Current Trend and Studies on Representations in Mathematics: The Case of Fractions

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Abstract

Relying on the fact that fractions are of a long time for researchers a challenging cognitive area for research, as students of all levels of education face particular difficulties on their understanding, this paper presents a review of contemporary literature on the subject of the representations of fractions. In particular, the international literature was investigated in order to study and record the results of researches which have been published in this time on the representations of fractions. In other words, this paper aiming to answers to what representations have been emerged by this research as the most appropriate or inappropriate for understanding the concept of fractions from the students. In this way, this study is expected to give to teachers of mathematics education and researchers who are examining the area of fractions a useful guide.

Keywords - *Fractions, representations, international literature.*

I. INTRODUCTION

The concept of rational number is one of the most important concepts of mathematics and taught from the beginning of primary school both in the Greek educational system and in the educational systems of other countries. However, it is clear that it is a topic which many teachers find difficult to understand and teach [22, 32, and 4] and many students find difficult to learn [50, 51, 52, 43, and 2]. So, these difficulties tend to maintain within the cognitive patterns of pupils for so long, and can reach adulthood. Thus, misconceptions which make manufactured prevent not only educational process, but also the students during the evolutionary trend school.

One of these difficulties are faced by students with fractions is the symbolism of the rational numbers that often helps create misunderstandings. Specifically, many students find it difficult to understand the fraction a/b as number. Thus, they tend to be treated as two different integers [27]. A typical example of the above treatment is the process of adding fractions. Many students when they want to add fractions such as $\frac{5}{6}+\frac{2}{3}$ adding the numerators and denominators leading to the result $\frac{7}{9}$. Consider, say, the numerator and the denominator as independent and not as associated entities and so when they add fractions, they add the numerators and denominators [31].

Another example that is associated with the symbolism of the rational numbers and address them as two different integers is when pupils have to compare fractions with numerator alike, for example $\frac{1}{3}$ and $\frac{1}{5}$. In this case, many pupils choose as larger fraction $\frac{1}{5}$, because 5 are larger than the 3.

In Addition, due to the sequence of natural numbers, whereby the students know that between two successive natural numbers there is not a third they adopt this sequence on rational numbers, and they assume that between two fractions there is not someone else. Thus, it is unable of the students to understand the existence of infinite numbers between two fractional numbers [46, 2, and 53].

Furthermore, students face difficulties in the ability to place fractions from smallest to largest, and vice versa. Also, the students find hard to divide a whole into equal portions even when teaching fractions in elementary school [18, 49, and 3]. Finally, considerable difficulty seems to be faced by students with equivalence and division of fractions and with the ability to translate from one system of representing fractions in another [2]. This ability is particularly important for the solving a mathematical problem and more generally for the learning of mathematics concepts [16, 54].

All these difficulties have been attributed by several researchers in a variety of factors. According to Janvier [16] most textbooks today include a variety of representations in order to promote understanding. However, Lo [21] in his research, evaluates the difficulties in understanding of fractions and ratios possible in inappropriate method of teaching in the classroom. Streefland [43] supports the same view, adding that the failure in teaching the concept of the fraction is due to the complexity of the concept and the traditional approach to fractions, which is typical and mechanics [39]. In addition, according to international literature, important factors are also the way of teaching fractions [7, 55, and 56], the use of representations of mathematical concepts [57, 55, 58, and 59] and students' and prospective teachers' attitudes and beliefs towards mathematics [20, 60, 61, 53, 62, 63, 73 and 64].

Thus, there is a common conception that the teaching approach is important factor which affects the understanding the concept of the fractions in the perceptions of the students. Hence, the present study was undertaken in order to record all teaching approaches and researches about fractions which have been published from 2006 until today in international journals providing useful information about the appropriate and inappropriate teaching and representations in fractions to the teachers and researchers.

II. LITERATURE REVIEW: REPRESENTATIONS IN MATHEMATICS

In education at least in some instances no understanding can be achieved without the aid of representation. Such a case is the notion of fractions. In mathematics education, the concept of representation is used as equivalent to a sign that shows and makes present a mathematical concept - a symbol or mark to think about the concept. Representations are those schemes or mental images with which the subjects work on mathematical ideas [65]. Particularly, it is usual to consider the duality, external and internal representations. To think about and to communicate mathematical ideas we need to represent them in some way. Communication requires that the representations be external, taking the variety of forms, including pictures (e.g., Drawing, charts, graphs), written symbols (e.g., numbers, equations, words), manipulative models, oral language (e.g., talk between pairs of students and whole class discussion), and real-world situations [35]. The multiple representations are the use variety of these external representations during teaching a mathematics concept.

According to some researches [66, 67, 68, 50] representing mathematical objects in multiple ways plays an important role in mathematical understanding and brings value to teaching processes. In addition, recent trends in curriculum standards, including standards developed by the National Council of Teachers of Mathematics [69], have highlighted the productive role that drawn models and other external representations can play in teaching and learning mathematics [70]. Although the representations add complexity, using a range of representations is necessary for developing children's fractions understanding because each provide links to the underlying fractions concepts and children require support to make active connections within and between the various representations [71].

However, Duval [72] called attention to a cognitive paradox hidden within in various representations. Handling these representations choosing he distinguishing features of the concept we must treat and convert, is not learnt automatically. This learning results from a process of explicit teaching in which the teacher must render the student co-responsible. Teachers often underestimate this aspect and passing from one register to another, believing that the student follows. The teacher is able to jump from one register to another without problems, because he has already conceptualized: while in fact the student does not so, the student follows at the level of semiotic representatives, but not of meanings [26].

III.THE RESEARCH

Many researchers have studied the issue of representation of fractions and the role they play in understanding the concept of fractions. This paper presents these studies of the contemporary international literature that show convergent and divergent elements of the findings. This study aims to present researches as a guide and not to analyze in depth which could have a large area and is beyond the aims of this survey.

A. Teaching Errors on Representation of Fractions

Your Muzheve and Capraro [25] talked about idiosyncratic representations in teaching of converting among fractions, decimals, and percent which often help make abstract mathematical concepts more approachable to students. Counters, pictures, imagery, drawings, cutouts, micro-worlds and beans are examples of idiosyncratic representations, unlike the mathematical representations which are characterized by The National Council of Teachers of Mathematics as formal, standard and internationally understood representations used to communicate mathematically. Examples include diagrams, graphical displays and symbolic expressions [69].

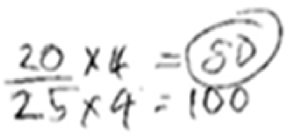

Fig. 1 Use of double equal signs

$$\frac{1}{2} \quad \frac{2}{4} \quad \frac{3}{6} \quad \frac{50}{100} \quad 0.5 \quad 50\%$$

$$\frac{5}{3} \quad 1\frac{2}{3}$$

Fig. 2 Two examples where equals signs were not used

Among idiosyncratic representations was used by teacher, negatively affected the students had the use of two equal signs and it appeared the numerator and denominator of a fraction as two separate entities (figure 1). In addition, some teachers were not explicitly expressing relationships among fractions, decimals and percent using the equal sign (figure 2).

