Existence and Uniqueness of a Fuzzy Solution for Some Fuzzy Neutral Partial Differential Equation with Nonlocal Condition

Atimad Harir[*#1], Said Melliani[#2], L.Saadia Chadli[#3]
[#]*Laboratory of Applied Mathematics and Scientific Computing,*
Sultan Moulay Slimane University,P.O. Box 523, Beni Mellal, 23000,Morocco

Abstract

In this work, we establish several results about the existence of fuzzy solutions for some Fuzzy Neutral partial Differential Equation with nonlocal condition. Our approach rest on the Banach fixed-point theorem.

Keywords - *Neutral partial Differential Equation, Fuzzy mild Solution, nonlocal condition, fuzzy semigroups of linear operators.*

I. INTRODUCTION

–In this work, we study the existence of fuzzy solutions for fuzzy neutral partial differential equations with nonlocal conditions of the following from:

$$\begin{cases} \frac{d}{dt}\left[x(t) \ominus F\left(t, x(h_1(t))\right)\right] = A\left[x(t) \ominus F\left(t, x(h_1(t))\right)\right] \oplus G\left(t, x(h_2(t))\right), \quad 0 \le t \le a, \\ x(0) \oplus g(x) = x_0 \in E^n \qquad (1) \end{cases}$$

Where $A\colon E^n \to E^n$ is fuzzy operator, is the infinitesimal generator of an C_0 -semigroup on E^n and E^n is the set of all upper semi continuous, convex, normal fuzzy numbers with bounded a-level intervals, called spaces of fuzzy numbers, or more general with values in E^n , where $(E^n, \oplus, \odot, D)$ represents any from the fuzzy number type spaces introduced by section 2, and

$F, G : [0, a] \times E^n \to E^n$, $g : C([0,a]; E^n) \to E^n$ and $h_i \in C([0,a];[0,a])$, $i = 1, 2$.

The very recent paper [10] is concerned with equation 1 in Banach space X, the theory of neutral differential equations has many applications, for the reader, we refer to [1,3]. Differential equations with nonlocal conditions have been studied extensively in the literature. The importance of nonlocal conditions in different fields has been discussed [4,5,9] and the references therein. In the past several years theorems about existence, uniqueness and stability of differential equations with nonlocal conditions have been studied by Byszewski and Lakshmikantham [6], Balachandran and Chandrasekaran [2], Lin and Liu [14]and Ntouyas and Tsamatos [15]. but it is known that the classical of neutral differential equations whose solution are real valued functions (or Banach space valued function, respectively) often represent an idealization of real situations, where imprecision may in fact play a significant role.

Generally, several systems are mostly related to uncertainty and inexactness. The problem of inexactness is considered in general exact science, and that of uncertainty is considered as vague or fuzzy and accident. Ding and Kandel [7,16-19] analyzed a way to combine differential equations with fuzzy sets to form a fuzzy logic system called a fuzzy dynamical system, which can be regarded to form a fuzzy neutral functional differential equation.

Note that with respect to Banach spaces, the fuzzy number type spaces $(E^n, \oplus, \odot, D)$ represent more general structures, in the sense that although the metric has similar properties with a metric derived from a norm of Banach space, however $(E^n, \oplus, \odot, D)$ with respect to the addition $\oplus$ is not a group and with respect to the scalar multiplication is not linear space .

The organization of this work is as follows:in Section 2, we call some fundamental results on fuzzy numbers. In Section 3 we study the existence of fuzzy mild solutions.

II. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let T = [c, d] ⊂ ℝ be a compact interval and denote $E^n = \{u| \ u : \mathbb{R}^n \to [0,1]$ satisfies (1)-(4) below $\}$ where

1. u is normal i.e, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$
2. u is fuzzy convex i.e for $x, y \in \mathbb{R}^n$ and $0 < \lambda \leq 1,\ u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$
3. u is upper semi-continuous on ℝ.
4. $[u]^0$ is a compact set.

