Quotient of Ordered Meet Hyper Lattices with a Regular Relation

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Abstract:
In this paper, we consider meet hyperlattices and we define ordered meet hyperlattices. It is already introduced that there exists product of hyperlattices [5]. We introduce the notion of product of two ordered meet hyperlattices in this paper. Moreover, we define the quotient of ordered meet hyperlattices with a regular relation. Also, we investigate isomorphism on the product of two ordered meet hyperlattices with a regular relation.

Keywords: Meet hyper lattice [1], regular relation, Quasi-ordered relation.

I. INTRODUCTION AND BASIC DEFINITIONS

We define a regular relation on Ordered meet hyperlattice such that its quotient [4] is an ordered hyperlattice and we study some properties of such relations.

Definition 1.1:

Let H be a non-empty set. A Hyper operation on H is a map ∘ from H×H to P*(H), the family of non-empty subsets of H. The Couple (H, ∘) is called a hypergroupoid. For any two non-empty subsets A and B of H and x Є H, we define A ∘ B = $\cup_{a\in A, b\in B}$ a ∘ b;

A ∘ x = A ∘ {x} and x ∘B = {x} ∘ B

A Hypergroupoid (H, ∘) is called a Semihypergroup if for all a, b, c of H we have (a ∘ b) ∘ c = a ∘ (b ∘ c). Moreover, if for any element a ∈ H equalities

A ∘ H = H ∘ a = H holds, then (H, ∘) is called a Hyper group.

Definition 1.2:

A Lattice is a partially ordered set L such that for any two elements x, y of L, glb {x, y} and lub {x, y} exists. If L is a lattice, then we define x V y = glb {x, y} and lub {x, y}.

Definition 1.3:

Let L be a non-empty set, **∧:** L × L → p* (L) be a hyper operation and V: L × L →L be an operation. Then (L, V, **∧**) is a Meet Hyperlattice if for all x, y, z Є L. The following conditions are satisfied:

1) x Є x **∧** x and x = x V x
2) x V (y V z) = (x V y) V z and x **∧** (y **∧** z) = (x **∧** y) **∧** z
3) x V y = y V x and x **∧** y = y **∧** x
4) x Є x **∧** (x V y) ∩ x V (x **∧** y)

Definition 1.4[3]:

Let $(L_1, V_1, \wedge_1, \leq_1)$ and $(L_2, V_2, \wedge_2, \leq_2)$ be two ordered hyperlattiice.
Give $(L_1 \times L_2, V', \wedge')$, he two hyperoperations V' and **∧**' on $L_1 \times L_2$ such that for any $(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$, we have

(x_1, y_1) **∧**'(x_2, y_2) = {(u, v); u Є $x_1 \wedge_1 x_2$, v Є $y_1 \wedge_2 y_2$},

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq_1 x_2, y_1 \leq_2 y_2$.

The Hyper operation **V'** is defined similar to **∧**'.

Definition 1.5:

Let $\mathcal{R}$ be an equivalence relation on a non-empty set L and A, B ⊆ L, A $\bar{\mathcal{R}}$ B means that

for all a Є A, there exists some b Є B such that a $\mathcal{R}$ b,

for all b' Є B, there exists a' Є A such that a' $\mathcal{R}$ b'.

Also, $\mathcal{R}$ is called a regular relation respect to **∧** if x $\mathcal{R}$ y implies that x ∧ z $\bar{\mathcal{R}}$ y ∧ z, or all x, y, z Є L. $\mathcal{R}$ is called a Regular relation if it is regular respect to V and ∧, at the same time.

Definition 1.6:

An Ideal P of a meet hyperlattice L is Prime [2] if for all x, y Є L and x V y Є P, we have x Є P and y Є P.

II. QUOTIENT OF ORDERED MEET HYPERLATTICES WITH A REGULAR RELATION.

In this section, we study special relation which is regular on ordered meet hyperlattices which has connection with order on L and we derive ordered meet hyperlattice from an ordered meet hyperlattice with such regular relation.