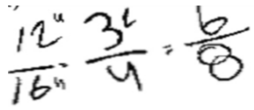

Fig.3 Writing numbers as superscripts

Researches [30, 44, and 2] on the other hand have shown that a common misconception among students is the thinking that there are no relationships among fractions, decimals and percent. Moreover, the use of equations in which numbers are written above and below the equal sign and equations in which numbers are written as superscripts or subscripts (figure 3) had no bearing on the correctness of solutions on the number test of students, the concern then becomes how using such representations can affect related on future learning. For instance, writing $\frac{3^2}{4^2}$ when one really means $\frac{3*2}{4*2}$ (figure 3) can lead to confusion when students are learning about exponents where 3^2 are supposed to be interpreted as 3×3. Therefore, there is a link between the teacher's idiosyncratic representation and student's expressed misconceptions.

Furthermore, the research [25] leads to admit that the hundreds grid was used by teachers to represent fractions and in discussions about converting fractions into percent for easier comparisons did not help as well half of the students were not able to represent 3/5 on a hundreds grid as required for one question on the number test. When teachers used the hundreds grid in the classroom, they emphasized finding a fraction with denominator hundred first and then representing the resulting fraction on the hundreds grid. Taking this approach fails to utilize the part-of-group meaning of fractions that would allow students to shade 3 out of every 5 squares to arrive at 60 out of the 100 squares which shows 3/5=60%, that is, use of the hundreds grid to facilitate converting fractions to percent by teaches in this study relies on the understanding of a fraction as part of a whole, and not as part of a group.

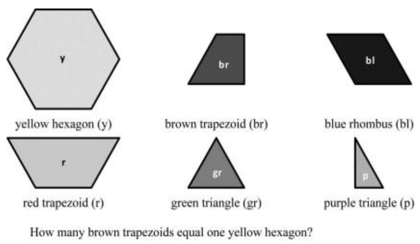

Fig. 4 Pattern blocks

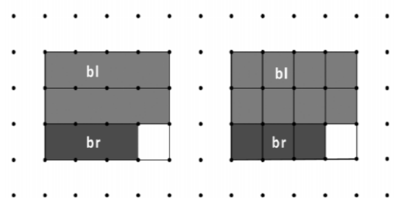

Fig. 5 Dot model

The other research [9], which survey of students fourth and fifth grades, also indicated that pattern blocks (figure 4), fraction chart, dot model (figure 5) and chip model (figure 6) were not effective on students' understanding of the part-whole construct for fractions.

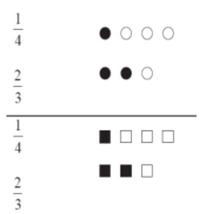

Fig. 6 Chip model

Fig. 7 Illustration for sharing eight pizzas among 10 people

In addition, the wrong drawing of representation can lead to misunderstandings as the study Olive and Vomvoridi [29] shows. In particular, for solving on a problem that involved sharing 8 pizzas among 10 people and determining how much pizza each person would get, it were used circles to represent fractions without partitioning of circle into ten equal parts to be equal in size (figure 7). This representation leads to lack of necessity for parts to be equal.

B. Representation of Fractions on Number Line

Brousseau et al. [5] carried out the experiment in the fourth grade classrooms, with the aim of leading the students day by day to invent, understand and become fluent with all the aspects of rationals and decimals. The lessons were reproduced in two parallel classes by different teachers over a period of more than 15 years, which means that more than 750 students took part in them.

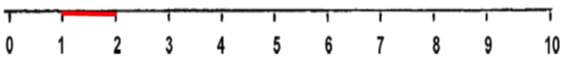

Fig. 8 Interval [1, 2)

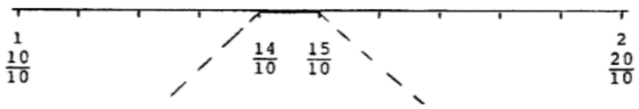

Fig. 9 14/10 and 15/10 on number line

Here paper we will present the Module 5, Lesson 3 which referred representation on the rational number line. The teaching approach concluded following game: The teacher chooses a fraction (145/100, for example), and writes it in a hidden place. The children work in groups of 2 or 3 and write the first intervals in their notebooks. Once the teacher is sure that all the groups have chosen an interval, he asks them one at a time. The children ask: "Is it between 0 and 5? It is between 0 and 3? And so on until they have found an interval of length 1 (in this case, [1, 2)). Teacher draws this interval [1, 2) in color (figure 8). Then Teacher asks the children to find shorter intervals. They start with intervals in tenths. Each time the children propose a new subdivision, the teacher has them come to the board and write the division points as fractions (figure 9). Then, they propose an enlargement of the interval which they will cut into 10 equal pieces. At that point a student will come up and mark both the end points and the intermediate points in hundredths (figure 10). The game continues until the interval [145/100, 146/100) is proposed, at which point the teachers says "Trapped!"

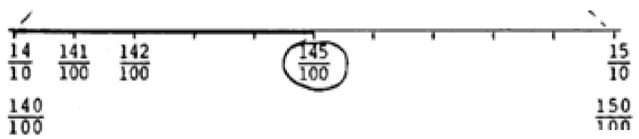

Fig. 10 Number line points in hundredths

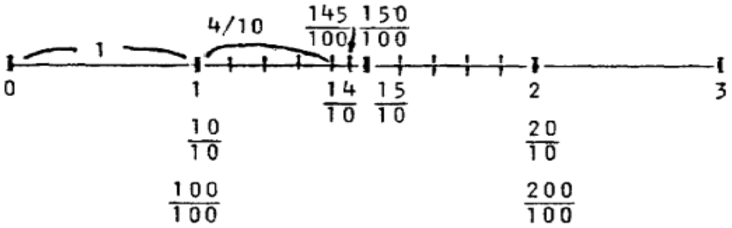

Fig. 11 Number line final form

In the second phase, a student comes up and colors the interval [0, 1) in red, then proposes to divide the interval [1, 2) in ten. He extends the red line to 14/10. The teacher asks what has to be cut in ten. The student shows to the interval [14/10, 15/10) and marks the fraction 145/100. He finishes by marking in red the interval [140/100, 145/100) (figure 11). The teacher then asks to measure this ribbon, how many units we do need, how many tenths beyond the 1 and how many hundredths. And teacher writes on the board: number of units 1, number of 1/10 4 (so 4/10), number of 1/100 5 (so 5/100). So, it must measure 1 + 4/10 + 5/100. In the end, asks a student to come to the board and carry out the addition.