For $0 < \alpha \leq 1$ denote $\{[u]^\alpha = x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$, then from (1) to (4),it follows that the $\alpha -$ level sets $[u]^\alpha$ for all $0 \leq \alpha \leq 1$ is a closed bounded interval which we denote by $[u]^\alpha = [u_1(\alpha), u_2(\alpha)]$,
If $g: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is any function, then, according to Zadeh's extension principle, we can extend $g: E^n \times E^n \to E^n$ by the function defined by $g(u,v)(z) = \sup_{\{z=g(u,v)\}} \min\{u(x), v(y)\}$
It is well known that $[g(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha)$for all $u, v \in E^n$,$0 \leq \alpha \leq 1$ and a continuos function g.
Especially for addition and scalar multiplication, we have $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$, $[k \odot u]^\alpha = k[u]^\alpha$ where $u, v \in E^n, k \in \mathbb{R}, 0 \leq \alpha \leq 1$. We say that there exists $a \ominus b$, if there exists $c \in E^n$ such that $a = b \oplus c$ and we denote $c = a \ominus b$ [8].
The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\{sup_{a \in A} inf_{b \in B} \parallel a - b \parallel, sup_{b \in B} inf_{a \in A} \parallel a - b \parallel\}$$

where ‖ . ‖ denotes the usual Euclidean norm in $\mathbb{R}^n$.

We define $D: E^n \times E^n \to \mathbb{R}^+ \cup \{0\}$ by the equation $D(u,v) = sup_{\alpha \in [0,1]} d_H([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n$ where d_H is the Hausdorff metric .

proposition1: Let u, v, w and $e \in E^n$

(i) $D(u \oplus w, v \oplus w) = D(u,v)$
(ii) $D(k \odot u, k \odot v) = |k| D(u,v), \ \forall k \in \mathbb{R}$
(iii) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$
(iv) If $u \ominus v$ and $w \ominus e$ exist, then $D(u \ominus v, w \ominus e) \leq D(u,v) + D(w,e)$.
(v) (E^n, D) is a complete metric space

Now, according to [11], with the aid of $(E^n, \oplus, \odot, D)$ we can define new spaces as follows. $C([a,b], E^n)$ the space of all continuous functions, endowed with the metric $H(u,v) = sup_{t \in T} D(u(t), v(t))$. and the natural operations induced by those in E^n, $(C([a,b], E^n), H)$ is a complete metric space.

Theoreme1 : [10]

1. If we denote $\tilde{0} = \chi_{\{0\}}$ then $\tilde{0} \in E^n$ is neutral element with respect to $\oplus$ i.e $u \oplus \tilde{0} = \tilde{0} \oplus u = u$ for all $u \in E^n$
2. for any $\lambda, \mu \in \mathbb{R}$ with $\lambda, \mu \geq 0$ or $\lambda, \mu \leq 0$ and any $u, v \in E^n$ we have $(\lambda + \mu) \odot u = \lambda \odot u \oplus \mu \odot u$ for general $\lambda, \mu \in \mathbb{R}$ the above property does not hold $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v, \lambda \odot (\mu \odot u) = (\lambda . \mu) \odot u$
3. If we denote $\parallel u \parallel_{E^n} = D(u, \tilde{0}), \forall u \in E^n$ then $\parallel . \parallel_{E^n}$ has the properties of a usual norm or E^n i.e $\parallel u \parallel_{E^n} = 0$ if $u = \tilde{0}$, $\parallel \lambda \odot u \parallel_{E^n} = |\lambda| . \parallel u \parallel_{E^n}$ and $\parallel u \odot v \parallel_{E^n} \leq \parallel u \parallel_{E^n} + \parallel v \parallel_{E^n}$ and $\leq | \parallel u \parallel_{E^n} - \parallel v \parallel_{E^n} | \leq D(u,v)$.

