^{in} x) > \epsilon, 1 \leq i < k$.

Proof:

The label vdW4 we have given this theorem suggests that it is another formulation of van der Waerden's theorem. We will now show that vdW4 implies vdW1.

Let $k, r \in$ N and suppose that N $= \cup_{i=1}^{r} C_1$. In order to apply vdW4 we need to use our partition to construct a topological space X and a continuous self-map T of X. Let

$$\Omega = \{1, \dots r\}^{\mathrm{N}} = \{\gamma : \mathrm{N} \rightarrow \{1, \dots r\}\}$$

Be the set of all sequence taking values in $\{1, \dots r\}$. For $\gamma \in \Omega$, define $\mathcal{T}\gamma \in \Omega$ by $T\gamma(n) = \gamma(n+1)$.T is called the shift on Ω.

For this we use the following corollaries

(a) Ω is a compact metric space with metric $\rho(\gamma, \xi) = \frac{1}{\min\{k : \gamma(k) \neq \xi(k)\}}$
(b) $T : \Omega \rightarrow \Omega$ is continuous
(c) $T^n\gamma(t) = \gamma(n+t)$for $n, t \in$ N.
(d) $\rho(\gamma, \xi) < \frac{1}{n}$ if and only if $\gamma(j) = \xi(j), 1 \leq j \leq n$.

The partition N $= \cup_{i=1}^{r} C_1$. May be used to define a point $a \in \Omega$ by $a(n) = i$ if and only if $n \in C_1$. Let $X = \overline{\{T^m a : m \in \mathrm{N}\}}$.

The pair (X,T) is an example of a dynamical system. (Actually, (Ω, T) is a dynamical system as well, but, having nothing whatsoever to do with our given partition. We are ready to apply vdW4 to the system (X, T).

By vdW4 there exists $x \in$ N such that $\rho(x, T^{in}x) < 1, 1 \leq i < k$. In particular, $x(1) = T^n x(1) = \cdots = T^{(k-1)n} x(1)$, or

$$x(1) = x(n+1) = x(2n+1) = \cdots = x\big((k-1)n+1\big).$$

But $x \in X = \overline{\{T^m a : m \in \mathrm{N}\}}$, meeting that for some $m \in$ N we have $\rho(T^{in} a, x) < \frac{1}{(k-1)n+1}$. In particular, if we let $i = x(1)$ this gives

$$i = T^m a(1) = T^m a(n+1) = T^m a(2n+1) = \cdots + T^m a(k-1)n+1),$$

Or

$$i = a(m+1) = a(m+1+2n) = a(m+1+2n) = \cdots = a(m+1+(k-1)n).$$

In other words, the k-term arithmetic progression

$$\{(m+1), (m+1)+n, (m+1)+2n, \dots, (m+1)+(k-1)n\}$$

is contained in C_i.

III. MULTIDIMENSIONAL VAN DER WAERDEN

Van der Waerden's theorem, which deals with colorings of N (or Z), admits a natural N^k (or N^k) generalization. This multidimensional van der Waerden theorem due to Gallai (also called Grunwald in the literature), has as van der Waerden's theorem does many equivalent formulations. First we give an affine form. Note for case $k = 1$ is just vdW3.

MvdW3. Let $k \in$ N. for any finite partition of N^k one of the cells contains affine images of every finite set.

Here now is a topological dynamics version of the theorem. It is this version whose proof we will sketch at the end of the section.

MvdW4. ([5].)

Let $k \in$ N and $\epsilon > 0$. If X is a compact metric space and $T_1, \dots. T_k$ are commuting homeomorphism of X then there exists $x \in$ X and $n \in Z, n \neq 0,$ *such that* $\rho(x, T_i^n x) < \epsilon, 1 \leq i \leq k$.

Proof:

In the section we showed how a topological recurrence result gave rise to a chromatic result and left reverse implication as an exercise. Here we will do the opposite showing that MvdW3 gives MvdW4 and proving the converse.

Suppose than that $k \in$ N, $\epsilon > 0, X$ is a compact metric space and $T_1, \dots . T_k$ are commuting homeomorphisms of X. Let $U_1, \dots . U_r$ be a covering of X by parwise disjoint sets of less than ϵ diameter. Let $y \in X$ and determine a partition $Z^k = \cup_{i=1} C_i$by the rule $(n_1, \dots . n_k) \in C_i$if and only if $T_1^{n1} \dots T_k^{mk} y \in U_i$. According to MvdW3, one of the cells C_i contains an affine image of the set

$$F = \{(0, \dots ,0), (1,0, \dots ,0), \dots ,0), \dots , (0, \dots ,0,1)\}. \quad (1.1)$$

That is, there exists $(n_1, \dots . n_k) \in \mathbf{Z}^k \ and \ n \in \mathbf{Z}, n \neq 0$, such that

$$(n_1, \dots . n_k) + F = \{(n_1, \dots . n_k), (n_1 + n, n_2, \dots . , n_k),$$

$$(n_1 + n_2 + n, n_3, \dots n_k), \dots , (n_1, \dots . n_{k-1}, n_k + n)\} \subset C_i.$$

Letting $x = T_1^{n1} \dots T_k^{mk} y$, we then have $\{x, T_1^{n1}x, \dots T_k^{n}x\} \subset U_i$.Since U_i. is of diameter less than ϵ, hence proving MvdW3 implies MvdW4.

IV. CONCLUSION

Thus in this paper a proper use of topological dynamics is used to characterize the combinatorial problems,which is then used to apply to a class of sets and made the necessary and sufficient impact on van der waerdan's theorem.

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