$b^{\#}$Closed Sets in Intuitionistic Fuzzy Topological Spaces

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Abstract:
In this paper, we have introduced a new class of closed sets namely $b^{\#}$ closed sets in intuitionistic fuzzy topological spaces and discussed some of their properties. Also we have studied the relation between the new class of set with the other existing closed sets.

***Keywords*: Intuitionistic fuzzy topology, intuitionistic fuzzy b closed sets, intuitionistic fuzzy $b^{\#}$closed sets.**

I. INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh [9]. After that Atanassov [1] introduced the notion of intuitionistic fuzzy sets and Coker [2] introduced the concept of intuitionistic fuzzy topological spaces and produced many interesting results and theorems. The concept of fuzzy $b^{\#}$ closed sets was introduced by Indhumathi and Jayanthi [6].In this paper, we have introduced the notion of intuitionistic fuzzy $b^{\#}$closed sets and investigated some of their properties and obtain some interesting characterizations.

II. PRELIMINARIES

Definition 2.1: [1]An **intuitionistic fuzzy set** (IFS for short) A is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$where the functions $\mu_A: X \longrightarrow [0,1]$ and $\nu_A : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X. An intuitionistic fuzzy set A in X is simply denoted by $A= \langle x, \mu_A, \nu_A\rangle$instead of denoting $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle: x \in X\}$.

Definition 2.2: [1] Let A and B be two IFSs of the form
$A=\{\langle x, \mu_A(x), \nu_A(x)\rangle: x \in X\}$ and
$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle: x \in X\}$.
Then,
(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,
(b) $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$,
(c) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle: x \in X\}$,
(d) $A\cup B= \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle: x \in X\}$,
(e) $A\cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \mu_A(x) \vee \nu_B(x)\rangle: x \in X\}$.
The intuitionistic fuzzy sets $0_{\sim} = \langle x, 0, 1\rangle$ and $1_{\sim} = \langle x, 1, 0\rangle$ are respectively the empty set and the whole set of X.

Definition 2.3: [2] An **intuitionistic fuzzy topology** (IFT in short) on X is a family τ of IFSs in X satisfying the following axioms:

(i) $0_{\sim}, 1_{\sim} \in \tau$,
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,
(iii) $\cup G_i \in \tau$ for any family $\{G_i: i \in I\} \subseteq \tau$.

In this case the pair (X, τ) is called the **intuitionistic fuzzy topological space** (IFTS in short) and any IFS in $\tau$is known as an **intuitionistic fuzzy open set** (IFOS in short) in X. The complement A^c of an IFOS A in an IFTS (X, τ) is called an **intuitionistic fuzzy closed set** (IFCS in short) in X.

Definition 2.4: [2] Let (X, τ) be an IFTS and $A= \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then the **intuitionistic fuzzy interior** and **intuitionistic fuzzy closure** are defined by

$$int(A) = \cup\{ G/ G \text{ is an IFOS in X and } G \subseteq A \},$$

$$cl(A) = \cap \{K/ K \text{ is an IFCS in } X \text{ and } A \subseteq K\}.$$

Definition 2.5: [4] An intuitionistic fuzzy set (IFS for short) $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in an IFTS (X, τ) is said to be an

1. **intuitionistic fuzzy semi open set** (IFSOS) if $A \subseteq cl(int(A))$, **intuitionistic fuzzy semi closed set** (IFSCS) if $int(cl(A)) \subseteq A$
2. **intuitionistic fuzzy pre open set** (IFPOS) if $A \subseteq int(cl(A))$, **intuitionistic fuzzy pre closed set** (IFPCS) if $cl(int(A)) \subseteq A$
3. **intuitionistic fuzzy α open set** (IFαOS) if $A \subseteq int(cl(int(A)))$, **intuitionistic fuzzy α closed set** (IFαCS) if $cl(int(cl(A))) \subseteq A$
4. **intuitionistic fuzzy β open sets (IFβOS)** if $A \subseteq cl(int(cl(A)))$, **intuitionistic fuzzy β closed set** (IFβCS) if $int(cl(int(A))) \subseteq A$
5. **intuitionistic fuzzy regular open set** (IFROS) if $A = int(cl(A))$,**intuitionistic fuzzy regular closed set** (IFRCS) if $A = cl(int(A))$.