In the end of Module, 5 lesson 3 the students have learned how to put decimal fractions on a number line. Many know how to place them quickly and surely. Some still have difficulties. They are aware that some fractions can't be put on a line subdivided in powers of 10.

In addition, Widjaja et al. [45] used the number line as representation for putting negative fractions and decimals (figure 12). Because of the large percentage of wrong answers in this case, researchers stressed in their study the most important implication for education at school is that the teaching of negative numbers and of the number line must not be confined to integers, as is frequently the case, but must also include negative fractions and decimals.

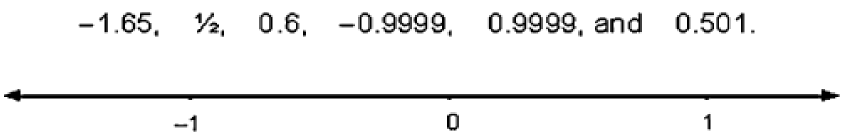

Fig. 12 Locate the following decimals and fractions on the number line

At this point they mention how important is familiarity with number line on development of conceptual understanding of fractions and placement them on number line as research of Hodgen et al. [14] suggested. The same conclusion, the correlation between success of students in a representation to familiarize students with it, and research of Jiang and Chua [17] suggested.

In their article Vamvakoussi and Vosniadou [47] focused on the density property of rational numbers. They present two experiments they explored the instructional value of a cross-domain mapping between "number" and "line" in secondary school students' understanding of density.

In the first experiment, the participants were 229 seventh to eleventh graders; they designed a multiple-choice questionnaire, consisting of three item-blocks of five items each. The items of the first block (hereafter, Numbers-block) asked how many numbers there are between two rational numbers, varying the interval endpoints (e.g., integers, decimals, fractions). The items of the second block (hereafter, Number Line-block) were similar with the items of the Numbers-block, but the two numbers were presented on the number line. The items of the third block (hereafter, Points-block) asked the number of points on a straight line segment, varying the length and the direction of the segment.

In the second experiment they drew on the findings of the first to design a short text-based instructional intervention with the purpose of testing whether the use of the rubber line bridging analogy and explicit use of the numbers-to-points correspondence could help students grasp not only the "infinity many intermediates" but also the "no successor" aspect of density. The participants were 149 eighth and tenth graders. All three texts used in the intervention had a common part T1 (Basic Text) referring to the number of numbers in the interval defined by 0 and 1, which provided the correct answer and reminded students of the numbers-to-points correspondence. T1 continued by evoking the notion of space between 0 and 1 on the number line and presented several examples of decimals lying in the interval. This last paragraph was varied in T2 (Figure Text) (figure 13), with the insertion of two figures illustrating the interval and the given examples of intermediate numbers. These figures were similar to the ones that are typically presented in our participants' textbooks, when presenting numbers on the number line. Finally, T3 (Rubber-line Text) begun with the common part of the text and continued with a paragraph introducing the "rubber line" anchor.

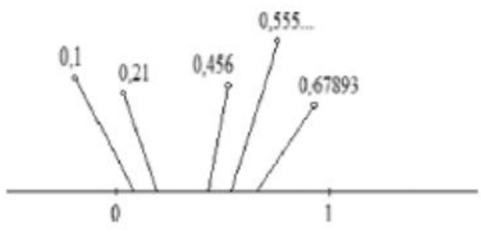

Fig. 13 Figure Text

The text intervention improved student performance in tasks regarding the infinity of numbers in an interval; the "rubber line" bridging analogy further improved performance successfully conveying the idea that these numbers can never be found one immediately next to the other.

The results of their studies show that the number and the ordering of points on a line segment were indeed challenging for the students. They indicated, however, that the infinity of points on a segment is more accessible for students than the infinity of numbers in an interval. In their intervention study they attempted to support students to build the idea of density in a geometrical context, by making explicit the numbers-to point's correspondence and by helping students re-represent the line segment as a dense array of points with the "rubber line" analogy. This bridging analogy proved successful in bringing within the grasp of students the "no successor" aspect of density, which is the most difficult aspect of this notion.

Rather, the merit of this study is that it demonstrates the added instructional value of a specific tool, which can be a valuable component of an efficient learning environment.

C. Representation of the Levels of Units

Conceptual units of various types have played a central role in research on children's understandings of whole and rational numbers. The most important point in the following discussion is the distinction between two and three levels of units, which are explained first in the context of whole numbers. According to Steffe [41, 42], a child who can form two levels of units can understand a whole number such as five simultaneously as five separate units (one level of unit) and as one group of five understood to be a single entity (the whole group is a second level of unit) (figure 14a). Steffe states that such a child has formed composite units. Through interiorization and coordination of composite units, a child can produce three levels of units by nesting composite units within composite units. As an example, a child with such an operation can assimilate a display of 20 blocks as a single group (the whole group is one level) composed of five units (a second level), where each of the five units is also a composite unit composed of four separate units (a third level) (figure 14b). Reasoning with three levels of units requires attending to all three levels simultaneously.

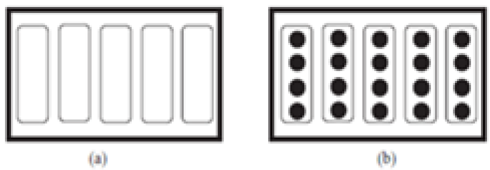

Fig. 14 (a) The number 5 understood with two levels of units (b) The number 20 understood with three levels of units

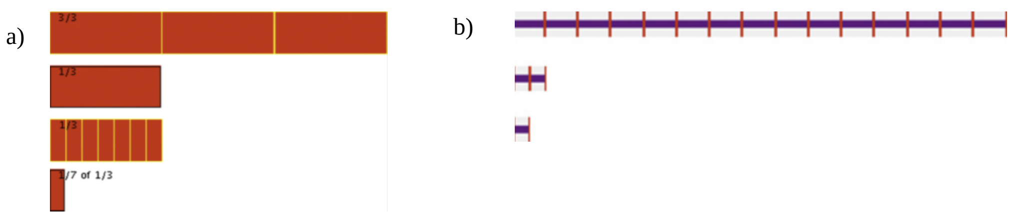

Fig. 15 a) JavaBars: Making 1/7of 1/3 b) Sticks

Hence, in their research Hackenberg and Tillema [13] addressed levels of units on fraction multiplication. For this aim used for representation of fraction multiplication two Microworlds, TIMA: Sticks and Java Bars were developed to allow students to enact key mental actions such as partitioning, iterating, and disembedding, in establishing fraction Microworlds. A central difference between the two programs is that in Sticks students work only on line segments (sticks), whereas in Java Bars students can draw rectangles of varying dimensions (figure 15). In this activity, two pairs of sixth grade students participated in an 8-month teaching experiment that investigated the students' construction of fraction composition schemes in particular, fraction multiplication. The results showed that the interiorization of two levels of units, a particular multiplicative concept, was found to be necessary for the construction of a unit fraction composition scheme, while the interiorization of three levels of units was necessary for the construction of a general fraction composition scheme. In addition, this software facilitated students to consolidate the use of two and three levels of the unit and therefore the concept of multiplication of fractions.

