On Fuzzy Retract of a Fuzzy Loop Space

Gülnur Haçat

MSc. & Department of Mathematics & Kastamonu University
Kastamonu, Turkey

Abstract

Homotopy theory is one of the main areas of algebraic topology. In [7] the concept of H-spaces (Hopf spaces) is introduced from the viewpoint of homotopy theory, then a grouplike space which is a group up to homotopy, called an H-group (Hopf group) is defined. An H-space is a pointed space (X, x_0) *with a constant map* $c: X \to X, x \to x_0$, *for all* $x \in X$ *and a continuous multiplication* $m: X \times X \to X$ *such that* x_0 *is a homotopy identity, that is, the diagram*

$$\begin{tikzcd}
X \arrow[r, "(c,1_X)"] \arrow[dr, "1_X"'] & X \times X \arrow[r, "(1_X,c)"] \arrow[d, "m"] & X \arrow[dl, "1_X"] \\
& X &
\end{tikzcd}$$

commutes up to homotopy: $m \circ (c, 1_X) \simeq 1_X \simeq m \circ (1_X, c)$. *Also if an H-space* (X, x_0) *has a homotopy commutative multiplication* m $(i.e. m \circ (m \times 1_X) \simeq m \circ (1_X \times m))$ *and has a homotopy invers* n *(i.e. there exist a function* $n: X \to X$ *such that* $m \circ (n, 1_X) \simeq c \simeq m \circ (1_X, n)$*), then* (X, x_0) *is called a H-group. If there exist a function* $T: X \times X \to X \times X$, $T(x,y) = (y,x)$ *such that* $m \circ T \simeq m$ *, then H-space* (X, x_0) *is called an abelian H-space.*

In [13] Zadeh introduced the concepts of fuzzy sets. After Chang developed the theory of fuzzy topological spaces [3], basic concepts from homotopy theory were discussed in fuzzy settings. In this direction, Zheng [4] introduced the concept of fuzzy paths. Also, in [2], fuzzy homotopy concepts in fuzzy topological spaces were conceived.

In this study firstly, I define fuzzy H-space and fuzzy H-group. Then I show that a fuzzy loop space ΩX is a fuzzy H-group with the continuous multiplication

$$m: \Omega X \times \Omega X \to \Omega X, \;\; m(\alpha(E), \beta(D)) = \gamma(E + D).$$

Then I show that a fuzzy retract of a fuzzy loop space is a fuzzy H-space. Besides, a fuzzy deformation retract is defined and it is shown that a fuzzy deformation retract of a fuzzy loop space is a fuzzy H-group. Also I show that if ΩX is an abelian fuzzy H-group, then the deformation retract of ΩX is an abelian fuzzy H-group.

Keywords— *Fuzzy Loop Space, Fuzzy H-space, Fuzzy Retract, Fuzzy H-homomorphism.*

I. Introduction and Preliminaries

The notation of fuzzy sets and fuzzy set operations were introduced by Zadeh (1965) in his paper. Subsequently, Chang (1968), Wong (1974) and Lowen (1976) applied some basic concepts from general topology to fuzzy sets. Chang & Wang Jin (1984) and Chuanlin (1985) defined the consept of fuzzy homotopy in fuzzy topological spaces. Besides they studied fuzzy topological. In the following years, the concepts of algebraic topology were generalized to fuzzy topology. Gumus and Yildiz (2007) gave the consept of the pointed fuzzy topological spaces.

Zadeh introduced the values of the membership function in the classical sets concepts, aiming to provide and generalize [0,1] multivalency instead of {0,1} and introduce fuzzy sets. Some concepts from general topology have been applied to fuzzy sets by many scientists [3, 6]. After the publication of Zadeh's Fuzzy Sets in 1965 and the definition of Chang's Fuzzy Topological Space in 1968, concepts in general topology began to move into fuzzy topological spaces.

The concept of Hopf space was introduced by Hopf and subsequently a lot of scientists studied in this field. An Hopf space consists of a pointed topological space P together with a continuous multiplication $m: X \times X \to X$ for which the constant map $c: X \to X$ is a homotopy identity, i.e., $m \circ (1_X, c) \simeq 1_X$ and $m \circ (c, 1_X) \simeq 1_X$.

A group structure can be established on an Hopf space by the homotopy group operations which are like to group operations. This group is called Hopf group. More precisely, an Hopf group is an Hopf space which has an homotopy associative multiplication and an homotopy identity. The classical example to Hopf group is topological groups.