Remark1 : from theorem 1, (2) we can deduce that for any $\lambda, \mu \in \mathbb{R}$ with $\lambda > \mu > 0$ and any $u \in E^n$ $\lambda \odot u \ominus \mu \odot u$ exists and $\lambda \odot u \ominus \mu \odot u = (\lambda - \mu) \odot u$

The following definitions and theorems are given in [12]

Definition1: A mapping $F: T \times E^n \to E^n$ is strongly measurable if, for all $\alpha \in [0,1]$ the multi-valued mapping $F_\alpha: T \to P_K(\mathbb{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d_H and T is a subinterval of real number ℝ where $P_K(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$

Definition2: A mapping is called levelwise continuous at $t_0 \in T$ if the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is continuous at$t = t_0$ with respect to the Hausdorff metric d_H for all $\alpha \in [0,1]$.
A mapping $F: T \to E^n$ is called integrably bounded if there exists an integrable function h such that $\parallel x \parallel \leq h(t)$ for all $x \in F_0(t)$

Definition3: Let $F: T \to E^n$.Then the integral of F over T, denoted by $\int_T F(t)dt$ or $\int_c^d F(t)dt$ is defined

$[\int_T F(t)\ dt]^\alpha = \int_T F_\alpha(t)\ dt = \left\{\int_T f(t)dt \,\middle|\, f: T \to \mathbb{R}^n\right.$ is a measurable selection for $F_\alpha(t)\}$ for all $0 < \alpha \leq 1$.

Also, a strongly measurable and integrably bounded mapping $F: T \times E^n \to E^n$ is said to be integrable over T if $\int_T f(t)dt \in E^n$

Theoreme2: If $F: T \to E^n$. is strangely mesurable and integrably bounded, then F is integrable. It is known that $[\int_T F(t)\ dt]^0 = \int_T F_0(t)\ dt$

Theoreme3: Let $F, G: T \to E^n$ be integrable and $\lambda \in \mathbb{R}$. Then

(i) $\int_T (F(t) + G(t))\, dt = \int_T F(t)\ dt + \int_T G(t)\ dt$

(ii) $\int_T \lambda F(t)\ dt = \lambda \int_T F(t)\ dt$

(iii)D(F,G) is integrable,

(iv)D($\int_T F(t)\ dt, \int_T G(t)\ dt$)$\leq \int_T D(F,G)(t)\ dt$

Definition4: A mapping $F: T \to E^n$ is Hukuhara differentiable at $t_0 \in T$ if there exists a $F'(t_0) \to E^n$ such that the limits

$$\lim_{h\to 0^+} \frac{F(x_0+h) \ominus F(x_0)}{h} = \lim_{h\to 0^+} \frac{F(x_0) \ominus F(x_0-h)}{h}$$

exist and are equal to $F'(t_0)$ ($F'(t_0)$ is called the Hukuhara derivative of F at $t_0 \in T$. Here the limit is taken in the metric space (E^n, D). At the end points of T, we consider only the one-site derivatives.

If $F: T \to E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at point t_0.

For the concepts of fuzzy measurability and fuzzy continuity we refer to [13].

Theoreme4: Let $F: T \to E^1$ be differentiable with level sets $F_\alpha(t) = [f_1{}^\alpha, f_2{}^\alpha]$. Then $f_1{}^\alpha, f_2{}^\alpha : [0,1] \to \mathbb{R}^1$ are differentiable and $[F'(t)]^\alpha = \left[f_1{}^{\alpha'}, f_2{}^{\alpha'}\right]$ for all $0 < \alpha \leq 1$.

Definition5: A mapping $F: T \times E^n \to E^n$ is called levelwise continuous provided that for any fixed $\alpha \in [0,1]$ and arbitrary $\varepsilon > 0$ there exists a $\delta(\varepsilon, \alpha) > 0$ such that $d_H([f(t,x)]^\alpha, [f(t_0,x_0)]^\alpha) < \varepsilon$ whenever $|t - t_0| < \delta(\varepsilon,\alpha)$ and $d_H([x]^\alpha, [x_0]^\alpha) < \delta(\varepsilon, \alpha)$ for all $t \in T, x \in E^n$.

Now, let us recall some elements of operator theory and semigroup of operators on E^n in [11]

Definition6: $A: E^n \to E^n$ is called linear operator if

$$A(\lambda \odot x \oplus \mu \odot y) = \lambda \odot A(x) \oplus \mu \odot A(y)$$

for all $\lambda, \mu \in \mathbb{R}$ with $\lambda > \mu > 0$ and all $x, y \in E^n$

Definition7: A family of functions $(T(t))_{t\geq 0}$ of continuous linear operators on E^n is called fuzzy C_0-semigroup if

1. For all $x \in E^n$ the mapping $T(t)(x): \mathbb{R}_+ \to E^n$ is continuous with respect to $t \geq 0$
2. $T(t+s) = T(t)[T(s)]$ for all $t, s \in \mathbb{R}_+$
3. $T(0) = I$ where I is the identity operator on E^n.

