Some Identities of Fibonacci-Like Based on Lucas Numbers

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Abstract

Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$ with $F_0 = 0$ and $F_1 = 1$. Lucas sequence is defined by $L_n = L_{n-1} + L_{n-2}$ for $n \ge 2$ with $L_0 = 2$ and $L_1 = 1$. While Fibonacci-Like sequence is a generalized of Fibonacci and Lucas Number that defined by $s_n = s_{n-1} + s_{n-2}$ for $n \ge 2$ with $s_0 = 2$ and $s_1 = 2$. In this paper, we determine some identities of Fibonacci-Like number based on Lucas number then Fibonacci-Like is just defined by Lucas number. The new identities of Fibonacci-Like can be proved by Binet's formula.

Keywords - Fibonacci, Fibonacci-Like, Lucas, Binet's formula

I. INTRODUCTION

Fibonacci is a sequence that starts from 0 and 1, where the next terms can be obtained by adding up the two previous consecutive terms. The Fibonacci numbers forms the sequence numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... [1, h. 287]. Lucas numbers is the sequence that starts from 2 and 1, where the next terms can also be obtained by adding up two previous consecutive terms. Lucas numbers forms the sequence number 2, 1, 3, 4, 7, 11, 18, 29, ... [5, h. 136].

Fibonacci and Lucas sequence have been many generalized became a new sequence, such as Fibonacci-Like with initial condition $s_0 = 2$ and $s_1 = 2$ that introduced by Singh et al. [11]. Singh et al. discussed the Binet's formula, generating function of Fibonacci-Like sequence, and using induction methods and Binet's formula to proven the property of Fibonacci-Like sequence. Singh et al. [10] has defined some identities of the relation between Fibonacci, Lucas and Fibonacci-Like number on another paper. Suvarmani [12] has defined some properties of the generalized (p,q)- Fibonacci-Like sequence by using Binet's formula. Gupta et al. [4] introduced some determinant identities of generalized Fibonacci-Lucas sequence. Rathore [9] have generalized the Fibonacci-Like sequence with initial condition $R_0 = 2b$ and $R_1 = a + b$, where *a* and *b* were non-zero real number.

In this paper, we present some of identities of Fibonacci-Like sequence with initial condition $s_0 = 2$ and $s_1 = 2$ based on the relation between Fibonacci-Like and Lucas number, then Fibonacci-Like can just be defined by Lucas number.

II. THE BINET'S FORMULA OF FIBONACCI, LUCAS AND FIBONACCI-LIKE SEQUENCE

In this section, the definition and Binet's formula of Fibonacci, Lucas and Fibonacci-Like sequence are given. Fibonacci sequence is defined by a recurrence relation as [2]

$$F_n = F_{n-1} + F_{n-2}, \ n \ge 2 \tag{1}$$

with $F_0 = 0$ and $F_1 = 1$. Lucas sequence is defined by a recurrence relation [2]:

$$L_n = L_{n-1} + L_{n-2} , \ n \ge 2$$
⁽²⁾

with $L_0 = 2$ and $L_1 = 1$.

Singh et al. [11] defined the Fibonacci-Like sequence as following

$$s_n = S_{n-1} + S_{n-2}, \ n \ge 2$$
 (3)

with $S_0 = 2$ and $S_1 = 2$.

The relation between Fibonacci, Fibonacci-Like and Lucas number can be defined by

$$S_n = F_n + L_n \tag{4}$$

Binet's formula is an explicit formula to determine the nth terms of the Fibonacci sequence. The Binet's formula of Fibonacci sequence is given by [7]

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$
(5)

with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The Binet's formula of Lucas sequence can be stated as

$$L_n = \alpha^n + \beta^n \tag{6}$$

According to equation (4), (5) and (6), the Binet's formula for Fibonacci-Like sequence can be written as

$$S_{n} = \frac{2}{\sqrt{5}} \left(\alpha^{n+1} - \beta^{n+1} \right)$$
(7)

III.IDENTITIES OF FIBONACCI-LIKE SEQUENCE BASED ON

In this section, some theorems of the relation of Fibonacci-Like and Lucas that given some identities of Fibonacci-Like numbers are presented.