Definition 1.1 [13] Let X be a set. A fuzzy set $A = \{\langle x, A(x)\rangle \mid x \in X\}$ in X is characterized by a membership function $A: X \to [0,1]$ which associates with each point $x \in X$ its "grade of membership" $A(x) \in [0,1]$.

I denote the fuzzy sets of X as $F(X)$, the fuzzy sets as empty set and the universal set X as 0_X and 1_X, respectively.

Definition 1.2 [9] Let A be a fuzzy set in X. The set

$$SuppA = \{x \in X \mid A(x) > 0\}$$

is called the support of fuzzy set A.

Definition 1.3 [6] A fuzzy point p_λ in X is a fuzzy set with the membership function,

$$p_\lambda(x) = \begin{cases} \lambda, & x = p \\ 0, & x \neq p \end{cases}$$

where $0 < \lambda \leq 1$. In other words, the fuzzy set consisting of a single value other than 0 is called a fuzzy point.

Definition 1.4 [11] A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfies the following conditions:

i) $0_X, 1_X \in \tau$
ii) $A, B \in \tau \Rightarrow A \wedge B \in \tau$
iii) If $A_j \in \tau$ for all $j \in J$ (where J is an index set) then $\bigvee A_j \in \tau$.

The pair (X, τ) is called fuzzy topological space. Every member of τ is called τ-fuzzy open set.

Definition 1.5 [1] Let (X, τ) be a fuzzy topological space and $Z \subset X$. Then

$$\tau_Z = \{A \wedge Z \mid Z \in \tau\}$$

is called induced fuzzy topology on Z and the pair (Z, τ_Z) is called fuzzy subspace of (X, τ).

Definition 1.6 [9] Let X and Y be two sets, $f: X \to Y$ be a function and A be a fuzzy set in X, B be a fuzzy set in Y. Then,

(1) the image of A under f is the fuzzy set $f(A)$ defined such that,

$$f(A)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} A(x), & if\ f^{-1}(y) \neq \emptyset, \\ 0 \quad, & otherwise. \end{cases}$$

for all $y \in Y$.

(2) the inverse image of B under f is the fuzzy set $f^{-1}(B)$ in X defined such that $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$.

Definition 1.7 [5] Let (X, τ) and (Y, τ') be two fuzzy topological spaces. A function $f: (X, \tau) \to (Y, \tau')$ is fuzzy continuous if the inverse image of any τ'-fuzzy open set in Y is a τ-fuzzy open set in X, i.e. $f^{-1}(V) \in \tau$, for all $V \in \tau'$.

The set of all fuzzy continuous functions from (X, τ) to (Y, τ') is denoted by $FC(X, Y)$.

Definition 1.8 [7] Let (X, τ) be a fuzzy topological space and p_λ be a fuzzy point in X. The pair (X, p_λ) is called a pointed fuzzy topological space (PFTS) and p_λ is called the base point of (X, p_λ).

Definition 1.9 [3] Let (X, T) be a topological space. Then

$$\tilde{\tau} = \{A \in F(X) \mid SuppA \in T\}$$

is a fuzzy topology on X, called the fuzzy topology on X introduced by T and $(X, \tilde{\tau})$ is called the fuzzy topological space introduced by (X, T). (Gumus and Yildiz, 2007)

Let ε_I denote Euclidean subspace topology on I and $(I, \widetilde{\varepsilon_I})$ denote the fuzzy topological space introduced by the topological space (I, ε_I).

II. FUZZY HOMOTOPY AND FUZZY LOOP SPACE

Definition 2.1 [12] Let $(X, \tau), (Y, \tau')$ be fuzzy topological spaces and f and g are fuzzy continuous functions from X to Y. If there exist a fuzzy continuous function

$$F: (X, \tau) \times (I, \widetilde{\varepsilon_I}) \to (Y, \tau')$$

such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$, then I say f and g are fuzzy homotopic. The mapping F is called fuzzy homotopy from f to g and I write $f \simeq g$. Also, if for a fuzzy point p_λ of $(X, \tau), F(p, t) = f(p) = g(q)$ then f and g are called fuzzy homotopic relative to p_λ. If $f = g$ then $f \simeq g$ with the fuzzy homotopy $F(x, t) = f(x) = g(x)$, for all $t \in [0,1]$.