Definition 2.6: [5] An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in an IFTS is said to be an

1. **intuitionistic fuzzy b open set** (IFbOS) if $A \subseteq int(cl(A)) \cup cl(int(A))$,
2. **intuitionistic fuzzy b closed set** (IFbCS) if $cl(int(A)) \cap int(cl(A)) \subseteq A$.

Definition 2.7: [7] An IFS A in (X, τ) is called **intuitionistic fuzzy nowhere dense** set if there exists no IFOS U such that $U \subseteq cl(A)$. That is $int(cl(A)) = 0_\sim$.

Remark 2.8: [7] Let A be an IFS in X. If A is an **intuitionistic fuzzy nowhere dense** set in X, then $int(A) = 0_\sim$.

Definition 2.9: [3] An intuitionistic fuzzy point (IFP in short), written as $p_{(\alpha,\beta)}$, is defined to be an IFS of X given by

$$p_{(\alpha,\beta)}(x) = \begin{cases} (\alpha, \beta) & \text{if } x = p, \\ (0, 1) & \text{otherwise.} \end{cases}$$

An intuitionistic fuzzy point $p_{(\alpha,\beta)}$ is said to belong to a set A if $\alpha \leq \mu_A$ and $\beta \geq \nu_A$.

Definition 2.10: [8] Two IFSs are said to be q-coincident($A\, _q\, B$ in short) if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$.

Definition 2.11: [8] Two IFSs are said to be not q-coincident ($A\, _{\bar{q}}\, B$ in short) if and only if $A \subseteq B^c$.

III. INTUITIONISTIC FUZZY B#CLOSED SETS

In this section we have introduced intuitionistic fuzzy $b^\#$closed sets and studied some of their properties.

Definition 3.1: An IFS A in an IFTS (X, τ) is said to be an intuitionistic fuzzy $b^\#$closed set (IF$b^\#$CS for short) if $int(cl(A)) \cap cl(int(A)) = A$.

The complement A^c of an IF$b^\#$CS A in an IFTS (X, τ) is called an intuitionistic fuzzy $b^\#$ open set (IF$b^\#$OS for short) in X.

The family of all IF$b^\#$CSs of an IFTS (X,τ) is denoted by IF$b^\#$C(X).

Example 3.2: Let $X = \{a, b\}$ and $\tau = \{0_\sim, G, 1_\sim\}$ where $G = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.6_b)\rangle$ is an IFT on X.

Let $A = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.6_b)\rangle$. Now $int(cl(A)) \cap cl(int(A)) = G \cap G^c = G = A$. Then A is an IF$b^\#$CS in X.

Remark 3.3: Every IFCS and every IF$b^\#$CS are independent to each other in general in (X,τ).

Example 3.4: In Example 3.2, A is an IF$b^\#$CS but not an IFCS in X as $cl(A) = G^c \neq A$.

Example 3.5: In Example 3.2, $A = \langle x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)\rangle$ is an IFCS as $cl(A) = G^c = A$ but not an IF$b^\#$CS in X as $int(cl(A)) \cap cl(int(A)) = G \neq A$.

Theorem 3.6: Every IF$b^\#$CS is an IFbCS in (X, τ) but not conversely in general in (X,τ).

Proof: Let A be an IF$b^\#$CS in X, then $int(cl(A)) \cap cl(int(A)) = A$. Now as $A \subseteq A$, $int(cl(A)) \cap cl(int(A)) \subseteq A$. Therefore A is an IFbCS in (X, τ).

Example 3.7: In Example 3.2, Let $A = \langle x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)\rangle$. Then $int(cl(A)) \cap cl(int(A)) = G \subseteq A$. Therefore A is an IFbCS but not an IF$b^\#$CS in X as $int(cl(A)) \cap cl(int(A)) \neq A$.

Remark 3.8: Every IFRCS and every IF$b^\#$CS are independent to each other in general in (X, τ).

Example 3.9: In Example 3.2, $A = \langle x, (0.4_a, 0.4_b), (0.4_a, 0.6_b)\rangle$ is an IF$b^\#$CS in X but not an IFRCS in X as $cl(int(A)) = G^c \neq A$.