Definition8: if $A: E^n \to E^n$ is a linear operator, then it is called generator of the C_0-semigroup if for all $x \in E^n$, there exists $T(t)(x) \ominus x$ and $\lim_{t\to 0^+} \frac{1}{t} \odot [T(t)(x) \ominus x] = A(x)$

Theoreme5: [11] if $A: E^n \to E^n$ is linear and continuous on $\tilde{0}$ then for all $x \in E^n$ we have

$$\| A(x) \|_{E^n} \leq |\ \| A \| \, |_{E^n} \| x \|_{E^n}$$

where $\| A \| \, |_{E^n} = \sup\{\| A(x) \|_{E^n}, x \in E^n, \| x \|_{E^n} \leq 1\} \in \mathbb{R}$, $\| A(x) \|_{E^n} = D(A(x), \tilde{0})$

if A is linear on E^n and continuous on $\tilde{0}$, then it does not follow the continuity of A on the whole space E^n.

$T(t)$ is generalized differentiable with respect to $t \in \mathbb{R}_+$, with the derivative equal to $A[T(t)]$. More exactly, it is Hukuhara differentiable with respect to $t \in \mathbb{R}_+$ i.e

$$\lim_{h\to 0} \frac{1}{h} \odot [T(t+h)(x) \ominus T(t)x] = A[T(t)(x)]$$

$$\lim_{h\to 0} \frac{1}{h} \odot [T(t)(x) \ominus T(t-h)x] = A[T(t)(x)]$$

Here, the limit is taken in the metric space (E^n, D).

Remark2 : By the linearity of $T(t)$ it easily follows that

$$T(t)[x(t) \ominus y(t)] = T(t)[x(t)] \ominus T(t)[y(t)] \quad t \geq 0$$

III. FUZZY NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

the Throughout the whole of this work, we assume that
(H_0) The linear and continuous operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on E^n such that
$\| T(t) \| |_{E^n} \leq M$ for all $t \geq 0$ with $M > 0$.

III.1 existence of mild solutions

Let $C([0,a],E^n)$ be the space of continuous functions. We assume that: (H_1), $F, G: [0,a] \times E^n \rightarrow E^n$ are levelwise continuous and lipschitzians with respect to the second argument there existe constants $L_1 > 0$ and $L_2 > 0$ such that

$$d_H([F(t,x)]^\alpha, [F(t,y)]^\alpha) \leq L_1\, d_H([x]^\alpha, [y]^\alpha) \qquad (2)$$

And

$$d_H([G(t,x)]^\alpha, [G(t,y)]^\alpha) \leq L_2\, d_H([x]^\alpha, [y]^\alpha) \qquad (3)$$

for any pairs $(t,x), (t,y) \in [0,a] \times E^n$.
(H_2), g: $C([0,a],E^n) \rightarrow E^n$ is lipschitz continuous : there existe constants $L_3 > 0$ such that

$$d_H([g(u_1)]^\alpha, [g(u_2)]^\alpha) \leq L_2\, d_H([u_1]^\alpha, [u_2]^\alpha) \qquad (4)$$

for $u_1, u_2 \in C([0,a],E^n)$.
(H_3) h_i: $C([0,a],[0,a])$, $i = 1,2$

Definition9: A continuous function $x(.)\colon [0,a] \rightarrow E^n$ is said to be a mild solution of equation (1) if

$$x(t) = T(t)\left[x_0 \ominus g(x) \ominus F\left(0, x\big(h_1(0)\big)\right)\right] \oplus F\left(t, x\big(\mathrm{h}_{\{1\}(t)}\big)\right) \oplus \int_0^t T(t-s)G\left(t, x\big(\mathrm{h}_{\{2\}(t)}\big)\right) ds \quad (5)$$

for all $t \in [0,a]$.
everywhere in this section when we refer to the equation (1) we mean that there + is replaced by the fuzzy addition $\oplus$ and - is replaced by the fuzzy subtraction $\ominus$.