The fuzzy homotopy relation "$\simeq$" is an equivalence relation. Thus, the set of the fuzzy continuous functions from X to Y is partitioned into equivalence classes under the relation "$\simeq$". The equivalence classes are called fuzzy homotopy classes and the set of all fuzzy homotopy classes of the fuzzy continuous functions from (X, τ) to (Y, τ') is denoted by $[(X, \tau); (Y, \tau')]$. The fuzzy homotopy class of a function f is denoted by $[f]$.

Let (X, p_λ) and (Y, q_η) be pointed fuzzy topological spaces. If $f: (X, p_\lambda) \to (Y, q_\eta)$ is a fuzzy continuous function then it is assumed that all subsets contain the basepoint, f preserves the base point, i.e. $f(p) = p$ and that all fuzzy homotopies are relative to base point.

Definition 2.2 [8] Let (X, p_λ) be a pointed fuzzy topological space, $m: X \times X \to X$ is a fuzzy continuous multiplication and $c: X \to X, c: x \to p$ is a constant function. If $m \circ (c, 1_X) \simeq 1_X \simeq m \circ (1_X, c)$ then (X, p_λ) is called a fuzzy H-space and c is called homotopy identity of (X, p_λ). Here, $(c, 1_X)(x) = (c(x), 1_X(x)) = (p, x)$ for all $x \in X$.

Definition 2.3 [10] Let the PFTS (X, p_λ) be a fuzzy H-space with the fuzzy continuous multiplication m. If there exist a funciton

$$T: X \times X \to X \times X, \;\; T(x, y) = (y, x)$$

which makes the diagram

$$\begin{tikzcd} X \times X \arrow[rr, "\tau"] \arrow[dr, "m"'] & & X \times X \arrow[dl, "m"] \\ & X & \end{tikzcd}$$

fuzzy homotopy commutative, i.e. $m \circ T \simeq m$, then m is called fuzzy homotopy abelian and (X, p_λ) is called an abelian fuzzy H-space.

Definition 2.4 [10] Let (X, p_λ) and (Y, q_η) be fuzzy H-spaces with the fuzzy continuous multiplications m and n, respectively. Then a function $f: (X, p_\lambda) \to (Y, q_\eta)$ is called fuzzy H-homomorphism if the square

$$\begin{tikzcd} X \times X \arrow[r, "m"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, "n"'] & Y \end{tikzcd}$$

is fuzzy homotopy commutative, i.e. $f \circ m \simeq n \circ (f \times f)$.

Definition 2.5 [10] Let the PFTS (X, p_λ) be a fuzzy H-space with the fuzzy continuous multiplication m. If

$$m \circ (m \times 1_X) \simeq m \circ (1_X \times m),$$

then m is called fuzzy homotopy associative. If there exist a fuzzy continuous funciton $\varphi: X \to X$ which makes the diagram

$$\begin{tikzcd} X \arrow[r, "(\varphi, 1_X)"] \arrow[dr, "c"'] & X \times X \arrow[d, "m"] & X \arrow[l, "(1_X, \varphi)"'] \arrow[dl, "c"] \\ & X & \end{tikzcd}$$

fuzzy homotopy commutative, i.e. $m \circ (\varphi, 1_X) \simeq c \simeq m \circ (1_X, \varphi)$, then φ is called fuzzy homotopy inverse.

Definition 2.6 [12] A fuzzy H-group is a fuzzy H-space which has a fuzzy homotopy associative multiplication and a fuzzy homotopy inverse.

Definition 2.7 [4] Let (X, τ) be a fuzzy topological space. If $\alpha: (I, \widetilde{\varepsilon_I}) \to (X, \tau)$ is a fuzzy continuous function and the fuzzy set E is connected in $(I, \widetilde{\varepsilon_I})$ with $E(0) > 0$ and $E(1) > 0$, then the fuzzy set $\alpha(E)$ in (X, τ) is called a fuzzy path in (X, τ). The fuzzy point $(\alpha(0))_{E(0)} = \alpha(0_{E(0)})$ and $(\alpha(1))_{E(1)} = \alpha(1_{E(1)})$ are called the initial point and the terminal point of the fuzzy path $\alpha(E)$, respectively.

Definition 2.8 [4] Let A be a fuzzy set in a fuzzy topological space (X, τ). If for any two fuzzy points $a_\lambda, b_\eta \in A$, there is a fuzzy path contained in A with initial point a_λ and terminal point b_η, then A is said to be fuzzy path connected in (X, τ).