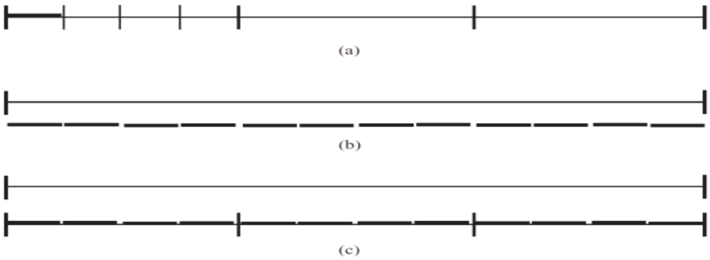

Fig. 16 Determining 1/4 of 1/3 (a) Constructing part of a part (b) Using iteration and two levels of units (c) Using recursive partitioning and three levels of units

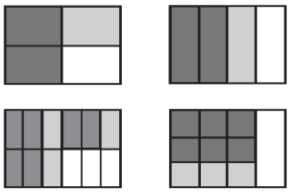

Fig. 17 Four ways to draw 2/3 of 3/4

Other researchers have found similar evidence for connections between students' whole number multiplicative concepts and students' fraction knowledge are Izsák [15], Hackenberg [12] and Empson et al. [10]. In study of Izsák for teaching of fraction multiplication were used the materials that use drawings to represent fractions as length or area quantities (figure 16 and 17). This study builds on the distinction made in past research between reasoning with two and with three levels of quantitative units (figure 14 and 16) and demonstrates that reasoning with three levels of units is necessary but insufficient if teachers are to use students' reasoning with units as the basis for constructing generalized numeric methods for fraction arithmetic. Generally, with using of multiple representations and strategies is providing alternatives from which each student can understand the three levels of the unit and the concept of multiplication of fractions.

Furthermore, Hackenberg approached improper fractions as interiorized three levels of units by using the same above Microworlds, JavaBars. Hackenberg indicates that the construction of improper fractions requires having interiorized three levels of units (figure 18).

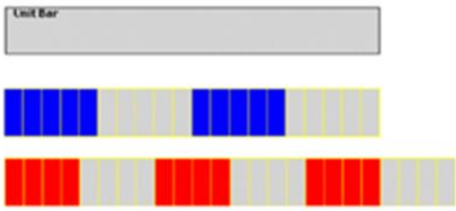

Fig. 18 Representation of 6/5

Students can construct the splitting operation without also interiorizing the coordination of three levels of units and this interiorized coordination appears to be necessary for constructing improper fractions and therefore an iterative fractional scheme.

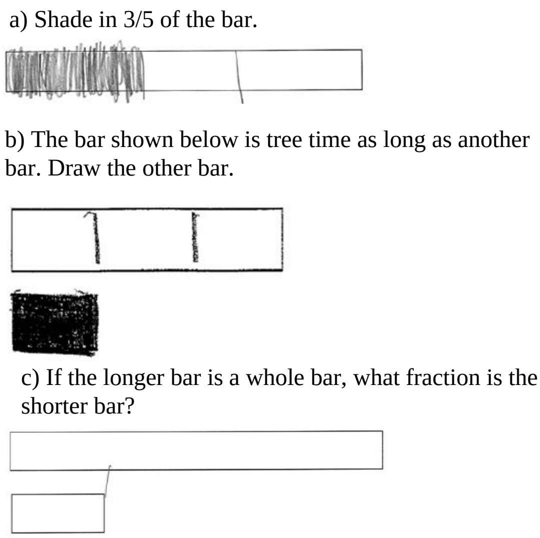

Fig. 19 a) Part–whole fraction scheme b) Splitting operation c) Partitive unit fraction scheme

Results of other research [28] support Hackenberg's hypothesis. In their research, two year professional development study, which took place from 2005 to 2007 involved one fifth-grade classroom and one sixth-grade classroom for each of the two years, with no students involved in both years; a total of 84 students completed tests. The study using representations of figure 19, affirms distinctions between part–whole and partitive reasoning with fractions by measuring significant differences between students' performance on unit and non-unit partitive items and by indicating no significant difference between students' performance on unit and non-unit part–whole items and it suggest that students' construction of a partitive fraction scheme facilitates the development of splitting. In addition, researchers stressed that student who had constructed splitting without even a partitive unit fraction scheme that the partitive fraction scheme and the splitting operation could be constructed independently.

D. Representation of Fractions and Cultural Influence

Interest are researches that compare the representations used by teachers of different countries hence, they have an impact on students' perceptions of the concept of the fraction. Such research is the study Moseley et al. [23] which investigated US and Japanese fourth-grade teachers' domain knowledge of key fraction representations in individual interviews. The results indicated that US and Japanese teachers possess very different facilities for working with an array of rational number perspectives and their representations. That is, Japanese teachers interpreted all rational number representations as conveying primarily mathematical information, whereas US teachers interpreted only some representations as conveying primarily mathematical information. The US teachers also focused more intently on part-whole relations than Japanese in their interpretations and Japanese teachers more easily linked rational number representations to more advanced upcoming content in the curriculum. These differents there are because US textbooks tend to foster limited understanding about mathematical representation with more space devoted to irrelevant pictures and the US teachers apparently received training to teach a narrower range of grades in their pre-service preparation and were more likely to stay at particular grade levels for a longer period of time than their Japanese counterparts who in their pre-service training are required to familiarize themselves with the mathematics curriculum in Grades 1 through 6.

These findings are consistent with Cai and Wang [6] who examined U.S. and Chinese teachers' conceptions and construction of representations to teach the concept of ratio and proportion. The analyses of the data collected showed that none of the U.S. plans has kind of in-depth mathematical analysis. Instead, most U.S. plans require students to directly enter into problem solving after initial introduction of the concept. By devoting the entire lesson to one concept, the Chinese lessons were designed in a more coherent way than were the U.S. lessons. Hence, research Cai and Wang confirms that these identified differences reflect their different cultural beliefs about the teaching and learning of mathematics in general and different cultural values about representations in specific.