Theoreme6: Assume that assumptions $(H_0 - H_3)$ hold. Then there existe a unique mild solution $x = x(t)$ of Eq (1) provided that

$$L_0 = ML_3 + (M+1)L_1 + a\, ML_3 < 1$$

Proof:

Consider the operator N defined on $C([0,a],E^n)$ by

$$Nx(t) = T(t)\left[x_0 \ominus g(x) \ominus F\left(0, x\big(h_1(0)\big)\right)\right] \oplus F\left(t, x\big(\mathrm{h}_{\{1\}}(\mathrm{t})\big)\right) \oplus \int_0^t T(t-s)G\left(t, x\big(\mathrm{h}_{\{2\}}(\mathrm{s})\big)\right) ds$$

for all $0 \leq t \leq a$
we shall that is N a contraction operator. Indeed, consider
$x, y \in C([0,a],E^n)$ and $\alpha \in (0,1]$ then

$$d_H([(Nx)(t)]^\alpha, [(Ny)(t)]^\alpha) =$$

$$d_H([T(t)\left[x_0 \ominus g(x) \ominus F\left(0, x\big(h_1(0)\big)\right)\right] \oplus F\left(t, x\big(\mathrm{h}_{\{1\}}(\mathrm{t})\big)\right) \oplus \int_0^t T(t-s)G\left(t, x\big(\mathrm{h}_{\{2\}}(\mathrm{s})\big)\right) ds\ ]^\alpha,$$
$$\left[T(t)\left[x_0 \ominus g(y) \ominus F\left(0, y\big(h_1(0)\big)\right)\right] \oplus F\left(t, y\big(\mathrm{h}_{\{1\}}(\mathrm{t})\big)\right) \oplus \int_0^t T(t-s)G\left(t, y\big(\mathrm{h}_{\{2\}}(\mathrm{s})\big)\right) ds]^\alpha\right)$$

$$\leq d_H(\left[[[T(t)x_0]^\alpha - [T(t)g(x)]^\alpha - [T(t)F\left(0, x\big(h_1(0)\big)\right)]^\alpha\right] + [F\left(t, x\big(\mathrm{h}_{\{1\}}(\mathrm{t})\big)\right)]^\alpha$$
$$+ [\int_0^t T(t-s)G\left(t, x\big(\mathrm{h}_{\{2\}}(\mathrm{s})\big)\right) ds\ ]^\alpha, [T(t)\,x_0]^\alpha - [T(t)g(y)]^\alpha$$
$$+ [T(t)F\left(0, y\big(h_1(0)\big)\right)]^\alpha + [F\left(t, y\big(\mathrm{h}_{\{1\}}(\mathrm{t})\big)\right)]^\alpha + [\int_0^t T(t-s)G\left(t, y\big(\mathrm{h}_{\{2\}}(\mathrm{s})\big)\right) ds\ ]^\alpha$$
$$\leq d_H([T(t)g(x)]^\alpha, [T(t)g(y)]^\alpha) + d_H([T(t)F\left(0, x\big(h_1(0)\big)\right)]^\alpha, [T(t)F\left(0, y\big(h_1(0)\big)\right)]^\alpha)$$

$$+d_H([F(t,x(h_{\{1\}}(t)))]^\alpha,[F(t,y(h_{\{1\}}(t)))]^\alpha)$$

$$+d_H([\int_0^t T(t-s)G(t,x(h_{\{2\}}(s)))ds\ ]^\alpha,[\int_0^t T(t-s)G(t,y(h_{\{2\}}(s)))ds\ ]^\alpha)$$

$$= d_H([T(t)g(x)]^\alpha,[T(t)g(y)]^\alpha)+d_H([T(t)F(0,x(h_1(0)))]^\alpha,[T(t)F(0,y(h_1(0)))]^\alpha)$$

$$+d_H([F(t,x(h_{\{1\}}(t)))]^\alpha,[F(t,y(h_{\{1\}}(t)))]^\alpha)$$

$$+d_H([\int_0^t T(t-s)G(s,x(h_{\{2\}}(s)))\ ]^\alpha ds,[\int_0^t T(t-s)G(s,y(h_{\{2\}}(s)))\ ]^\alpha ds)$$