Definition 2.9 [13] Let (X, p_λ) be a pointed fuzzy topological space. A fuzzy path $\alpha(A)$ which the initial point and the terminal point are $p_\lambda \in X$, is called a fuzzy loop in (X, p_λ) . The set of all the fuzzy loops in (X, p_λ) is called fuzzy loop space and this space is a fuzzy topological space having the fuzzy compact open topology.

We can think the fuzzy loop space of (X, p_λ) as a pointed fuzzy topological space with a constant path $\omega_0(A)$ which is equal to the base point p_λ of (X, p_λ) at any point. Let denote this space by ΩX.

Definition 2.10 Let $\alpha(E)$ and $\beta(D)$ be fuzzy loops in ΩX at p_λ, then we define

$$(E + D)(t) = \begin{cases} E(2t), & 0 \le t \le \frac{1}{2} \\ D(2t-1)\,, & \frac{1}{2} \le t \le 1. \end{cases}$$

By Lemma 3 in [6], $E + D$ is a connected fuzzy set in $(I, \widetilde{\varepsilon_I})$ with $(E + D)(0) > 0$ and $(E + D)(1) > 0$. Let $\gamma: (I, \widetilde{\varepsilon_I}) \to (X, \tau)$ be a fuzzy continuous function defined by

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

There is a fuzzy continuous function

$$m: \Omega X \times \Omega X \to \Omega X$$

defined by

$$m(\alpha(E), \beta(D)) = \gamma(E + D).$$

Theorem 2.11 Fuzzy loop space ΩX is a fuzzy H-space.

Proof. Let be $\alpha(E), \beta(D) \in \Omega X$ and $m: \Omega X \times \Omega X \to \Omega X$ be a fuzzy continuous multiplication function such that,

$$m(\alpha(E), \beta(D))(t) = \begin{cases} \alpha((2t)_{E(2t)}), & 0 \le t \le \frac{1}{2} \\ \beta\big((2t-1)_{D(2t-1)}\big), & \frac{1}{2} \le t \le 1. \end{cases}$$

m can be taken as a fuzzy continuous multiplication of X. Let $c: \Omega X \to \Omega X$ be a constant function that takes all fuzzy loops to $\omega_p(A)$ and let $1_{\Omega X}: \Omega X \to \Omega X$ be the unit function.

Now let show that the diagram,

$$\begin{tikzcd} \Omega X \arrow[r, "(c,1_{\Omega X})"] \arrow[dr, "1_{\Omega X}"'] & \Omega X \times \Omega X \arrow[r, "(1_{\Omega X},c)"] \arrow[d, "\gamma"] & \Omega X \arrow[dl, "1_{\Omega X}"] \\ & \Omega X & \end{tikzcd}$$

provides $m \circ (1_{\Omega X}, c) \simeq 1_{\Omega X} \simeq m \circ (c, 1_{\Omega X})$.

$$\begin{aligned} (m \circ (1_{\Omega X}, c))(\alpha(E))(t) &= m(\alpha(E), \omega_p(A))(t) \\ &= \begin{cases} \alpha((2t)_{E(2t)}), & 0 \le t \le \frac{1}{2} \\ p_\lambda, & \frac{1}{2} \le t \le 1. \end{cases} \end{aligned}$$

$$\begin{aligned} (m \circ (c, 1_{\Omega X}))(\alpha(E))(t) &= m(\omega_p(A), \alpha(E))(t) \\ &= \begin{cases} p_\lambda, & 0 \le t \le \frac{1}{2} \\ \alpha((2t-1)_{E(2t-1)}), & \frac{1}{2} \le t \le 1. \end{cases} \end{aligned}$$

and $(1_{\Omega X})(\alpha(E))(t) = \alpha(E)(t) = \alpha(t_{E(t)})$. Let $F: \Omega X \times (I, \widetilde{\varepsilon_I}) \to \Omega X$ be a function such that,

$$F(\alpha(E), t)(t') = \begin{cases} \alpha\left(\left(\frac{2t'}{1+t}\right)_{E\left(\frac{2t'}{1+t}\right)}\right), & 0 \le t' \le \frac{1+t}{2} \\ p_\lambda, & \frac{1+t}{2} \le t' \le 1. \end{cases}$$

$$= (\gamma \circ (1_{\Omega X}, c))(\alpha(E))$$

Then F is fuzzy continuous and $m \circ (1_{\Omega X}, c) \simeq 1_{\Omega X}$ since,

$$F(\alpha(E),0)(t') = \begin{cases} \alpha((2t')_{E(2t')}), & 0 \le t' \le \frac{1}{2} \\ p_\lambda\,, & \frac{1}{2} \le t' \le 1. \end{cases}$$
$$= (\gamma \circ (1_{\Omega X}, c))(\alpha(E))$$

and
$$F(\alpha(E),1)(t') = \alpha((t')_{E(t')}) = (1_{\Omega X})(\alpha(E))(t).$$