The above findings imply that strong cultural differences may help explain established cross-national differences in student reasoning and for responding, for example, why there is the tendency for Chinese students to choose abstract strategies and symbolic representations and for U.S. students to choose concrete strategies and drawing representations in their solving mathematical problems. In particular, the results of research of Moseley and Okamoto [24] indicated that top-performing US students scored significantly higher in problem solving and showed more effectively linked rational number representations than the other groups. The results imply that successful rational number problem solving is intertwined with representational knowledge for a wide range of rational numbers and that the bulk of US students do not possess effective skills for working with rational number representations. This means that the type of cultural influence such as, curricula, pressures and support that are provided to both teachers and students, as the results of Yang et al. [48] also indicated who examined the presentation of fractions in textbooks used by fifth and sixth graders in Singapore, Taiwan, and the United States.

E. Representation of Fraction Operations

Interest As far as the four operations of fraction are concerned, Chen and Li [7] case study examined the features of instructional coherence in one Chinese teacher classroom both within individual lessons and across a sequence of four lessons on fraction division. There were over 50 students in the class. Each lesson lasted about 40 min. He taught at the sixth grade level and teaching was familiar with the textbook.

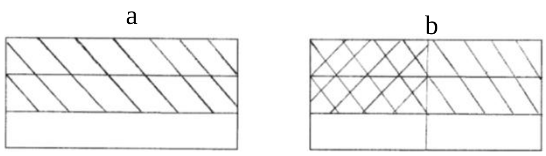

Fig. 20 Pictorial representation to review the relationship between fraction multiplication and fraction division

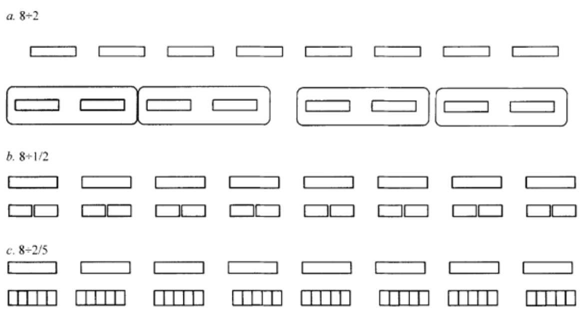

Fig. 21 Unit-changing interpretation

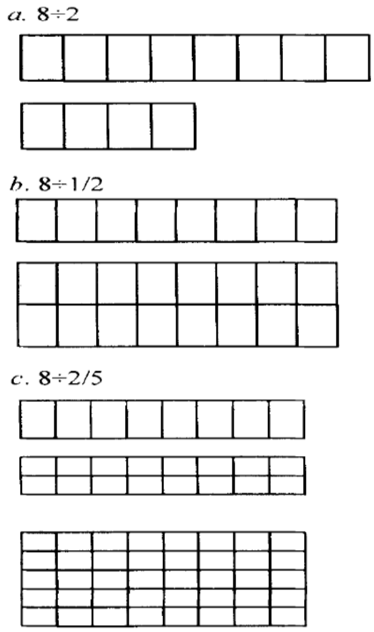

Fig. 22 Unit-keeping interpretation

Through teachings, the teacher and students provided multiple representations for understanding, mainly used line segment and the lesson content was developed within and across activity segments (figure 20). In particular, this pictorial representation of figure 20 used to review the relationship between fraction multiplication and fraction division. Teacher drew a pictorial representation to show 2/3 (figure 20a). Teacher continued the drawing (figure 20b) to finding how much is 1/2 of 2/3 and he confirmed that these two expressions (e.g. 2/3x 1/2=2/3÷2) have the same answer. In this way, a connection was made among the fraction multiplication and even division.

Hence, results suggest that coherent curriculum and the teacher's perception of the knowledge coherence facilitated the teacher's construction of coherent classroom instruction. In addition, the use of the pictorial representation helped students understand the real-world problem. For example, for problem "a car runs 18 km in 3/10 h, what is the speed of this car?" the teacher designs a line segment was divided into ten equal parts, three parts were used to represent that the car runs 18 km in 3/10 h. So, in the third and fourth lessons, the majority of students were able to use at least one way to present and justify the algorithm of fraction division.

Lee and Sztajn [19] also involved division of fractions. In particular, in their study addresses the redefinition of the measurement and partitive interpretations of division from whole number to fractional contexts. That is, they suggest that focusing on the idea of unit can be considered as unit-changing (figure 21) and unit-keeping (figure 22). The unit-keeping and unit-changing interpretations allow for a discussion of why one inverts and multiplies. To understand division in fractional contexts, attention to the unit indicates that, in the process of "transforming" division into multiplication we are searching for the unit of the divisor or the unit of the dividend.

In measurement problems, the dividend and the quotient do not have the same unit; therefore, we call it the unit-changing interpretation of division. Figure 21 shows three problems and representations for the unit changing interpretation of division. In each case, we are counting the divisor and the dividend using the same unit. The quotient, however, is based on a new counting unit that is two, one half, or two-fifths of the dividend's unit and the results of the measuring are four, 16**,** and 20, respectively. One way to think about measurement problems is that we want to count the dividend using the new counting unit defined by the divisor.

In partitive problems, the dividend and the quotient have the same unit leading to what we call the unitkeeping interpretation. Figure 22 also shows the unit-keeping interpretation for the three problems being examined. In these cases, the divisor indicates the relation of the dividend with respect to what we want to find and our goal is to find a new counting unit for the divisor and count (or measure) it in terms of the unit of dividend. So, in the examples we have that eight is two, one half or two-fifths of what we want to find. To solve each of the unit-keeping problems, we construct a counting unit for the divisor and then we count (or measure) the new unit using the unit of the dividend. The task in a partitive problem is to use the counting unit of the dividend to compose and count what is in a counting unit for the divisor.

Hence, Lee and Sztajn [19] contend that the unit-keeping and unit-changing interpretations of division of fractions support development of conceptual understanding of fractions and division.

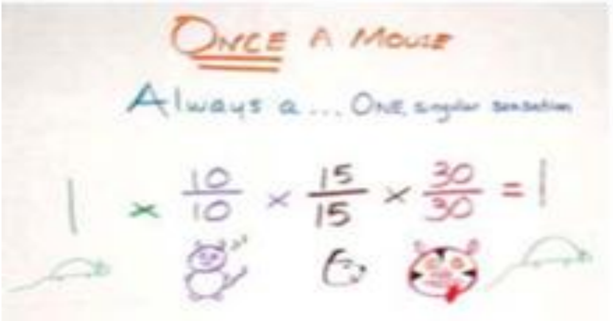

Fig. 23 Mathematical model representations: Fraction equation model

The other research [35] examined the role of visual representations in the learning of mathematics. The study emphasizes the role of multiple representations in creating and recreating new understandings. In the case of fraction, it presents example for multiplication equations (figure 23). In particular, a fraction equation is used to explore the concept of equality and to emphasize that even though the mouse goes through a series of transformations, the new animal is ‘‘still a mouse.’’ In this representation the mouse is assigned an initial value of 1 and each transformation is assigned a fraction expression that is equal to one. Thus the students’ mathematical model express both equality (all animals are assigned fraction values equal to one) and inequality (by the use of different colors). So, mathematical models via multiple representations help students learn a key process used by mathematicians and provide an opportunity to explore how assumptions impact problem solving.