$$\le d_H([T(t)g(x)]^\alpha,[T(t)g(y)]^\alpha)+d_H([T(t)F(0,x(h_1(0)))]^\alpha,[T(t)F(0,y(h_1(0)))]^\alpha)$$

$$+d_H([F(t,x(h_{\{1\}}(t)))]^\alpha,[F(t,y(h_{\{1\}}(t)))]^\alpha)$$

$$+\int_0^t (d_H([T(t-s)G(s,x(h_{\{2\}}(s)))\ ]^\alpha,[T(t-s)G(s,y(h_{\{2\}}(s)))])ds$$

$$\le \max\{|T_1^\alpha(t)(g_1^\alpha(x)-g_1^\alpha(y))|,T_2^\alpha(t)(g_2^\alpha(x)-g_2^\alpha(y))|\}$$

$$+\max\{|T_1^\alpha(t)(F_1^\alpha(0,x(h_1(0)))-F_1^\alpha(0,y(h_1(0))))|,T_2^\alpha(t)(F_2^\alpha(0,x(h_1(0)))-F_2^\alpha(0,y(h_1(0))))|\}$$

$$+\max\{|(F_1^\alpha(t,x(h_1(t)))-F_1^\alpha(t,y(h_1(t))))|,(F_2^\alpha(t,x(h_1(t)))-F_2^\alpha(t,y(h_1(t))))|\}$$

$$+\int_0^t max\{|T_1^\alpha(t-s)(G_1^\alpha(t,x(h_{\{2\}}(s)))-G_1^\alpha(t,y(h_{\{2\}}(s))))|,T_2^\alpha(t-s)(G_2^\alpha(t,x(h_{\{2\}}(s)))$$

$$-G_2^\alpha(t,y(h_{\{2\}}(s))))\Big|\}ds$$

$$\le \max\{T_1^\alpha(t),T_2^\alpha(t)\}\max\{|g_1^\alpha(x)-g_1^\alpha(y)|,|g_2^\alpha(x)-g_2^\alpha(y)|\}$$

$$+\max\{T_1^\alpha(t),T_2^\alpha(t)\}\max\{|F_1^\alpha(0,x(h_1(0)))-F_1^\alpha(0,y(h_1(0)))|,|F_2^\alpha(0,x(h_1(0)))-F_2^\alpha(0,y(h_1(0)))|\}$$

$$+\max\{|F_1^\alpha(t,x(h_1(t)))-F_1^\alpha(t,y(h_1(t)))|,|F_2^\alpha(t,x(h_1(t)))-F_2^\alpha(t,y(h_1(t)))|\}$$

$$+\int_0^t \max\{|T_1^\alpha(t-s),T_2^\alpha(t-s)|\}\max\{\Big|G_1^\alpha(t,x(h_{\{2\}}(s)))-G_1^\alpha(t,y(h_{\{2\}}(s)))\Big|,|G_2^\alpha(t,x(h_{\{2\}}(s)))$$

$$-G_2^\alpha(t,y(h_{\{2\}}(s)))|\}ds$$

$$\le d_H([T(t)]^\alpha,0)d_H([g(x)]^\alpha,[g(y)]^\alpha)+d_H([T(t)]^\alpha,0)d_H([F(0,x(h_1(0)))]^\alpha,[F(0,y(h_1(0)))]^\alpha)$$

$$+d_H([F(t,x(h_{\{1\}}(t)))]^\alpha,[F(t,y(h_{\{1\}}(t)))]^\alpha)$$

$$\int_0^t d_H([T(t-s)]^\alpha,0)\,d_H([G(s,x(h_{\{2\}}(s)))\ ]^\alpha,[G(s,y(h_{\{2\}}(s)))\ ]^\alpha ds$$

$$\le L_3 d_H([T(t)]^\alpha,0)d_H([x]^\alpha,[y]^\alpha)+L_1 d_H([T(t)]^\alpha,0)d_H([x]^\alpha,[y]^\alpha)+L_1 d_H([x]^\alpha,[y]^\alpha)$$