Example 3.10: In Example 3.2, A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)⟩ is an IFRCS in X as cl(int(A)) = G^c = A but not an IFb$^{\#}$CS in X as int(cl(A)) ∩ cl(int(A)) ≠ A.

Remark 3.11: Every IFPCS and every IFb$^{\#}$CS are independent to each other in general in (X,τ).

Example 3.12: In Example 3.2, A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)⟩ is an IFPCS in X as cl(int(A)) = $G^c \subseteq$ A but not an IFb$^{\#}$CS in X as int(cl(A)) ∩ cl(int(A)) ≠ A.

Example 3.13: In Example 3.2, A = ⟨x, (0.4_a, 0.4_b), (0.4_a, 0.6_b)⟩ is an IFb$^{\#}$CS in X but not an IFPCS in X as cl(int(A)) = $G^c \nsubseteq$ A.

Theorem 3.14: Every IFb$^{\#}$CS is an IFSCS in (X, τ) but not conversely in general in (X,τ).

Proof: Let A be an IFb$^{\#}$CS in X. Then int(cl(A)) ∩ cl(int(A)) = A. Now as int(cl(A)) = int(cl(int(cl(A) ∩ cl(int(A)))) ⊆ int(cl(cl(A) ∩ cl(int(A)))) ⊆ int(cl(A)) ∩ cl(int(A))) = A. Hence A is an IFSCS in (X, τ).

Example 3.15: In Example 3.2, A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)⟩ is an IFSCS in X as int(cl(A)) = G ⊆ A but not an IFb$^{\#}$CS in X as int(cl(A)) ∩ cl(int(A)) = G ≠ A.

Remark 3.16: Every IFαCS and every IFb$^{\#}$CS are independent to each other in general in (X, τ).

Example 3.17: In Example 3.2, A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)⟩ is an IFαCS in X as cl(int(cl(A))) = $G^c \subseteq$ A but not an IFb$^{\#}$CS in X as int(cl(A)) ∩ cl(int(A)) = G ≠ A.

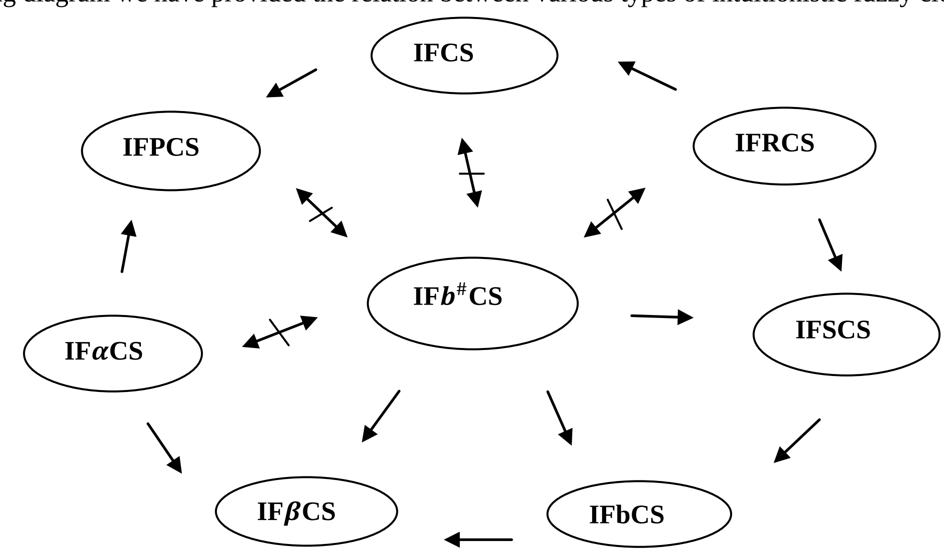
Example 3.18: In Example 3.2, A = ⟨x, (0.4_a, 0.4_b), (0.4_a, 0.6_b)⟩ is an IFb$^{\#}$CS in X as int(cl(A)) ∩ cl(int(A)) = G = A but not an IFαCS in X as cl(int(cl(A))) = $G^c \nsubseteq$ A.