F. Digital Representation of Fractions

The study [36] a six-month ethnographic study of young people’s game play that was conducted as part of a broader effort to understand how people think and learn across settings such as school, work, and home, indicated the positive implications on concept of fractions from use of Video Games’ representations. More specifically, during the game, the player comes into contact with important mathematical topics like rate of change and proportion are the basis of how many games function (figure 24).

Fig. 24 Quantitative representations across video games

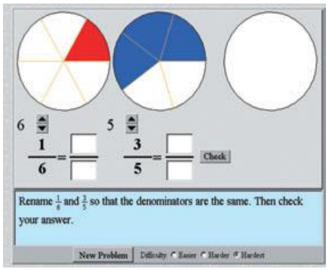

Fig. 25 Example of the interactive websites

In doing so, they use the represented quantities to make predictions about future states of the game and to determine their actions. In other words, in this particular context young people are not just learning to use quantitative representations but using quantitative representations to learn. Furthermore, games are open-ended and allow for multiple ways of achieving success. What’s more, open-ended games often come with well-defined tasks and explicit markers of achievement. This balance allows for the emergence of many different kinds of goal-directed quantitative practices.

Lin [20] indicated also the positive effects of use of web-based in learning fractions (figure 25). In particular, he compares the effectiveness of web-based instruction (WBI) with the traditional lecture in mathematics content and methods for the elementary school course. The results of this study suggest that the use of WBI is significantly more effective as method in providing students with the opportunity to promote their procedural and conceptual knowledge on fractions. Likewise, the interactive websites used for web-based instruction provide a dynamic and animated tool for improving students’ visual and conceptual abilities in learning fractions. These findings are consistent with Fuchs et al., [11] and Schorr and Goldin [37].

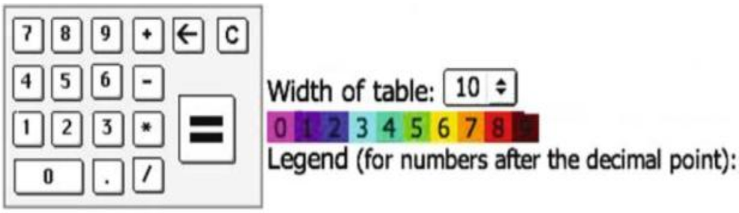

Fig. 26 The Colour Calculator

Another study [40] investigated the possibilities of using a micro world, a web-based colour calculator; both enriched pre-service teachers' experiences with rational numbers and challenged their understandings of several properties related to fractions and their decimal representations. The Colour Calculator is an internet-based calculator that provides numerical results, but that also offers its results in a colour-coded table. Conventional operations are provided, as shown in Figure 26. Each digit of the result corresponds to one of 10 distinctly colored swatches – reflected in a legend – in the table. The calculator operates at a maximum precision of 100 decimals digits, and thus each result is simultaneously represented by a (long) decimal string and a table, or grid of colour swatches. It is possible to change the dimension, or the width, of colour table to values between 1 and 30. Figure 27 shows the result of typing 1/7 into the calculator with the grid width set at 10, thus generating the associated colored table.

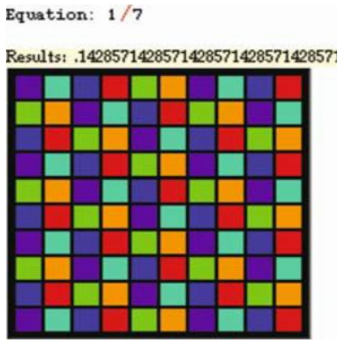

Fig. 27 The Colour Calculator showing 1/7

The findings showed that pre-service teachers' interactions with a web-based colour calculator both enriched their experiences with rational numbers and challenged their understandings of several properties related to fractions and their decimal representations. The novelty and aesthetic appeal of the Colour Calculator – in synchronization with the tasks we chose – helped the participants overcome their reluctance to reengage with properties and relationships associated with these concepts. Through colour, speed, and size of display, previously opaque qualities such as repetition and length of the period became transparent, and thus available for exploration and problem solving. More importantly perhaps, this research participants had the opportunity to engage with representations of fractions in decimal form as objects, rather than as final steps in a procedure, and to construct vivid, memorable, and positive images of important mathematical ideas.

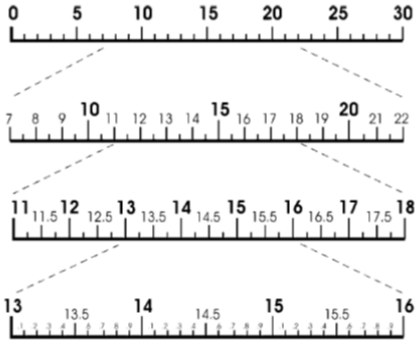

Fig. 28 Several levels of conceptual zooming of a real number line

Likewise, Sedig and Sumner [38] talked about the importance of visual mathematical representations (VMRs) by characterization of computer-based interactions by which learners can explore and investigate this VMRs. Interacting with VMRs means allowing learners to act upon VMRs and receive some form of reactive feedback. Interaction mediates also between the VMR and the thinking, reasoning and intentions of the learner and is often intended to support the cognitive tasks that the learner may want to perform on or with the representation. In case of the fractions, it is presented two micro-level interaction techniques that can be used to support VMR-based mathematical asks.

The first is the zoom on the number line (figure 28). Conceptual zooming increases or decreases the level of detail shown regarding an abstract concept the learner is studying. While discovering the concepts of rational numbers, children benefit from exploring the number line which represents a conceptually deep structure. As children select a region into which to zoom, concepts such as the division of numbers into equal parts can be divided are demonstrated. This can be most effectively shown when the flow of zooming is continuous, allowing learners to see the transition into the number space.

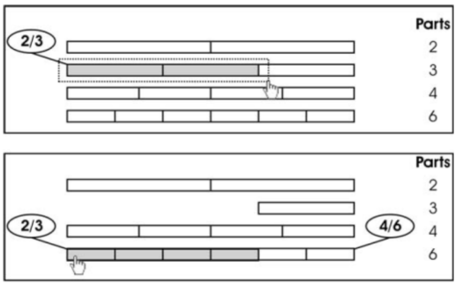

Fig. 29 Fragmenting line segments to discover equivalent fractions

The second interaction technique is fragmenting (figure 29). Fragmenting, whose variants are dissecting, decomposing, partitioning, segmenting, splitting, and unitizing, refers to interacting with a VMR to break it into its component or elemental parts. Fragmenting can allow the learner to see component parts of a VMR and to further interact and reason with those parts. What's more, fragmenting can be used to help learners in understanding concepts such as fractions. For example, a child may apply fragmenting to a set of equal line segments to explore the idea of equivalent fractions.