$$+L_2\int_0^t d_H([T(t-s)]^\alpha,0)d_H([x]^\alpha,[y]^\alpha)\ ds$$

$$D((Nx)(t),(Ny)(t)) \le L_3 D(T(t),\tilde{0})D(x(t),y(t))+L_1 D(T(t),\tilde{0})D(x(t),y(t))+L_1 D(x(t),y(t))$$

$$+L_2\int_0^t D(T(t-s),\tilde{0})D(x(s),y(s))ds$$

$$\le L_3|\parallel T(t)\parallel|_{E^n}D(x(t),y(t))+L_1|\parallel T(t)\parallel|_{E^n}D(x(t),y(t))+L_1 D(x(t),y(t))$$

$$+L_2\int_0^t \parallel T(t-s)\parallel|_{E^n}D(x(s),y(s))ds$$

Then we obtain

$$D((Nx)(t),(Ny)(t)) \le ML_3 D(x(t),y(t))+(M+1)L_1 D(x(t),y(t))+ML_2\int_0^t D(x(s),y(s))ds$$

$$H(Nx,Ny)\le \sup_{0\le t\le a}\{ML_3 D(x(t),y(t))+(M+1)L_1 D(x(t),y(t))+ML_2\int_0^t D(x(s),y(s))ds\}$$

$$\le (ML_3+(M+1)L_1+aML_2)H(x,y)$$

Since $ML_3+(M+1)L_1+aML_2<1$, N is a strict contraction mapping. By Banach fixed point theorem we conclude that N has a unique fixed point
$x=Nx\ \in\in\ C([0,a],E^n)$.

Theoreme7: Suppose that F, G, g and h_i, $i = 1,2$ are the same as in theorem6. let $x(.,x_0)$, $y(.,y_0)$ be a mild solution of Eq (1) corresponding to x_0 y_0 respectively. Then there exists a constant $\xi > 0$ such that

$$H\big(x(.,x_0),\ y(.,y_0)\big) \le \xi\ D(x_0,y_0) \text{ for any } x_0,y_0 \in E^n \text{ and } \xi = {}^{M}/_{(1-L_0)}$$

Proof : Let $x(t,x_0)$, $y(t,y_0)$ be a mild solution to Eq.(1) corresponding to x_0, y_0 respectively. Then $d_H(\ [x(t,x_0)]^\alpha, [y(t,x_0)]^\alpha) =$

$$d_H([T(t)\left[x_0 \ominus g(x) \ominus F\left(0, x\left(h_1(0)\right)\right)\right] \oplus F\left(t, x\left(\mathrm{h}_{\{1\}}(\mathrm{t})\right)\right) \oplus \int_0^t T(t-s)G\left(t, x\left(\mathrm{h}_{\{2\}}(\mathrm{s})\right)\right)ds\ ]^\alpha,$$

$$\left[T(t)\left[x_0 \ominus g(y) \ominus F\left(0, y\left(h_1(0)\right)\right)\right] \oplus F\left(t, y\left(\mathrm{h}_{\{1\}}(\mathrm{t})\right)\right) \oplus \int_0^t T(t-s)G\left(t, y\left(\mathrm{h}_{\{2\}}(\mathrm{s})\right)\right)ds]^\alpha\right)$$

$$\le d_H([T(t)]^\alpha,0)d_H([x_0]^\alpha,[y_0]^\alpha) + L_3 d_H([T(t)]^\alpha,0)d_H([x]^\alpha,[y]^\alpha) + L_1 d_H([T(t)]^\alpha,0)d_H([x]^\alpha,[y]^\alpha) + L_1 d_H([x]^\alpha,[y]^\alpha) + L_2 \int_0^t d_H([T(t-s)]^\alpha,0)d_H([x]^\alpha,[y]^\alpha)\ \mathrm{ds}$$

Then we obtain

$$D\big(x(t,x_0),y(t,x_0)\big) \le |\ \| T(t) \| |_{E^n} D(x_0,y_0) + L_3|\ \| T(t) \| |_{E^n} D\big(x(t),y(t)\big) + (M+1)\ L_1|\ \| T(t) \| |_{E^n} D\big(x(t),y(t)\big) + L_2\ \| T(t) \| |_{E^n} \int_0^t D\big(x(s),y(s)\big)ds$$