By the same way $1_{\Omega X} \simeq m \circ (c, 1_{\Omega X})$. So $m \circ (1_{\Omega X}, c) \simeq 1_{\Omega X} \simeq m \circ (c, 1_{\Omega X})$. Consequently, the fuzzy loop space ΩX is an H-space.

Theorem 2.12 ΩX fuzzy loop space is a fuzzy H-group.

Proof.
i) $m: \Omega X \times \Omega X \to \Omega X$ fuzzy continuous multiplication is defined as in Teorem 2.11.
Let show that the,

$$\begin{array}{ccc} \Omega X \times \Omega X \times \Omega X & \xrightarrow{m \times 1_{\Omega X}} & \Omega X \times \Omega X \\ {\scriptstyle 1_{\Omega X} \times m} \downarrow & & \downarrow {\scriptstyle m} \\ \Omega X \times \Omega X & \xrightarrow[m]{} & \Omega X \end{array}$$

diagram provides $m \circ (m \circ 1_{\Omega X}) \simeq m \circ (1_{\Omega X} \circ m)$ homotopies to show that ΩX fuzzy loop space is homotopy associative.

$$(m \circ (m, 1_{\Omega X}))(\alpha(E), \beta(D), \zeta(C))(t) = m(m(\alpha(E), \beta(D)), \zeta(C))(t)$$

$$= \begin{cases} m(\alpha(E), \beta(D))(2t), & 0 \le t \le \frac{1}{2} \\ \zeta\big((2t-1)_{C(2t-1)}\big), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \begin{cases} \alpha((4t)_{E(4t)}), & 0 \le t \le \frac{1}{4} \\ \beta\big((4t-1)_{D(4t-1)}\big), & \frac{1}{4} \le t \le \frac{1}{2} \\ \zeta\big((2t-1)_{C(2t-1)}\big), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$(m \circ (1_{\Omega X}, m))(\alpha(E), \beta(D), \zeta(C))(t) = m(\alpha(E), m(\beta(D), \zeta(C)))(t)$$

$$= \begin{cases} \alpha((2t)_{E(2t)}), & 0 \le t \le \frac{1}{2} \\ m(\beta(D), \zeta(C))(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \begin{cases} \alpha((2t)_{E(2t)}), & 0 \le t \le \frac{1}{2} \\ \beta\big((4t-2)_{D(4t-2)}\big), & \frac{1}{2} \le t \le \frac{3}{4} \\ \zeta\big((4t-3)_{C(4t-3)}\big), & \frac{3}{4} \le t \le 1 \end{cases}$$

Let a function $F: \Omega X \times \Omega X \times \Omega X \times (I, \widetilde{\varepsilon_I}) \to \Omega X$ be defined such that,

$$F(\alpha(E), \beta(D), \zeta(C), t)(t') = \begin{cases} \alpha\left(\left(\frac{4t'}{1+t}\right)_{E\left(\frac{4t'}{1+t}\right)}\right), & 0 \le t' \le \frac{1+t}{4} \\ \beta\left((4t'-t-1)_{D(4t'-t-1)}\right), & \frac{1+t}{4} \le t' \le \frac{t+2}{4} \\ \zeta\left(\left(\frac{4t'-t-2}{2-t}\right)_{C\left(\frac{4t'-t-2}{2-t}\right)}\right), & \frac{t+2}{4} \le t' \le 1. \end{cases}$$

$F(\alpha(E), \beta(D), \zeta(C), 0) = (m \circ (m, 1_{\Omega X}))(\alpha(E), \beta(D), \zeta(C))$,
$F(\alpha(E), \beta(D), \zeta(C), 1) = (m \circ (1_{\Omega X}, m))(\alpha(E), \beta(D), \zeta(C))$.

So $m \circ (m, 1_{\Omega X}) \simeq m \circ (1_{\Omega X}, m)$ and m is homotopy associative.

