Summation of Terms in Exponential Arithmetic Series through Integration Process and Derivation of an Approximate General Formula

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Abstract

The manuscript provides a process for summation of arithmetic series through integration process and derivation of approximate General Formula for the same.

Keywords - *Series Summation, Integration ,General Pth level Formula*

I. INTRODUCTION

The exponential Arithmetic series comprises of sequence of continuous numbers exponentially raised to any integer power, being added together. The said series is represented under Sigma Notation as here –in - below shown through the examples.

Examples of exponential Arithmetic series are as hereunder-

A. $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n^2 \quad \ldots Exp.(1)$$

B. $$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n^3 \quad \ldots Exp.(2)$$

C. $$\sum_{i=1}^{n} i^p = 1^p + 2^p + 3^p + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n^p \quad \ldots Exp.(3)$$

Where p is any integer to which the term in aforesaid series is raised.

This manuscript provides the general formula for summation of series having terms exponentially raised to some powers. The Formula for summing up the above said Arithmetic series given in Exp(3) above is—

$$\sum_{i=1}^{n} i^p = p \int \sum_{i=1}^{n} i^{p-1}\, dn \quad \ldots\ldots\ldots Exp.(I)$$

Here computation of $\sum_{i=1}^{n} i^p$ requires knowledge of $\sum_{i=1}^{n} i^{p-1}$.

The use of this formula is explained through following examples commencing with p=1. Let n be the variable that represents number of terms in the series .

A. Now, the series comprising of sum of n- terms can be computed as under

$$\sum_{i=1}^{n} i^1 = 1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots\ldots\ldots + n = 1 \int \sum_{i=1}^{n} i^{1-1}\, dn = 1 \int \sum_{i=1}^{n} 1 \; dn \ldots\ldots\ldots \text{Exp.A(i)}$$

$$= \int n \, dn$$

$$= \frac{n^2}{2} + C_1 \quad \ldots\ldots\ldots\ldots \text{Exp..A(ii)}$$

Where C_1 is constant of integration;

<u>Computation of Constant of Integration C_1</u>:

Thus we have

$$\sum_{i=1}^{n} i = \left(\frac{n^2}{2}\right) + C_1 \ldots\ldots\ldots\ldots\ldots\ldots \text{Exp.A(ii)}$$

Putting n = 1 in above expression , we get

L.H.S: $\sum_{i=1}^{1} i^1 = 1^1 = 1$

R.H.S.: $\left(\frac{1^2}{2}\right) + C_1$

$$= \frac{1}{2} + C_1$$

Since L.H.S = R.H.S.

Thus $1 = \frac{1}{2} + C_1$

$$C_1 = 1 - \frac{1}{2} = \frac{1}{2} \ldots\ldots\ldots\ldots\ldots \text{Exp..A(iii)}$$

If there are n terms then,

$$C_1 = n \times \frac{1}{2} = \frac{n}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{Exp..A(iv)}$$

Thus -

$$\sum_{i=1}^{n} i^1 = 1^1 + 2^1 + 3^1 + 4^1 + \ldots\ldots\ldots\ldots + n^1 = 1 \int \left(\sum_{i=1}^{n} i^0\right) dn$$

$$= 1 \int n \; dn$$

$$= \left(\frac{n^2}{2}\right) + C_1$$

$$= \frac{n^2}{2} + \frac{n}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{Exp.A(v)}$$

$$= \frac{(n^2 + n)}{2}$$

$$= \frac{n\,(n+1)}{2} \quad \ldots\ldots\ldots\ldots\ldots \text{Exp..A(vi)}$$

It is well known in elementary mathematics that --

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots\ldots\ldots + n = \frac{n\,(n+1)}{2}$$

B. Now the series comprising of sum of squared n- terms can be computed as under

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots\ldots + n^2 = 2\int \left(\sum_{i=1}^{n} i^1\right) dn$$

$$= 2\int \left(\frac{n^2}{2} + \frac{n}{2}\right) dn \quad \ldots\ldots\ldots\ldots\ldots \text{Exp..B(i)} \qquad (\ldots\ldots\ldots \text{From Exp..A(v)})$$

$$= 2\left(\frac{n^3}{6} + \frac{n^2}{4}\right) + C_1$$

$$= \left(\frac{n^3}{3} + \frac{n^2}{2}\right) + C_1 \quad \ldots\ldots\ldots\ldots \text{Exp..B(ii)}$$