Let (L, V, ∧,≤) be an ordered meet hyperlattice and ν be a relation which is transitive and contains the relation ≤. Moreover, for any x, y Є L, if x ν y, we have x ∧ z $\bar{\nu}$ y ∧ z and x V z $\bar{\nu}$ y V z, for all z Є L and x Є y ∧ z implies that xνy, yνx, xνz, zνx, we call such relations as quasi-ordered relations. We know that if ν is a regular relation, the quotient L/ν is a hyperlattice. But this relaton is not equivalence relation. So, we define ν* = {(a, b) Є$\nu\times\nu$; a ν b, b ν a}.

Theorem 2.1:

Let (L, V, ∧,≤) be an ordered strong meet hyperlattice and ν* be a relation defined by

ν* = {(a, b) Є$\nu\times\nu$; a ν b, b ν a}. Thus, L/ ν* is an ordered hyperlattice.

Proof:

We can easily show that ν* is an equivalence relation.
Now we show that ν* is a regular relation.
Let xν*y, z ∈ L and x' ∈ x ∧z.
Thus, x ν y and y ν x.
Therefore x ∧z $\bar{\nu}$ y ∧z, y ∧z $\bar{\nu}$ x ∧z and we conclude that there exists y'∈ y ∧z such that x' ν y'.
By the property ofν, we have y' ν z and z ν x'.
So, y' ν x' and x ∧z $\bar{\nu}$* y ∧z.
Since ∨ is a binary operation, we show that x ∨ z $\bar{\nu}$* y ∨ z.
So, ν* is a regular relation and L/ ν* is a hyperlattice.
We have to show that L/ ν* is an ordered hyperlattice.
Let ν*(x) ≼ ν*(y).
Since L/ ν* is a hyperlattice, we have ν*(x) ⋏ ν*(z) = ν*(z') where z' ∈ x ∧z and ν*(y) ⋏ ν*(z) = ν*(w), w ∈ y ∧z.
Thus, there exists x' ∈ ν*(x) and y' ∈ ν*(y) such that x' ≤ y'.
Therefore, we have (x' ∧z) ≤ (y' ∧z) and so (x' ∧z) ν (y' ∧z).
So, ν*(x') ⋏ ν*(z) ≼ ν*(y') ⋏ ν*(z).
Since ν*(x') = ν*(x) and ν*(y') = ν*(y), we have ν*(x) ⋏ ν*(z) ≼ ν*(y) ⋏ ν*(z).
Therefore, L/ ν* is an ordered hyperlattice.

Theorem 2.2:

Let (L,∨, ∧, ≤) be an ordered strong meet hyperlattice and ν be a quasi-ordered relation. There is one to one correspondence between quasi-ordered relations on L which contain ν and quasi ordered relations on L/ ν*.

Proof:

Let η be a quasi-ordered relation on L/ ν*.
We have to prove that τ = {(x, y); (ν*(x), ν*(y)) η} is a quasi-ordered relation on L which containsν.
Let x ≤ y. So, x ν y and (ν*(x), ν*(y)) ∈ L/ ν*.
Since η is a quasi-ordered relation, it is clear that (ν*(x), ν*(y)) $\epsilon\eta$ and so (x, y) $\epsilon\tau$ and ≤ ⊆ τ.
We can prove that τ has the transitive property.
Now, let x ϵ y ∧z. Therefore, ν*(x) $\epsilon\nu$*(y) ⋏ ν*(z) where ⋏ is a hyper operation on L/ ν*.
Therefore, (ν*(x), ν*(y)) $\epsilon\eta$ and (ν*(x), ν*(z)) $\epsilon\eta$.
So, (x, y) ϵ τ, (x, z) ϵ τ, (y, x) ϵ τ, (z, x) ϵ τ and let (x, y) ϵ τ, a ϵ x ∧z.
So, ν*(a) $\epsilon\nu$*(x) ⋏ ν*(z) and since η is a quasi-ordered relation, there exists ν*(b) $\epsilon\nu$*(y) ⋏ ν*(z) such that (ν*(a), ν*(b)) η.
Hence (a, b) τ.
Also, we can show for ⋁ that τ is a quasi-ordered relation on L.
Similarly, if we have a quasi-ordered relation on L which contains ν, then there exists a quasi-ordered relation on L/ ν*.