(ii) To show that ΩX has fuzzy homotopy invers, let $\phi: \Omega X \to \Omega X$ be a function be defined as $\phi(\alpha(E))(t) = \alpha(E)(1-t)$. Let show that the diagram,

$$\begin{tikzcd} \Omega X \arrow[r, "(\phi, 1_{\Omega X})"] \arrow[dr, "1_{\Omega X}"'] & \Omega X \times \Omega X \arrow[r, "(1_{\Omega X}, \phi)"] \arrow[d, "\gamma"] & \Omega X \arrow[dl, "1_{\Omega X}"] \\ & \Omega X & \end{tikzcd}$$

provides $m \circ (\phi, 1_{\Omega X}) \simeq 1_{\Omega X} \simeq m \circ (1_{\Omega X}, \phi)$.

$$(m \circ (\phi, 1_{\Omega X}))(\alpha(E))(t) = m(\alpha(E)(1-t), \alpha(E)(t)) = m(\alpha^{-1}(E)(t), \alpha(E)(t))$$

$$= \begin{cases} \alpha^{-1}\left((2t)_{E(2t)}\right), & 0 \le t \le \frac{1}{2} \\ \alpha\left((2t-1)_{E(2t-1)}\right), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \begin{cases} \alpha\left((1-2t)_{E(1-2t)}\right), & 0 \le t \le \frac{1}{2} \\ \alpha\left((2t-1)_{E(2t-1)}\right), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$(m \circ (1_{\Omega X}, \phi))(\alpha(E))(t) = m(\alpha(E)(t), \alpha(E)(1-t)) = m(\alpha(E)(t), \alpha^{-1}(E)(t))$$

$$= \begin{cases} \alpha\left((2t)_{E(2t)}\right), & 0 \le t \le \frac{1}{2} \\ \alpha^{-1}\left((2t-1)_{E(2t-1)}\right), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \begin{cases} \alpha\left((2t)_{E(2t)}\right), & 0 \le t \le \frac{1}{2} \\ \alpha\left((2-2t)_{E(2-2t)}\right), & \frac{1}{2} \le t \le 1. \end{cases}$$

Let $G: \Omega X \times (I, \widetilde{\varepsilon_I}) \to \Omega X$ be a function such that,

$$G(\alpha(E), t)(t') = \begin{cases} \alpha\left((1-2t'+t)_{E(1-2t'+t)}\right), & 0 \le t' \le \frac{1}{2} \\ \alpha\left((2t'+t-1)_{E(2t'+t-1)}\right), & \frac{1}{2} \le t' \le 1. \end{cases}$$

$G(\alpha(E),0)(t') = \big(m \circ (\phi, 1_{\Omega X})\big)\big(\alpha(E),(t')\big)$, $G(\alpha(E),1)(t') = \alpha(E)(t') = 1_{\Omega X}(\alpha(E),(t'))$. Consequently $m \circ (\phi, 1_{\Omega X}) \simeq c$.

Let $G': \Omega X \times (I, \widetilde{\varepsilon_I}) \to \Omega X$ be a function such that,

$$G'(\alpha(E),t)(t') = \begin{cases} \alpha\Big((2t'-t)_{E(2t'-t)}\Big), & 0 \le t' \le \frac{1}{2} \\ \alpha\Big((2-2t'-t)_{E(2-2t'-t)}\Big), & \frac{1}{2} \le t' \le 1. \end{cases}$$

$G(\alpha(E),0) = \big(m \circ (1_{\Omega X}, \phi)\big)\big(\alpha(E)\big)$, $G(\alpha(E),1) = 1_{\Omega X}(\alpha(E))$. Consequently $m \circ (1_{\Omega X}, \phi) \simeq c$. So ϕ is fuzzy homotopy inverse. By (i) and (ii) ΩX is a fuzzy H-group.

III. FUZZY RETRACT OF A FUZZY LOOP SPACE

In this part, I show that fuzzy retract of a fuzzy loop space is a fuzzy H-space and fuzzy deformation retract of a fuzzy loop space is a fuzzy H-group.

Definition 3.1 A fuzzy subspace (Z, τ_Z) of a fuzzy topological space (X, τ) is called a fuzzy retract of (X, τ) if there exists a fuzzy continuous map

$$r: (X, \tau) \to (Z, \tau_Z)$$

such that $r(a) = a$, for all $a \in Z$. The map r is called a fuzzy retraction.

Theorem 3.2 A fuzzy retract of ΩX is an H-space.