Remark 3.19: Every IFb$^{\#}$CS is an IFβCS in (X, τ) but not conversely in general in (X,τ).

Proof: Let A be an IFb$^{\#}$CS in X. Then int(cl(A)) ∩ cl(int(A)) = A. Now int(cl(int(A))) = int(cl(int(A))) ∩ cl(int(A)) ⊆ int(cl(A)) ∩ cl(int(A)) = A. Therefore we have int(cl(int(A))) ⊆ A. Hence A is an IFβCS in (X, τ).

Example 3.20: In Example 3.2, A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)⟩ is an IFβCS in X as int(cl(int(A))) = G ⊆ G^cbut not an IFb$^{\#}$CS as int(cl(A)) ∩ cl(int(A)) ≠ A.

In the following diagram we have provided the relation between various types of intuitionistic fuzzy closed sets.



Theorem 3.21: If A is both an IFROS and an IFRCS then A is an IFb$^{\#}$CS in (X, τ).

Proof: Let A be both IFRO and IFRC in (X, τ). Then int(cl(A)) ∩ cl(int(A)) = A ∩ A = A. This implies A is an IFb$^{\#}$CS in (X, τ).

Theorem 3.22: If A is both an IFOS and an IFCS then A is an IFb$^{\#}$CS in (X, τ).

Proof: Let A be both an IFOS and an IFCS in (X, τ). Then int(cl(A)) ∩ cl(int(A)) = int(A) ∩ cl(A) = int(A) = A. Therefore A is an IFb$^{\#}$CS in (X, τ).

Theorem 3.23: If A is both an IFb$^{\#}$CS and an IFCS in (X, τ) then A is an IFOS in (X, τ).

Proof: If A is both an IFb$^{\#}$CS and an IFCS in (X,τ). Then A = int(cl(A)) ∩ cl(int(A)). Now A = int(cl(A)) ∩ cl(int(A)) = int(A) ∩ cl(int(A)) = int(A). Therefore A is an IFOS in (X, τ).

Theorem 3.24: For an IFb$^{\#}$CS A in an IFTS (X, τ), the following conditions hold:

(i) If A is an IFROS then scl(A) is an IFb$^{\#}$CS,

(ii) If A is an IFRCS then sint(A) is an IFb$^{\#}$CS.

Proof:(i) Let A be an IFROS in (X, τ). Then int(cl(A)) = A. By definition, we have scl(A) = A ∪int(cl(A)) = A ∪ A = A, by hypothesis. Since A is an IFb$^{\#}$CS in X, scl(A) is an IFb$^{\#}$CS in X.

(ii) Let A be an IFRCS in (X, τ). Then cl(int(A)) = A. By definition, we have sint(A) = A ∩ cl(int(A)) = A ∩ A = A, by hypothesis. Since A is an IFb$^{\#}$CS in X, sint(A) is an IFb$^{\#}$CS in X.

Theorem 3.25: For an IFS A in (X, τ), the following conditions are equivalent:

(i) A is both an IFOS and an IFb$^{\#}$CS

(ii) A is an IFROS.

Proof: (i) ⇒ (ii) Let A be an IFOS and an IFb$^{\#}$CS in X. Then cl(int(A)) ∩int(cl(A)) = A and int(cl(A)) ∩ cl(A) = A, since int(A) = A. Therefore A = int(cl(A)). Hence A is an IFROS in X.

(ii)⇒ (i) Let A be an IFROS in X. Since every IFROS is an IFOS, A is an IFOS in X and A = int(cl(A)). Now int(cl(A)) ∩ cl(int(A)) = A ∩ cl(int(A)) = A ∩ cl(A) = A. Hence A is an IFb$^{\#}$CS in X.

Theorem 3.26: Let A be an IFb$^{\#}$CS in (X, τ) and let $p_{(\alpha,\beta)}$ be an IFP in X such that $p_{(\alpha,\beta)\, q}$ int(cl(A))∩ cl(int(A)). Then cl($p_{(\alpha,\beta)}$) $_q$ A.