<u>Computation of Constant of Integration C_1:</u>

$$\sum_{i=1}^{n} i^2 = \left(\frac{n^3}{3} + \frac{n^2}{2}\right) + C_1 \quad \ldots\ldots\ldots\ldots\ldots \text{Exp..B(ii)}$$

Putting n = 1 in above expression , we get

L.H.S: $\sum_{i=1}^{1} i^2 = 1^2 = 1$

R.H.S.: $\left(\frac{n^3}{3} + \frac{n^2}{2}\right) + C_1$

$$= \left(\frac{1^2}{3} + \frac{1^2}{2}\right) + C_1$$

$$= \frac{5}{6} + C_1$$

Since L.H.S. = R.H.S , we get,

Thus $1 = \frac{5}{6} + C_1$

$$C_1 = 1 - \frac{5}{6} = \frac{1}{6} \quad \ldots\ldots\ldots \text{Exp.B(iii)}$$

If there are n terms then,

$C_1 = n \times \frac{1}{6} = \frac{n}{6}$ ……Exp.B(iv)

Thus ,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = 2\int \left(\sum_{i=1}^{n} i^1\right) dn$$

$$= 2\int \left(\frac{n^2}{2} + \frac{n}{2}\right) dn$$

$$= 2\left(\frac{n^3}{6} + \frac{n^2}{4}\right) + C_1$$

$$= \left(\frac{n^3}{3} + \frac{n^2}{2}\right) + \frac{n}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{Exp.B(v)}$$

$$= \frac{(2n^3 + 3n^2 + n)}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{Exp.B(vi)}$$

It is well known that -

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

C. Now the series comprising of sum of cubed n- terms can be computed as under

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3 = 3\int \left(\sum_{i=1}^{n} i^2\right) dn \quad \ldots\ldots\text{Exp.C(i)}$$

$$= 3\int \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) dn \quad \ldots\ldots\text{Exp.C(ii)} \qquad (\ \ldots\ldots\text{From B(v)}\ )$$

$$= 3\left(\frac{n^4}{12} + \frac{n^3}{6} + \frac{n^2}{12}\right) + C_1 \text{ ; Where } C_1 \text{ is constant of Integration;}$$

$$= \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) + C_1$$

<u>Computation of Constant of Integration C_1:</u>

$$\sum_{i=1}^{n} i^3 = \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) + C_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{Exp..C(iii)}$$

Putting n = 1 in above expression , we get

L.H.S: $\sum_{i=1}^{1} i^2 = 1^2 = 1$

R.H.S.: $\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) + C_1$

$= \left(\frac{1^4}{4} + \frac{1^3}{2} + \frac{1^2}{4}\right) + C_1$

$= 1 + C_1$

Since L.H.S. = R.H.S , we get,

Thus $1 = 1 + C_1$

$C_1 = 1 - 1 = 0$ …….Exp.C(iv)

If there are n terms then,

$C_1 = n \times 0 = 0$ …….Exp..C(v)

Thus ,

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots + n^3 = 3\int \left(\sum_{i=1}^{n} i^2\right) dn$$

$$= 3\int \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) dn$$

$= 3\left(\frac{n^4}{12} + \frac{n^3}{6} + \frac{n^2}{12}\right) + C_1$; Where $C_1 = 0$, is constant of Integration;

$= \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right)$ ……Exp..C(vi)

$$= \frac{(n^4 + 2n^3 + n^2)}{4}$$

$$= \frac{n^2(n^2 + 2n + 1)}{4}$$

$= \frac{n^2(n+1)^2}{4}$ ……Exp.C(vii)