Theorem 2.3:

Let (L,∨, ∧, ≤) be an ordered strong meet hyperlattice and ν, τ be two quasi-ordered relations on L such that ν ⊆ τ and ν*(x) τ/ $\nu\nu$*(y) if and only if there exists a ∈ ν*(x), then there exists b ∈ ν*(y), a ν b. Then, τ/ ν is a quasi-ordered relation on L/ν*.

Proof:

Let (ν*(x), ν*(y)) ∈ τ/ ν.
Thus, there exists x ∈ τ*(a), y ∈ ν*(b) such that x ν y.

Hence, aν x and y ν b.
So, a ν b and since $\nu \subseteq \tau$, we have a τ b.
We can easily prove that τ/ν contains $\preccurlyeq$on L/ ν* and has the transitive property.
Now, we let (ν*(x), ν*(y)) ∈ τ/ν and ν*(z) ∈ L/ ν*, ν*(c) ∈ ν*(x) ⋏ ν*(z).
Thus, (x, y) ∈ ν and c ∈ x ∧z.
Since τ is a quasi-ordered relation, there exists u ∈ y ∧z such that c τ u.
Therefore, there exists ν*(u) ∈ ν*(y) ⋏ ν*(z) such that (ν*(c), ν*(u)) ∈ τ/ν.
Also, let ν*(z) ∈ ν*(x) ⋏ ν*(y).
Thus, z ∈ x ∧y and z τ x, z τ y, x τ z, y τ z.
Therefore, (ν*(z), ν*(x)) ∈ τ/ν, (ν*(z), ν*(y)) ∈ τ/ν, (ν*(x), ν*(z)) ∈ τ/ν and (ν*(y), ν*(z)) ∈ τ/ν.
Similarly, we can prove for ∨.

In the following theorem we are going to investigate quasi-ordered relation on the product of two ordered meet hyperlattices.

Theorem 2.4:
Let (L_1,$\vee_1$,$\wedge_1$, $\leq_1$) and (L_2,$\vee_2$,$\wedge_2$, $\leq_2$) be two ordered strong meet hyperlattices and ν_1, ν_2 be quasi-ordered relations on $L_1$and L_2. Then, (L_1×L_2) / ν* is isomorphic to L_1/ν_1* ×L_2/ν_2*.
Proof:
We define f: (L_1×L_2) / ν* ⟶ L_1/ν_1* ×L_2/ν_2* by f(ν*(a), ν*(b)) = (ν_1*(a), ν_2*(b)).
We can show that f is well defined and one to one.
Now, we claim that f is a homomorphism between two ordered meet hyperlattices.
f (ν*(a_1, b_1) ⋏ ν*(a_2, b_2)) = f(ν*(u, v)) where u ∈ $a_1 \wedge a_2$, v ∈ $b_1 \wedge b_2$.
So, we have f (ν*(a_1, b_1) ⋏ ν*(a_2, b_2)) = (ν*(a_1) $\wedge_1$ ν*(a_2), ν*(b_1) $\wedge_2$ ν*(b_2))
= f (ν*(a_1, b_1)) × f(ν*(a_2, b_2)).
Similarly, these relations also hold for the binary operation ⋎ and by the definition of order on(L_1×L_2), if ν*(a_1, b_1) ≼ ν*(a_2, b_2), we have (a_1, b_1) ν (a_2, b_2).
Therefore, (a_1, a_2) ∈ ν_1 and (b_1, b_2) ∈ ν_2.
Thus, ν_1*(a_1) $\leq_1$ ν_1*(a_2) and ν_2*(b_1) $\leq_2$ ν_2*(b_2).
Therefore, f is an order preserving map and it is clear that f is onto.
So, f is an isomorphism and the proof is completed.

III. CONCLUSION

In this paper, we have successfully derived a ordered meet hyperlattice from a ordered meet hyperlattice with a regular relation induced in it. We have also investigated quasi-ordered relations on the product of two ordered meet hyperlattices.

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