Product of Composition Operators

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Abstract

In this paper it is proved that the product of two composition operators C_1 and C_2 with $h_2 h_1 = 1$ and $h_1 = h_1 \circ T_2$ is quasi isometry and 2-isometry. Also it I s proved that this condition is not necessary for C_1 and C_2 with an example.

Key words: Isometric operators, Composition Operators

I. INTRODUCTION

Let (X, Σ, λ) be a sigma finite measure space and let $T : X \to X$ be a non singular measurable transformation. The equation $Cf = f \circ T$, $f \in L^2(\lambda)$ defines a composition transformation on $L^2(\lambda)$.

When the measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and the Radon Nikodym derivative $\dfrac{d\lambda T^{-1}}{d\lambda} = f_0 = h$ is essentially bounded, T induces a composition operator C on $L^2(\lambda)$.

The Radon Nikodym derivative of $\lambda(T \circ T)^{-1}$ with respect to λ is denoted by $f_0^{(2)}$ (ie) $\dfrac{d\lambda(T \circ T)^{-1}}{d\lambda} = f_0^{(2)}$ and the Radon Nikodym derivative of $\lambda(T \circ T)^{-1}$ with respect to λT^{-1} is denoted by g_0 (ie) $\dfrac{d\lambda(T \circ T)^{-1}}{d\lambda T^{-1}} = g_0$.

Every essentially bounded complex-valued measurable function f_0 induces the bounded operator M_{f_o} on $L^2(\lambda)$ which is defined by $M_{f_o} f = f_0 f$ for every $f \in L^2(\lambda)$.

Further from [6], we have $C^* C = f_0$ and in [5] it is proved that $C^{*2} C^2 = f_0^{(2)}$. Also we have $C^{*k} C^k = f_0^{(k)}$ from [2].

II. ISOMETRIC COMPOSITION OPERATORS

In [4], S. M. Patel has introduced quasi-isometric operator and studied some of its properties.

Definition: [4]: A bounded linear operator T is called a quasi-isometry if $T^{*2} T^2 = T^* T$.

Theorem : [10]: A composition operator C on $L^2(\lambda)$ is quasi-isometry iff $f_0^{(2)} = f_0$ a.e.

Corollary :[10]: If C is an quasi-isometric operator then $g_0 = 1$ a.e.

Definition. [1]: A bounded linear operator T is m-isometry if $\sum_{k=0}^{m}(-1)^k\, mC_k\, T^{*^{m-k}}T^{m-k} = 0$.

Definition. [3]: A bounded linear operator T is called 2-isometry if $T^{*^2}T^2 - 2T^*T + I = 0$

Lemma : Let $C, M_\theta \in B(L^2(\lambda))$. Then $CM_\theta = M_\theta C$ iff $\theta = \theta \circ T$ a.e where M_θ is the multiplication operator induced by θ. [7]

III. PRODUCT OF COMPOSITION OPERATORS

Theorem 1: Let C_1 and C_2 be two composition operators with $h_2h_1 = 1$ and $h_1 = h_1 \circ T_2$. Then the product C_1C_2 is quasi isometry.

Proof:

Consider $(C_1C_2)^{*2}(C_1C_2)^2$

$= (C_2^*C_1^*)^2(C_1C_2)^2$

$= C_2^*C_1^*C_2^*C_1^*C_1C_2C_1C_2$

$= C_2^*C_1^*C_2^*M_{h_1}C_2C_1C_2$

$= C_2^*C_1^*C_2^*C_2M_{h_1}C_1C_2$

$= C_2^*C_1^*M_{h_2}M_{h_1}C_1C_2$

$= C_2^*C_1^*C_1C_2$

$= (C_1C_2)^*(C_1C_2).$

$\Rightarrow$ C_1C_2 is quasi-isometry.

Theorem 2: Let C_1 and C_2 be two composition operators with $h_2h_1 = 1$ and $h_1 = h_1 \circ T_2$. Then the product C_1C_2 is 2- isometry.

Proof:

Consider $(C_1C_2)^{*2}(C_1C_2)^2 + I$

$= (C_2^*C_1^*)^2(C_1C_2)^2 + I$

$= C_2^*C_1^*C_2^*C_1^*C_1C_2C_1C_2 + I$

$= C_2^*C_1^*C_2^*M_{h_1}C_2C_1C_2 + I$

$= C_2^*C_1^*C_2^*C_2M_{h_1}C_1C_2 + I$

$= C_2^*C_1^*M_{h_2}M_{h_1}C_1C_2 + I$

$= C_2^*C_1^*C_1C_2 + I$

$= C_2^*C_1^*C_1C_2 + M_{h_2}M_{h_1}$

$= (C_1C_2)^*(C_1C_2) + C_2^*C_2M_{h_1}$

$= (C_1C_2)^*(C_1C_2) + C_2^*M_{h_1}C_2$

$= (C_1C_2)^*(C_1C_2) + C_2^*C_1^*C_1C_2$

$= (C_1C_2)^*(C_1C_2) + (C_1C_2)^*(C_1C_2)$

$= 2(C_1C_2)^*(C_1C_2).$

$\Rightarrow C_1C_2$ is 2-isometry.

Corollary: The following example shows that the conditions of Theorem 2 are not necessary for the composition operators C_1 and C_2.