It is well known that---

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

D. Now the summation of series comprising of n- terms , where each term is raised to its 4th power , is computed as hereunder

$$\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + \ldots\ldots\ldots + n^4 = 4\int \left(\sum_{i=1}^{n} i^3\right) dn$$

$$= 4\int \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) dn \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{Exp.D(i)} \quad \ldots\text{(From Exp. C(vi))}$$

$$= 4\left(\frac{n^5}{20} + \frac{n^4}{8} + \frac{n^3}{12}\right) + C_1 \text{ ; Where } C_1 \text{ is constant of Integration;}$$

$$= \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}\right) + C_1$$

Computation of Constant of Integration C_1

$$\sum_{i=1}^{n} i^4 = \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}\right) + C_1 \quad \ldots\ldots\ldots\ldots\ldots\text{Exp.D(ii)}$$

Putting n = 1 in above expression , we get

L.H.S: $\sum_{i=1}^{1} i^4 = 1^4 = 1$

R.H.S.: $\left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}\right) + C_1$

$$= \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right) + C_1$$

Since L.H.S. = R.H.S , we get,

$$= \left(\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right) + C_1$$

$$= \frac{(6+15+10)}{30} + C_1$$

Thus $1 = \frac{31}{30} + C_1$

$$C_1 = 1 - \frac{31}{30} = \frac{-1}{30} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{Exp.D(iii)}$$

If there are n terms then,

$$C_1 = n \times \frac{-1}{30} = \frac{-n}{30} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{Exp.D(iv)}$$

Thus,

$$\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + \ldots\ldots\ldots + n^4 = 4\int \left(\sum_{i=1}^{n} i^3\right) dn$$

$$= 4\int \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) dn$$

$$= 4\left(\frac{n^5}{20} + \frac{n^4}{8} + \frac{n^3}{12}\right) + C_1$$; Where C_1 is constant of Integration;

$$= \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}\right) - \frac{n}{30}$$ ……Exp.D(v)

$$= \frac{(6n^5 + 15n^4 + 10n^3 - n)}{30}$$ ……Exp.D(vi)

Verification of above equation

Let n =2 in Exp.D(vi);

$$\sum_{i=1}^{n} i^4 = \frac{(6n^5 + 15n^4 + 10n^3 - n)}{30}$$

L.H.S: $\sum_{i=1}^{2} i^4 = 1^4 + 2^4 = 17;$

R.H.S $= \frac{(6n^5 + 15n^4 + 10n^3 - n)}{30}$

$= \frac{(6x32 + 15x16 + 10x8 - 2)}{30}$

$= \frac{(192+240+80-2)}{30}$

$= \frac{(510)}{30}$

$= 17$

Thus L.H.S. = R.H.S……..Exp. D(vii)

E. Now the summation of series comprising of n-continuous terms , where each term is raised to its 5th power , is computed as hereunder

$$\sum_{i=1}^{n} i^5 = 1^5 + 2^5 + 3^5 + 4^5 + \ldots\ldots + n^5 = 5\int \left(\sum_{i=1}^{n} i^4\right) dn$$

$$= 5\int \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right) dn$$ ……Exp.E(i) …(From Exp. D(v))

$$= 5\left(\frac{n^6}{30} + \frac{n^5}{10} + \frac{n^4}{12} - \frac{n^2}{60}\right) + C_1$$; Where C_1 is constant of Integration;

$$= \left(\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}\right) + C_1 \quad \text{……..Exp.E(ii)}$$

Computation of Constant of Integration C_1:

$$\sum_{i=1}^{n} i^5 = \left(\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}\right) + C_1$$

Putting n = 1 in above expression , we get

L.H.S: $\sum_{i=1}^{1} i^5 = 1^5 = 1$

R.H.S.: $= \left(\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}\right) + C_1$

$$= \left(\frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12}\right) + C_1$$

Since L.H.S. = R.H.S , we get,

$$1 = \left(\frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12}\right) + C_1$$

$$= \frac{(2+6+5-1)}{12} + C_1$$

Thus $1 = 1 + C_1$

$C_1 = 1 - 1 = 0$ …….Exp.E(iii)

If there are n terms then,

$C_1 = n \times 0 = 0$ ……..Exp. E(iv)