Theorem 1 (The First Relation) Let s_n and L_n be the *n*th term of Fibonacci-Like and Lucas numbers, then s_n can be stated as

$$S_n = \frac{2}{5} (L_{n+2} + L_n).$$
(8)

Proof. By using the Lucas Binet's formula (6), we have

$$\begin{aligned} \frac{2}{5}(L_{n+2}+L_n) &= \frac{2}{5} \left(\alpha^{n+2} + \beta^{n+2} + \alpha^n + \beta^n \right) \\ &= \frac{2}{5} \left(\alpha^{n+2} + \beta^{n+2} + \alpha^{-2} \alpha^{n+2} + \beta^{-2} \beta^{n+2} \right) \\ &= \frac{2}{5} \left(\alpha^{n+2} + \beta^{n+2} + \alpha^{-2} \alpha^{n+2} + \beta^{-2} \beta^{n+2} \right) \\ &= \frac{2}{5} \left((1 + \alpha^{-2}) \alpha^{n+2} + (1 + \beta^{-2}) \beta^{n+2} \right) \\ &= \frac{2}{5} \left((1 + \beta^2) \alpha^{n+2} + (1 + \alpha^2) \beta^{n+2} \right) \end{aligned}$$

Since $(1 + \alpha^2) = \sqrt{5} \alpha$ and $(1 + \beta^2) = -\sqrt{5} \beta$, we obtain

$$\frac{2}{5}(L_{n+2} + L_n) = \frac{2}{5} \left(\left(\sqrt{5} \alpha^{n+1} - \sqrt{5} \beta^{n+1} \right) \right)$$
$$= \frac{2}{\sqrt{5}} \left(\left(\alpha^{n+1} - \beta^{n+1} \right) \right)$$
$$\frac{2}{5} \left(L_{n+2} + L_n \right) = S_n.$$

Theorem 2 (The Second Relation)) Let s_{n-1} be the (n-1)th term of Fibonacci-Like and L_n be the *n* th term of Lucas numbers, then s_{n-1} can be stated as

$$S_{n-1} = \frac{2}{5} (2L_{n+1} - L_n).$$
⁽⁹⁾

Proof. By using the Lucas Binet's formula (6), we have

$$\frac{2}{5}(2L_{n+1} - L_n) = \frac{2}{5}\left(2\alpha^{n+1} + 2\beta^{n+1} - \alpha^n - \beta^n\right)$$
$$= \frac{2}{5}\left(2\alpha\alpha^n + 2\beta\beta^n - \alpha^n - \beta^n\right)$$

$$\frac{2}{5}(2L_{n+1} - L_n) = \frac{2}{5}\left((2\alpha - 1)\alpha^n + (2\beta - 1)\beta^n\right)$$

Since $(2\alpha - 1) = \sqrt{5}$ dan $(2\beta - 1) = -\sqrt{5}$, we have

$$\frac{2}{5}(2L_{n+1} - L_n) = \frac{2}{5}\sqrt{5}\left((\alpha^n - \beta^n)\right)$$
$$= \frac{2}{\sqrt{5}}\left((\alpha^n - \beta^n)\right)$$
$$\frac{2}{5}(2L_{n+1} - L_n) = S_{n-1}$$

Theorem 3 (The Third Relation) Let S_{2n} be the 2*n*th term of Fibonacci-Like and L_n be the *n*th term of Lucas numbers, then S_{2n} can be stated as

$$S_{2n} = \frac{6L_{2n} + 2L_{2n-1}}{5}.$$
 (10)

Proof. By using the Lucas Binet's formula (6), we have

$$\frac{6L_{2n} + 2L_{2n-1}}{5} = \frac{6\alpha^{2n} + 6\beta^{2n1} + 2\alpha^{-1}\alpha^{2n} + 2\beta^{-1}\beta^{2n}}{5}$$
$$= \frac{(6 + 2\alpha^{-1})\alpha^{2n} + (6 + 2\beta^{-1})\beta^{2n}}{5}$$
$$\frac{6L_{2n} + 2L_{2n-1}}{5} = \frac{(6 - 2\beta)\alpha^{2n} + (6 - 2\alpha)\beta^{2n}}{5}$$