Example: Let $X = N$, the set of all natural numbers and λ be the counting measure on it.

Define $T_1 : N \to N$ by $T_1(n) = \begin{cases} 1, & n = 1, 2 \\ n-1, & n \geq 3 \end{cases}$ and $n \in N$.

Define $T_2 : N \to N$ by $T_2(n) = \begin{cases} 1, & n = 1, 2, 3 \\ n-2, & n \geq 4 \end{cases}$ and $n \in N$.

Now $T_1 \circ T_2 : N \to N$ is defined as $(T_1 \circ T_2)(n) = \begin{cases} 1, & n = 1, 2, 3, 4 \\ n-3, & n \geq 5 \end{cases}$.

This operator induces a composition operator C_1C_2 which is 2-isometry since $f_0^{(2)} - 2f_0 + 1 = 0, \ \forall \ n \in N$. But $h_2h_1(1) \neq 1$.

Lemma: Let $C \in B(L^2(\lambda))$ be quasinormal. Then $M_h{}^n C = C M_h{}^n, \ n \in N$. [8]

Lemma: Let C_1 and C_2 are quasinormal with $h_i \circ T_j = h_i$ a.e. and $h_j \circ T_i = h_j$ a.e. for $i, j = 1, 2$. Then $M_{h_i}{}^m C_j{}^n = C_j{}^n M_{h_i}{}^m, \ i, j = 1, 2$. [8]

Theorem 7: Let C_1 and C_2 are quasinormal with $h_2{}^n h_1{}^m = 1$ and $h_1 = h_1 \circ T_2$. Then the product of the powers of C_1 and C_2 (i.e.) $C_1{}^m C_2{}^n$ is 2- isometry for all $m, n \in N$.

Proof:

Consider $(C_1{}^m C_2{}^n)^{*^2}(C_1{}^m C_2{}^n)^2 + I$

$= C_2{}^{n^*} C_1{}^{m^*} C_2{}^{n^*} C_1{}^{m^*} C_1{}^m C_2{}^n C_1{}^m C_2{}^n + I$

$= C_2{}^{n^*} C_1{}^{m^*} C_2{}^{n^*} M_{h_1}{}^m C_2{}^n C_1{}^m C_2{}^n + I$

$= C_2{}^{n^*} C_1{}^{m^*} C_2{}^{n^*} C_2{}^n M_{h_1}{}^m C_1{}^m C_2{}^n + I$

$= C_2{}^{n^*} C_1{}^{m^*} M_{h_2}{}^n M_{h_1}{}^m C_1{}^m C_2{}^n + I$

$= C_2{}^{n^*} C_1{}^{m^*} C_1{}^m C_2{}^n + I$

$= (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n) + I$

$= (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n) + M_{h_2}{}^n M_{h_1}{}^m$

$= (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n) + C_2{}^{n^*} C_2{}^n M_{h_1}{}^m$

$= (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n) + C_2{}^{n^*} M_{h_1}{}^m C_2{}^n$

$= (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n) + C_2{}^{n^*} C_1{}^{m^*} C_1{}^m C_2{}^n$

$= (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n) + (C_1{}^m C_2{}^n)^*(C_1{}^m C_2{}^n)$

$= 2(C_1^m C_2^n)^*(C_1^m C_2^n).$

$\Rightarrow C_1^m C_2^n$ is 2- isometry for all $m, n \in N$.

Theorem 8: Let C_1 and C_2 are quasinormal with $h_2^n h_1^m = 1$ and $h_1 = h_1 \circ T_2$. Then the product of the powers of C_1 and C_2 (i.e) $C_1^m C_2^n$ is quasi-isometry for all $m, n \in N$.

Proof:

Consider $(C_1^m C_2^n)^{*^2}(C_1^m C_2^n)^2$

$= (C_2^{n^*} C_1^{m^*})^2 (C_1^m C_2^n)^2$

$= C_2^{n^*} C_1^{m^*} C_2^{n^*} C_1^{m^*} C_1^m C_2^n C_1^m C_2^n$

$= C_2^{n^*} C_1^{m^*} C_2^{n^*} M_{h_1}^m C_2^n C_1^m C_2^n$

$= C_2^{n^*} C_1^{m^*} C_2^{n^*} C_2^n M_{h_1}^m C_1^m C_2^n$

$= C_2^{n^*} C_1^{m^*} M_{h_2}^n M_{h_1}^m C_1^m C_2^n$

$= C_2^{n^*} C_1^{m^*} C_1^m C_2^n$

$= (C_1^m C_2^n)^*(C_1^m C_2^n).$

$\Rightarrow C_1^m C_2^n$ is quasi-isometry.

Corollary: Normaloid does not imply 2-isometry and quasi-isometry.

Example: Let $X = N$, λ be the counting measure on it and $L^2 = l^2$. Define $T : N \to N$ by $T(n) = n + 1,\ n \in N$. It is already proved in [9], that $\left\| f_0^{(k)} \right\| = 1,\ \forall\ k \in N$.

$\Rightarrow C$ is normaloid.

But since $f_0^{(2)} - 2 f_0 + 1 \neq 0$ for $n = 1$, C is not 2- isometry.

Also since $f_0^{(2)} \neq f_0$ for $n = 2$, C is not quasi- isometry.

Hence normaloid does not imply 2-isometry and quasi-isometry.

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