Another study [34] examined the students' normalizing activity, as they use this kind of dynamic manipulation to modify 'buggy' geometrical figures while developing meanings for ratio and proportion. In particular, students worked in groups of two using 'Turtle worlds', a piece of geometrical construction software which combines symbolic notation, through a programming language (Logo), with dynamic manipulation of geometrical objects by dragging on sliders representing variable values (figure 30). The study was carried out in a Greek secondary school with two 7th grade mixed ability classes with 26 13-year-old students in each class. The results indicate that most of the groups of students had a multiplicity of ways by which they developed part or all of these schemes and accessed different layers of complexity at different times of their engagement with the task. In other words, Logo environments offer possibilities for students to relate the symbolization to algebraic objects and procedures, hence, the integrated use of programming and dragging seemed to play a critical role in enhancing students' shift from visual to mathematical practices to determine proportional relations.

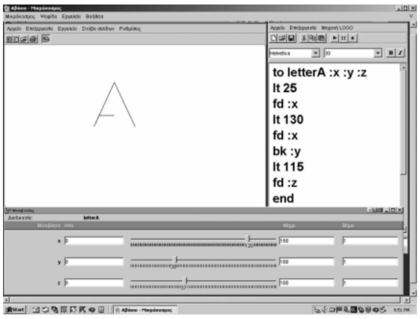

Fig. 30 Example of Turtleworlds

Finally, other study [1] addresses the analysis of the complexity of ratio problems at Grades 6 and 7 and reports a two-year experiment related to the teaching and learning of rational numbers and proportionality in these grades using precise guidelines and a specific computer environment. Two classes were followed and observed. Part of the teaching material was common to classes, mainly the objectives and the corpus of ratio problems in a physical context. But in one class, here called "Partial-experiment" (PEx), the learning environment was exclusively a paper-pencil one and the teacher followed his usual method in designing and conducting teaching sequences. In the other class, here called "Full-experiment" (FEx), the teaching was based on a framework involving precise guidelines and a specific computer environment. Therefore coal, they designed and developed the software series ORATIO and NewOra. ORATIO is designed for introducing rational numbers. It is composed of twenty computer-programmes in two sets and a database (figure 31), while NewOra (figure 32) is about quotients and proportionality in the linear scale register. It is presented after ORATIO, when pupils have been trained to "treatment" and "conversion" tasks.

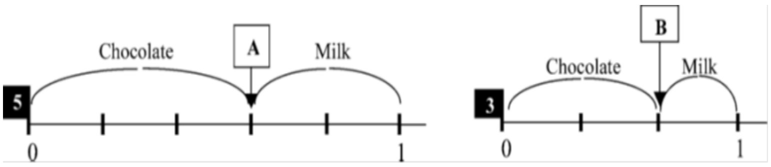

Fig. 31 One interpretation of the mixture-problem according to Gradu4 in ORATIO

This comparative pupil-oriented study indicates more complete improvement in the FEx class, i.e., a better acquisition of fractions and their use for solving usual proportionality problems. In other words, the learning in FEx appears to be more efficient than in PEx. FEx-pupils also give most obvious signs of recognizing proportionality as a mathematical structure underlying the different ratio problems in context. This leads to admit that above software and, generally, systematically working the separations and the articulations between and within the physical and the mathematical domains involved in ratios helps pupils to discern invariants and access the proportionality model.

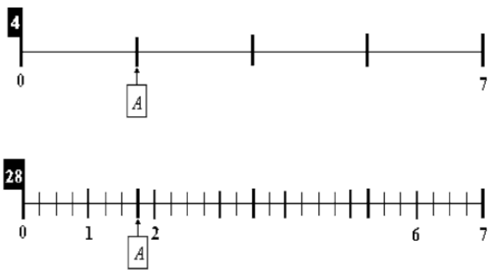

Fig. 32 Using the single scale in NewOra

Hansen et al. [33] report their article the iTalk2Learn. ITalk2Learn is a Fraction Lab with the aim of developing an open-source intelligent tutoring platform that supports math's learning for students aged 5 to 11.

It allows students to learn from a system in a more natural way than ever before. This empowers educators to deliver the right lesson at the right time for every child, enabling personalized learning at scale. In addition, this Fractions Lab utilized a variety of fractions representations including continuous and discrete fractions and fractions in one, two and three dimensions (number line, area/region, and liquid measures, respectively), developing, in this way, children's conceptual understanding with a virtual manipulative (see Fig. 33).

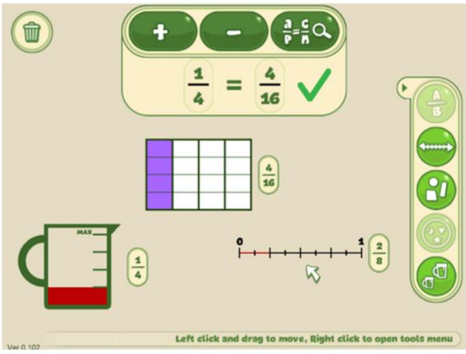

Fig. 33 Fractions Lab showing three representations for 1/4

Other research [3] used the application of Fraction Battles Software which was created by the research team (Fig. 34). This software was about the concepts of the equal parts of a unit, of improper fractions and of the classification of rational numbers on the geometric model of the number line. The software's target was to familiarize students with rational numbers and help them reduce difficulties they face with fractions with the assistance of multiple representations on which the added value of the software through a variety of activities of a dynamic multimedia environment.

Fig. 34 Home page of Fraction Battles software

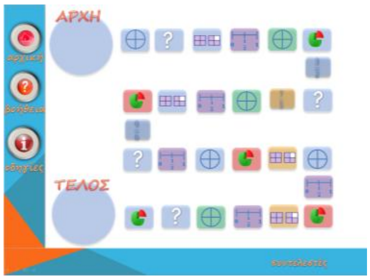

Fig. 35 Digital dashboard of Fraction Battles

In order to win in Fraction Battles, students must arrive at the finish, throwing the dice and following the route shown in Figure 35. The route includes 27 points/activities. Each time a student stops at some of these points, he is asked to answer the question/activity by clicking on the corresponding position. If he answers correctly, he continues, otherwise he waits for his turn again. Each activity is designed to refute some of the difficulties students have in rational numbers, as highlighted by previous research. In addition, activities are graded by difficulty (see Fig. 36).