Proof: Let A be an IFb$^{\#}$CS in (X, τ) and let $p_{(\alpha,\beta)\, q}$ (int(cl(A)) ∩ cl(int(A))). Suppose that cl(p(α, β))$_{\bar{q}}$ A, then by Definition A ⊆ [cl($p_{(\alpha,\beta)}$)]^c = int($p_{(\alpha,\beta)}$)^c⊆ {$p_{(\alpha,\beta)}$}^{c}. Since A is an IFb$^{\#}$CS, int(cl(A)) ∩ cl(int(A)) = A ⊆ [$p_{(\alpha,\beta)}$]^c. This implies that p(α, β))$_{\bar{q}}$int(cl(A)) ∩ cl(int(A)), which is a contradiction to the hypothesis. Hence cl($p_{(\alpha,\beta)}$) $_q$ A.

IV. INTUITIONISTIC FUZZY B# OPEN SETS

In this section we have introduced intuitionistic fuzzy b$^{\#}$ open sets and studied some of their properties. We have provided some of the characterizations.

Definition 4.1: An IFS A in an IFTS (X,τ) is said to be an intuitionistic fuzzy b$^{\#}$ open set (IFb$^{\#}$OS for short) if A = int(cl(A)) ∪ cl(int(A)).

The complement A^c of an IFb$^{\#}$CS A in an IFTS (X,τ) is called an intuitionistic fuzzy b$^{\#}$ open set (IFb$^{\#}$OS for short) in X.

The family of all IFb$^{\#}$OSs of an IFTS (X,τ) is denoted by IFb$^{\#}$O(X).

Example 4.2: In Example 3.2, let A = ⟨x, (0.4_a, 0.6_b), (0.4_a, 0.4_b)⟩ be an IFS in (X, τ).Now int(cl(A^c)) ∩ cl(int(A^c))= G ∩ G^c = G = A^c. This implies that A^c is an IFb$^{\#}$CS in X. Hence A is an IFb$^{\#}$OS in X.

Theorem 4.3: Every IFb$^{\#}$OS are IFbOS, IFSOS, IFβOS but not conversely in general.

Proof: Straight forward.

Example 4.4: Obvious from Example 3.7, Example 3.15, Example 3.20 by taking complement of A in the respective examples.

Theorem 4.5: Every IFOS, IFROS, IFPOS and IF α OS are independent to IFb$^{\#}$OS in (X, τ) and vice versa in general.

Example 4.6: Obvious from Example 3.4 and Example 3.5, Example 3.9 and Example 3.10, Example 3.12 and Example 3.13, Example 3.17 and Example 3.18, by taking complement of A in the respective examples.

Theorem 4.7: If A is an IFb$^{\#}$OS and intuitionistic fuzzy nowhere dense in X then A is an IFRCS in X.

Proof: Let A be an IFb$^{\#}$OS and intuitionistic fuzzy nowhere dense in X. Then A = int(cl(A)) ∪ cl(int(A)) = $0_\sim$ ∪ cl(int(A)) = cl(int(A)). Therefore cl(int(A)) = A. Hence A is an IFRCS in X.

Theorem 4.8: If A is both an IFb$^{\#}$OS and an IFOS then A is an IFCS in (X, τ).

Proof: Let A be both an IFb$^{\#}$OS and an IFOS in X. Then A = int(cl(A)) ∪ cl(int(A)). Now A = int(cl(A)) ∪ cl(int(A)) = int(cl(A)) ∪ cl(A) = cl(A). Hence A is an IFCS in X.

Theorem 4.9: Let A be an IFb$^{\#}$OS in an IFTS in X such that int(A) = $0_\sim$, then A is an IFPOS in X.

Proof: Let A be an IFb$^{\#}$OS then A is an IFbOS in X. Now A ⊆ int(cl(A)) ∪ cl(int(A)) ⊆ int(cl(A)) ∪ $0_\sim$ ⊆ int(cl(A)). Hence A ⊆int(cl(A)). Hence A is an IFPOS in X.

V. CONCLUSIONS

In this paper , the concepts of intuitionistic fuzzy b$^{\#}$ closed sets and intuitionistic fuzzy b$^{\#}$ open sets are introduced and studied. They are compared with the already existing intuitionistic fuzzy closed sets in intuitionistic fuzzy topological spaces.

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