Since $(6 - 2\alpha) = -2\sqrt{5}\beta$ dan $(6 - 2\beta) = 2\sqrt{5}\alpha$, we have

$$\frac{6L_{2n} + 2L_{2n-1}}{5} \cdot = \frac{2\sqrt{5}}{5} \left(\alpha^{2n+1} - \beta^{2n+1} \right)$$
$$= \frac{2}{\sqrt{5}} \left(\alpha^{2n+1} - \beta^{2n+1} \right)$$
$$\frac{6L_{2n} + 2L_{2n-1}}{5} \cdot = S_{2n}$$

Theorem 4 (The Fourth Relation) Let s_n and L_n be the *n*th term of Fibonacci-Like Lucas numbers, then the square of s_n for any integer *n* is given by

$$S_n^2 = \frac{4}{5} \left(L_{2n+2} + 2(-1)^{n+1} \right)$$
(11)

Proof. By using the Fibonacci-Like Binet's formula (7), we have

$$S_n^{2} = \left(\frac{2}{\sqrt{5}} \left(\alpha^{n+1} - \beta^{n+1}\right)\right)^{2}$$
$$= \frac{4}{5} \left(\alpha^{2n+2} + \beta^{2n+2} - 2(-1)^{n+1}\right)$$
$$= \frac{4}{5} \left(L_{2n+2} + 2(\alpha\beta)^{n+1}\right)$$
$$S_n^{2} = \frac{4}{5} \left(L_{2n+2} + 2(-1)^{n+1}\right)$$

Teorema 5 (The Fifth Relation) For n integer, the sum of first n terms is given by

$$\sum_{k=1}^{n} S_{k} = \frac{2}{5} (L_{n+4} + L_{n+2}) - 4$$

Proof. By using the Fibonacci-Like Binet's formula (6), we have

$$\sum_{k=1}^{n} S_{k} = \frac{2}{\sqrt{5}} \left(\alpha^{2} - \beta^{2} + \alpha^{3} - \beta^{3} + \dots + \alpha^{n} - \beta^{n} \right)$$
$$= \frac{2\sqrt{5}}{5} \left(\alpha^{2} + \alpha^{3} + \dots + \alpha^{n} - (\beta^{2} + \beta^{3} + \dots + \beta^{n}) \right)$$
$$= \frac{2\sqrt{5}}{5} \left(\frac{\alpha^{2} (1 - \alpha^{n})}{1 - \alpha} - \frac{\beta^{2} (1 - \beta^{n})}{1 - \beta} \right)$$
$$\sum_{k=1}^{n} S_{k} = \frac{2}{5} \left((\alpha - \beta)(-\alpha)\alpha^{2}(1 - \alpha^{n}) - (\alpha - \beta)(-\beta)\beta^{2}(1 - \beta^{n}) \right)$$
(12)

By solving (12), we obtain

$$\sum_{k=1}^{n} S_{k} = \frac{2}{5} \left(\left(\alpha^{n+4} + \beta^{n+4} - \left(\alpha^{4} + \beta^{4} \right) + \alpha^{3} \beta + \alpha \beta^{3} - \left(\alpha^{n+3} \beta + \alpha \beta^{n+3} \right) \right) \\ = \frac{2}{5} \left(L_{n+4} - L_{4} - \left(\alpha^{2} + \beta^{2} \right) + \alpha^{n+2} + \beta^{n+2} \right) \\ = \frac{2}{5} \left(L_{n+4} - 7 - 3 + L_{n+2} \right) \\ = \frac{2}{5} \left(L_{n+4} + L_{n+2} \right) - \frac{20}{5} \\ \sum_{k=1}^{n} S_{k} = \frac{2}{5} \left(L_{n+4} + L_{n+2} \right) - 4$$

IV.CONCLUSIONS

In this article, the author discusses the relationship between Fibonacci -Likes and Lucas numbers so that Fibonacci-Like numbers can just be expressed in Lucas numbers. This relationship is obtained from the first, second, third, fourth and fifth relations theorems. This relationship is been proven by using the Binet formula Fibonacci-Like and Lucas numbers. For further research, other relationships can be determined based on the identity of the Fibonacci-Like and Lucas relationship in this article.

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