Thus,

$$\le MD(x_0,y_0) + ML_3 D\big(x(t),y(t)\big) + (M+1)L_1 D\big(x(t),y(t)\big) + ML_2 \int_0^t D\big(x(s),y(s)\big)ds$$

$$H(x(.,x_0),y(.,y_0)) \le MD(x_0,y_0) + (ML_3 + (M+1)L_1 + aML_2)H(x(.,x_0),y(.,y_0))$$

So

$$H\big(x(.,x_0),y(.,y_0)\big) \le {}^{M}/_{1-(ML_3+(M+1)L_1+aML_2)}\ D(x_0,y_0)$$

Were $l_0 = ML_3 + (M+1)L_1 + aML_2$

ACKNOWLEDGMENT

The authors express their sincere thanks to the anonymous referees for numerous helpful and constructive suggestions which have improved the manuscript.

REFERENCES

[1] M. Adimy, K. Ezzinbi, A class of linear partial functional differential equations with nondense domain, J. Differential Equations 147 (1998) 285-332.
[2] K. Balachandran, M. Chandrasekaran, Existence of solutions of a delay differential equation with nonlocal condition, Indian J. Pure Appl. Math. 27 (1996) 443-449.
[3] J.K. Hale, K.R. Mayer, A class of functional equations of neutral type, Memoirs of the American Mathematical Society 76 (1967).
[4] L. Byszewski, Theorems about existence and uniqueness of a solution of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl. 162 (1991) 494-505.
[5] L. Byszewski, Existence of solutions of semilinear functional-differential evolution nonlocal problem, Nonlinear Anal. Theory Methods Appl. 34 (1998) 65-72.
[6] L. Byszewski, V. Lakshimikantham, Theorem about existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, Appl. Anal. 40 (1990) 11-19.
[7] z. Ding and A. Kandel, On the controllability of fuzzy dynamical systems (I), J. Fwzy Maths. 8, 203-214,(2000).
[8] D.Dubois and Prade, fuzzy numbers; on overview in : Analysis of fuzzy Information, vol.1 Math .logic, CRC press, boca Raton, 1987,pp.3-29
[9] K. Ezzinbi, J.H. Liu, Nondensely defined evolution equations with nonlocal conditions, Math. Comput. Model. 36 (2002) 1027-1038.
[10] K. Ezzinbi, X.Fu, Existence and regularity of solutions for some neutral partial differential equations with nonlocal conditions, Nonlinear Analysis 57 (2004) 1029-1041
[11] C.S.Gal,S.G.Gal,Semigroups of operators os spaces of fuzzy-number-valued functions with applications to fuzzy differential equations, J. FuzzyMath.13(2005)647-682.
[12] O. Kaleva, Fuzzy differential equations Fuzzy Sets and Systems 24 (1987) 301-317.
[13] Y. K. Kim, Measurability for fuzzy valued functions. Fuzzy Sets and Systems 129(2002),No. 1, 105-109.
[14] Y. Lin, H. Liu, Semilinear integrodifferential equations with nonlocal Cauchy problem, Nonlinear Anal. Theory Methods Appl. 26 (1996) 1023-1033.
[15] S.K. Ntouyas, P.Ch. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, J. Math. Anal. Appl. 210 (1997) 679-687.

[16] L. S. Chadli, A. Harir, S. Melliani,Fuzzy Euler Differential Equation,Annals of Fuzzy Mathematics and Informatics,ISSN(Print): 2373-8472,Volume 2, Number 1, January 2015.

[17] A.harir, S. Melliani, L. S. Chadli, Fuzzy Space-Time Fractional Telegraph Equations, *International Journal of Mathematics Trends and Technology (IJMTT) – Volume64 Number 2 – December 2018*

[18] L. S. Chadli, A. Harir and S. Melliani, Solutions of fuzzy wave-like equations by variational iteration method. International Annals of Fuzzy Mathematics and Informatics, volume 8, number 4, (2014), 527-547 .

[19] L. S. Chadli, A. Harir and S. Melliani, Solutions of fuzzy heat-like equations by variational iterative method , Annals of Fuzzy Mathematics and Informatics, Volume 10, number 1, (2015), pp 29-44.