Fig. 36 Indicative activities of Fraction Battles Software. a) (Left side) activity for translating from one representation of the concept of fraction to another. b) *(Middle side)* activity for improper fractions. c) (*Right side*) activity for the equidivision of a unit into parts

G. *Musician Representation of Fractions*

Courey and Siker [8] examined the effects of an academic music intervention on conceptual understanding of music notation, fraction symbols, fraction size, and equivalency of third graders (ages 8.5–10.11) from a multicultural, mixed socio-economic public school setting. Students which were 67 were assigned by class to their general education mathematics program or to receive academic music instruction two times/week, 45 min/session, for 6 weeks.

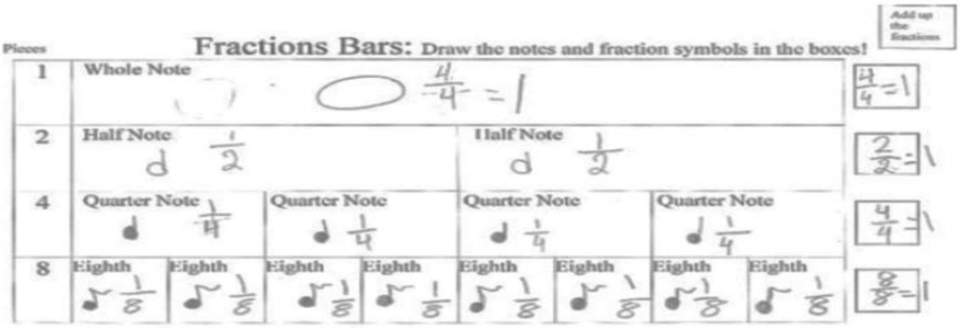

Fig. 33 Sample of student moving between fraction bars, music notes, and fraction symbols

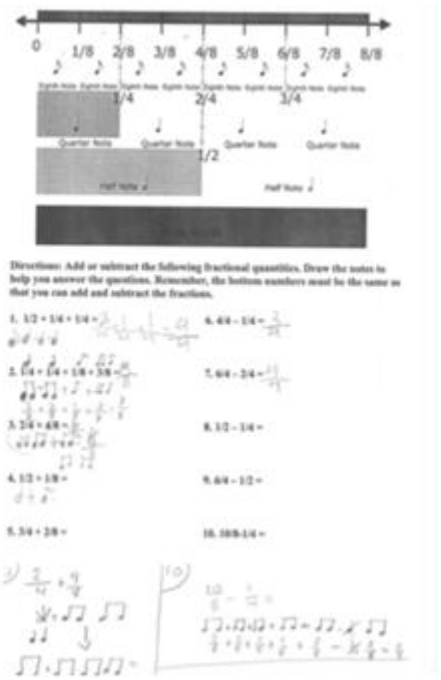

Fig. 34 Sample of student moving between the number line, music notes and fraction symbols, and sample of student using music notes to add fractions with unlike denominators

The first six lessons focused on music notation and the temporal value of music notes in four fourths time. By temporal value, we refer to the relative time duration of each note. The last six lessons focused on connecting the proportional values of the music notes to other signs or fraction representations and then to formal mathematical fraction symbols. The sequence of instruction was as follows: first, students were taught basic music notation for measures in four fourths time and how many beats of each note was equal to a whole note, the largest quantity that could fill a measure in four fourths time. Second, students were taught to connect the fraction symbol with the music note (figure 33). Third, students were taught to add and subtract the fraction quantities, often with unequal denominators, represented by different notes to create measures of four fourths (figure 34). Fourth, students were introduced to other representations of fractional quantities (i.e., fraction circles, fraction tiles, and the number line), taught to compare music notation and fraction symbols, and move comfortably between these representations (figure 35). Fifth, students were taught to add and subtract fractional quantities written as a number sentence by using a representation of choice (e.g., music note and number line) as a conceptual guide.

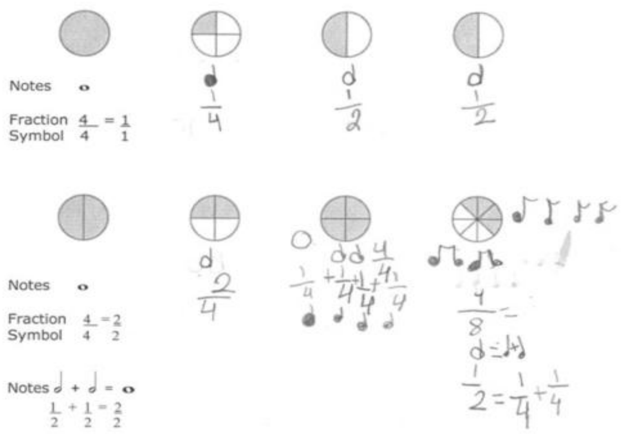

Fig. 35 Sample of student moving between circles, music notes, and fraction symbols

The results show promise for the use of music to teach fraction concepts in the elementary curriculum. We have a compelling reason to view music instruction as an integral part of the elementary curriculum, due to its utility in teaching beginning fraction concepts and related fraction computation to elementary students. Furthermore, this intervention appears to be particularly effective for students who are coming to instruction with a lower than average understanding of fractions. Academic music appears to have strengthened their conceptual understanding of both the magnitude and equivalency of fractions via a semiotic game.

IV. CONCLUSIONS

Our study examined the representation of fractions which have been used by researchers and have been published in international journals from 2006 until today in order to compose a useful guide for teachers and researchers who are investigating the area of fractions.

The results of our research indicated that visual fractional representations by computer and software, such as programming language, Microworlds, web-based, help the participants overcame many cognitive difficulties on fractions reported in the literature. In other words, Computer environment provide a dynamic and animated tool for improving students' visual and conceptual abilities in learning fractions allowing for multiple ways of achieving success.

In addition, use of idiosyncratic representations such as equations with missing equal signs, equations with double equal signs, equations in which numbers are written above and below the equal sign, equations in which numbers are written as superscripts or subscripts and partitioning of a whole into not equal parts can lead students to misunderstandings.

Furthermore, data analyses indicated that representation on the rational number line helps students to learn how to put decimal fractions on a number line, but also it is important there is a familiarity with number line because in doing so develop the conceptual understanding of fractions and placement fractions on number line.

Moreover, students must consolidate the use of two and three levels of the unit. This interiorized coordination appears to be necessary for understanding of multiplication of fractions, partitive fraction scheme, for constructing improper fraction, splitting and an iterative fractional scheme. This is able to achieve with use multiple representations and strategies is providing alternatives from which student can understand the above notions.

Likewise, the use of appropriate representation and mainly visual representations such as the unit-keeping and unit-changing interpretations (partitive interpretations of division from whole number to fractional contexts), pictorial representation to review the relationship between fraction multiplication and fraction division, help students understand the fraction's operations.

Last but not least, strong cultural differences and type of cultural influence such as, curricula, pressures and support that are provided to both teachers and students, affect the understanding of rational number representation, have an impact on students' perceptions of the concept of the fraction and that these differences help explain established cross-national differences in student reasoning on fractions.

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