Proof. Let ΩA be a fuzzy subspace of fuzzy loop space ΩX and $r: \Omega X \to \Omega A$ be the fuzzy retraction. It is clear that for the inclusion map $i: \Omega A \to \Omega X, r \circ i = 1_{\Omega A}$. Let m be the fuzzy continuous multiplication of ΩX. Then,

$$\Omega A \times \Omega A \xrightarrow{i \times i} \Omega X \times \Omega X \xrightarrow{m} \Omega X \xrightarrow{r} \Omega A.$$

Let $n = r \circ m \circ (i \times i)$, then n is a fuzzy continuous multiplication of ΩA. Let show that,

$$n \circ (c', 1_{\Omega A}) \simeq 1_{\Omega A} \simeq n \circ (1_{\Omega A}, c')$$

for the constant map $c'(\alpha(E)) = c \mid_{\Omega A} (\alpha(E)) = c(\alpha(E)) = \omega_p(A)$ for all $\alpha(E) \in \Omega A$. It is clear that

$$(i \times i) \circ (1_{\Omega A}, c') = (1_{\Omega A}, c) \circ i.$$

Since ΩX is a fuzzy loop space, then

$$m \circ (c, 1_{\Omega X}) \simeq 1_{\Omega X} \simeq m \circ (1_{\Omega X}, c)$$

Therefore

$$\begin{aligned} n \circ (1_{\Omega A}, c') &= (r \circ m \circ (i \times i)) \circ (1_{\Omega A}, c') \\ &= (r \circ m \circ (1_{\Omega X}, c) \circ i) \simeq r \circ 1_{\Omega X} \circ i \\ &= r \circ i = 1_{\Omega A}. \end{aligned}$$

In a similar way $n \circ (c', 1_{\Omega A}) = r \circ m \circ (c, 1_{\Omega X}) \circ i \simeq r \circ i = 1_{\Omega A}$. Consequently ΩA is a fuzzy H-space.

Theorem 3.3 A fuzzy retract of a fuzzy topological space (X, τ) is called a fuzzy deformation retract of (X, τ) if there exists a fuzzy homotopy such that $i \circ r \simeq 1_X$ for the inclusion map i and the fuzzy retraction r.

Therefore, we obtain the following corollary.

Corollary 3.4 A fuzzy deformation retract of a fuzzy loop space is a fuzzy H-space.

Theorem 3.5 A fuzzy deformation retract of a fuzzy loop space is a fuzzy H-group.

Proof. Let a fuzzy loop space ΩA be a fuzzy deformation retract of ΩX and m and $n = r \circ m \circ (i \times i)$ be fuzzy continuous multiplications of ΩX and ΩA, respectively, for the inclusion map i and the fuzzy retract r. Then by Theorem 3.2 ΩA is a fuzzy loop space. Since ΩX is a fuzzy H-group, it has a fuzzy homotopy inverse, m is fuzzy homotopy associative. As $r \circ i = 1_{\Omega A}$,

$$\begin{aligned} n \times 1_{\Omega A} &= (r \circ m \circ (i \times i)) \times 1_{\Omega A} \\ &= (r \circ m \circ (i \times i)) \times (r \circ i) \\ &= (r \circ m \circ (i \times i)) \times (r \circ 1_{\Omega X} \circ i) \\ &= (r \times r) \circ (m \times 1_{\Omega X}) \circ (i \times i \times i) \end{aligned}$$

As a similar way $1_{\Omega A} \times n = (r \times r) \circ (1_{\Omega X} \times m) \circ (i \times i \times i)$. Then,

$$\begin{aligned} n \circ (n \times 1_{\Omega A}) &= r \circ m \circ (i \times i) \circ (r \times r) \circ (m \times 1_{\Omega X}) \circ (i \times i \times i) \\ &\simeq r \circ m \circ 1_{X \times \Omega X} \circ (m \times 1_{\Omega X}) \circ (i \times i \times i) \\ &= r \circ (m \circ (m \times 1_{\Omega X})) \circ (i \times i \times i) \\ &\simeq r \circ (m \circ (1_{\Omega X} \times m)) \circ (i \times i \times i) \\ &= r \circ m \circ 1_{X \times \Omega X} \circ (1_{\Omega X} \times m) \circ (i \times i \times i) \\ &\simeq r \circ m \circ (i \times i) \circ (r \times r) \circ (1_{\Omega X} \times m) \circ (i \times i \times i) \\ &= n \circ (1_{\Omega A} \times n). \end{aligned}$$

Therefore n is fuzzy homotopy associative.