Thus

$$\sum_{i=1}^{n} i^5 = 1^5 + 2^5 + 3^5 + 4^5 + \ldots\ldots\ldots + n^5 = 5\int \left(\sum_{i=1}^{n} i^4\right) dn$$

$$= 5\int \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right) dn$$

$= 5\left(\frac{n^6}{30} + \frac{n^5}{10} + \frac{n^4}{12} - \frac{n^2}{60}\right) + C_1$; Where C_1=0, is constant of Integration;

$$= \left(\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}\right) \quad \text{……Exp.E(v)}$$

$$= \frac{(2n^6 + 6n^5 + 5n^4 - n^2)}{12} \quad \text{……Exp.E(vi)}$$

Verification

Let n =2 in above Exp.E(vi)

$$\sum_{i=1}^{n} i^5 = \frac{(2n^6 + 6n^5 + 5n^4 - n^2)}{12}$$

L.H.S: $\sum_{i=1}^{2} i^5 = 1^5 + 2^5 = 33;$

R.H.S: $\frac{(2n^6 + 6n^5 + 5n^4 - n^2)}{12}$

$= \frac{(2x64 + 6x32 + 5x16 - 4)}{12}$

$= \frac{(128+192+80-4)}{12}$

$= \frac{(396)}{12}$

$= 33$

Thus L.H.S. = R.H.S……..Exp.E (vii).

1.1. Approximation of General Pth Level Formula

The Formula for the summation of terms in series, where each term is raised to any power P , can be derived by using the abovesaid Formula as under-

$$\sum_{i=1}^{n} i^p = p \int \sum_{i=1}^{n} i^{p-1}\, dn \quad \ldots\ldots Exp.(I)$$

Thus we have--

1. $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots + n = \frac{n^2}{2} + \frac{n}{2}$

2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots\ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$

3. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots + n^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n}{4}$

4. $\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + \ldots\ldots\ldots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{6} - \frac{n}{30}$

5. $\sum_{i=1}^{n} i^5 = 1^5 + 2^5 + 3^5 + 4^5 + \ldots\ldots\ldots + n^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$

and so on.

1.1.1 Determination of Pth level General Formula from above process:

Now , General Pth Level Formula for any Series with each term of the n-terms series being raised to power P, is determined as—

$$\sum_{i=1}^{n} i^p = 1^p + 2^p + 3^p + \ldots\ldots\ldots\ldots + n^p = \frac{n^{p+1}}{(p+1)} + \frac{n^p}{2} + \frac{p(n^{p-1})}{12} + \ldots\ldots\ldots$$

It is essential to note certain features of this Pth Level General Formula. There are positive and negative terms both in abovesaid General Formula depending upon value of Power P. The abovesaid Pth Level General formula for series summation contains (p+4)/2 terms where P is even and (P+3)/2 terms if P is odd. It is hereby submitted that P is the power to which, each of the n terms in the given Series are raised. Thus, for example, summation formula of given Series with each term being raised to odd power of 3, shall contain three terms and similarly in Series, with each term being raised to power of four, there shall be four terms in the said summation formula and these results can be verified from the above submission.

It is difficult to calculate complete General Formula for summation of n terms in such Pth Level Arithmetic series. However the sum of first three terms alone in abovesaid Pth level General formula gives an upper bound beyond which the sum of n-terms in a series, where each term is raised to power P, cannot exceed. The said General Formula , gives exact result for P<=3 where P>0 and is an upper bound for Series with P>3 , differing from the exact sum of the terms in the series by less than 0.01% , when the number of terms in series exceed or equal 2P, where P is the power to which the terms in Series are raised to. The said General Formula can therefore be limited to sum of first three terms and expressed as --

$$\sum_{i=1}^{n} i^p = 1^p + 2^p + 3^p + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n^p = \frac{n^{p+1}}{(p+1)} + \frac{n^p}{2} + \frac{p(n^{p-1})}{12}$$

where n is the number of terms in the given Arithmetic series with each of its term being raised to Power P such that P>0.

II. CONCLUSION

In this manner, the sum of any Pth level exponential arithmetical series, where each term is raised to power P, can be computed through process of integeration and besides that , Pth level General Formula , evolved as above, can also be used to approximate sum of the terms of the Arithmetic series where each of the term is raised to Power P, to the precision of 99.99% when the relation n = number of terms in the series = 2P is satisfied. This general formula shall be immensely useful in applications involving computation of terms of complex arithmetical series where each of the term of the series is raised to a Power P.

REFERENCES

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