Let ϕ be fuzzy homotopy inverse for ΩX and $\phi' = r \circ \phi \circ i$. Since ΩA is a fuzzy deformation retract of ΩX, $i \circ r \simeq 1_{\Omega X}$. Then

$$\begin{aligned} (i \times i) \circ (1_{\Omega A}, \phi') &= (i, i \circ \phi') \\ &= (i, i \circ (r \circ \phi \circ i)) \simeq (i, \phi \circ i) = (1_{\Omega X}, \phi) \circ i \\ &\Rightarrow r \circ m \circ (i \times i) \circ (1_{\Omega A}, \phi') \simeq r \circ m \circ (1_{\Omega X}, \phi) \circ i. \end{aligned}$$

Thus $n \circ (1_{\Omega A}, \phi') \simeq r \circ m \circ (1_{\Omega X}, \phi) \circ i \simeq r \circ c \circ i = c'$, where $c' = c \mid_{\Omega A}$, because $(r \circ c \circ i)(a(E)) = (r \circ c)(a(E)) = r(c(a(E))) = r(\omega_p(A)) = \omega_p(A) = c'(a(E))$, for all $a \in A$.

In a similar way $n \circ (\phi', 1_{\Omega A}) \simeq c'$. Hence ϕ' is a fuzzy homotopy inverse for ΩA. Consequently ΩA is a fuzzy H-group.

Then we have the following corollary.

Corollary 3.6 A fuzzy deformation retract of an abelian fuzzy loop space is an abelian fuzzy H-group.

IV. CONCLUSIONS

In this study firstly, I defined fuzzy H-space and fuzzy H-group. Then I show that a fuzzy loop space ΩX is a fuzzy H-group with the continuous multiplication m. Then I show that a fuzzy retract of a fuzzy loop space is a fuzzy H-space. Besides, a fuzzy deformation retract is defined and it is shown that a fuzzy deformation retract of a fuzzy loop space is a fuzzy H-group. Also, I show that if ΩX is an abelian fuzzy H-group, then the deformation retract of ΩX is an abelian fuzzy H-group.

ACKNOWLEDGMENT

I would like to express my thanks to dear S. Demiralp for valuable discussions on the problems considered here.

REFERENCES

[1] A.R. Salleh, The homotopy property of the induced homomorphisms on homology groups of fuzzy topological spaces, Fuzzy Sets and Systems 56 (1993),111-116.
[2] C. G. Aras and S. Bayramov, On fuzzy exact homotopy sets, Southeast Asian Bulletin of Mathematics, vol. 34, (2010), pp. 1009-1022.
[3] C.L. Chang, Fuzzy topological space, J. Math. Anal. Appl., 24(1968), 182-190.
[4] C. Zheng, "Fuzzy path and fuzzy connectedness", Fuzzy Sets and Systems 14(1984), 273-280.
[5] D.H. Foster, Fuzzy topological groups, J. Math. Anal. Appl. 67 (1979), 549-564.
[6] E. Güner, Fuzzy Contractibility, Commun. Fac. Sci. Univ. Ank. Series A1 56(2) (2007),11-16.
[7] J. Stashe§, H-Spaces from a Homotopy Point of View, Springer-Verlag (1970), 1-16.
[8] S. Demiralp and E. Güner, Fuzzy Loop Spaces Having Hopf Group Structure, Journal of Selçuk University Natural and Applied Science 2(4),(2013), 21-28 .
[9] S. Demiralp and E. Güner, Some Characterizations of Hopf Group on Fuzzy Topological Space, Iranian Journal of Fuzzy Systems 11(6),(2014),111-121.
[10] S. Demiralp, E.H. Hamouda and E. Güner, Some Properties of Fuzzy H-spaces, JP Journal of Algebra, Number Theory and Applications 40(4),(2018), 429-448.
[11] S. Gumus and C. Yildiz, The fuzzy sheaf of the groups formed by fuzzy topological space, Int. J. Pure Appl. Math. 37(1) (2007),101-112.
[12] Y. Z. Chang, and L. Wang Jin. "Fuzzy homotopy type invariance of fundamental group of fuzzy topological space." Fuzzy Math 4 (1984): 53-56.
[13] Zadeh, Lotfi A. "Fuzzy sets." Fuzzy Sets, Fuzzy Logic, And Fuzzy Systems: Selected Papers by Lotfi A Zadeh. 1996. 394-432.