Dhage Iteration Method For Nonlinear First Order Abstract Measure Differential Equations With A Linear Perturbation

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ABSTRACT: *In this paper the authors prove for the existence and approximation of the solutions for an initial and a periodic boundary value problem of nonlinear first order ordinary abstract measure differential equations with a linear perturbation via Dhage iteration method. Also we have solved an examples for the applicability of given results in the paper.*

Keywords and Phrases: *Abstract measure differential equations, Dhage iteration methods, existence theorem, extremal solutions, approximation of solution, hybrid fixed theorem.*

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1. INTRODUCTION

Initially Sharma [16] study of nonlinear abstract measure differential equations and studied some basic results concerning the existence and uniqueness of solutions for such equations. Later, such equations were studied by various authors for different aspects of the solutions under continuous and discontinuous nonlinearities. The abstract measure differential equations involve the derivative of the unknown set-function with respect to the σ-finite complete measure. Some of the abstract measure differential equations have been studied in a series of papers by Joshi [17], Dhage [2–3], Dhage et al. [8] and Dhage and Bellale [9] for different aspects of the solutions.

The perturbed ordinary differential equations have been treated in Dhage [14] and it is mentioned that the inverse of such equations yields the sum of two operators in appropriate function spaces. The Dhage [15] fixed point theorem is useful for proving the existence results for such perturbed differential equations under mixed geometrical and topological conditions on the nonlinearities involved in them.

The fixed point theorems so far used in the above papers of Dhage [1], Joshi [17], Bellale [4-6] study the abstract measure integro differential equation and existence theorem. This is a stringent condition and recently, the authors in Dhage [8] proved the existence and uniqueness results for abstract measure differential equations. Here our approach is different from that of Sharma [16] and Joshi [11]. The results of this paper complement and generalize the results of the above-mentioned papers on abstract measure differential equations under weaker conditions.

2. PRELIMINARIES

A mapping $T: X \to X$ is called D-Lipschitz if there exists a continuous and nondecreasing function $\phi: R^+ \to R^+$ such that

$$\| Tx - Ty \| \leq \varphi(\| x - y \|)$$

for all $x, y \in X$, where $\phi(0) = 0$. In particular if $\phi(r) = \alpha r, \alpha > 0$, T is called a Lipschitz with a Lipschitz constant α. Further if $\alpha < 1$, then T is called a contraction on X with the contraction constant α.

Let X be a Banach space and let $T: X \to X$, T is called compact if $\overline{T(X)}$ is a compact subset of X. T is called totally bounded if for any bounded subsets S of X, $T(S)$ is a totally bounded subset of X. T is called completely continuous if T is continuous and totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of X. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [19].

To prove the main existence result of this section, we need the following nonlinear alternative proved in Dhage [2].

Theorem 2.1. Let U and $\overline{U}$ denote respectively the open and closed bounded subset of a Banach algebra X such that $0 \in U$. Let $A: X \to X$ and $B: \overline{U} \to X$ be two operators such that
(a) A is nonlinear D -contraction, and

(b) B is completely continuous.
Then either

(i) the equation $Ax + Bx = x$ has a solution in $\overline{U}$, or

(ii) there is a point $u \in \partial U$ such that satisfying $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u.$

for some $0 < \lambda < 1$, where ∂U is a boundary of U in X.

An interesting corollary to Theorem 2.1 in the applicable form is

Corollary 2.1. Let $B_r(0)$ and $\overline{B}_r(0)$ denote respectively the open and closed balls in a Banach algebra X centered at origin 0 of radius r for some real number $r > 0$. Let $A: X \to X, B: \overline{B}_r(0) \to X$ be two operators such that

(a) A is contraction, and
(b) B is completely continuous.
Then either
(i) the operator equation $Ax + Bx = x$ has a solution x in X with $\| x \| \le r$, or
(ii) there is an $u \in X$ such that $\| u \| = r$ satisfying $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u$ for some $0 < \lambda < 1$.

In the following section we state our perturbed abstract measure differential equations to be discussed qualitatively in the subsequent part of this paper.

3. STATEMENT OF THE PROBLEM

Let X be a real Banach algebra with a convenient norm $\|.\|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \le r \le 1\} \tag{3.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{rx \mid -\infty < r < 1\}, \tag{3.2}$$

and

$$\overline{S}_x = \{rx \mid -\infty < r \le 1\} \tag{3.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let ca(X, M) be the space of all vector measures (real signed measures) and define a norm $\|.\|$ on ca(X, M) by

$$\|p\| = |p|(X), \tag{3.4}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup \sum_{i=1}^{\infty} |p(E_i)|, \quad E_i \subset X, \tag{3.5}$$

where the supremum is taken over all possible partitions $\{E_1 : i \in N\}$ of X. It is known that $ca(X,M)$ is a Banach space with respect to the norm $\|.\|$ given by (3.4).

Let μ be a σ-finite positive measure on X, and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ - algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $S_x, x \in \overline{x_0 z}$.

Given a $p \in ca(X,M)$ with $p << \mu$, consider the abstract measure differential equation (AMDE) of the form

$$\frac{dp}{d\mu} = f(x, p(\bar{S}_x)) + g(x, p(\bar{S}_x)) \quad a.e.[\mu] \text{ on } \overline{x_0 z}. \tag{3.6}$$

and $$p(E) = q(E), \quad E \in M_0, \tag{3.7}$$

where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Randon-Nikodym derivate of p with respect to μ, $f, g : S_z \times R \to R$, and $f(x, p(S_x))$ and $g(x, p(S_x))$ is μ-integrable for each $p \in ca(S_z, M_z)$.

Definition 3.1. Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)(z > x_0)$ is said to be a solution of AMDE (3.6)–(3.7) if

(i) $p(E) = q(E), E \in M_0$

(ii) $p << \mu$ on $\overline{x_0 z}$ and

(iii) p satisfies (3.6) a.e.[μ] on $\overline{x_0 z}$.

Remark 3.1. The AMDE (3.6)–(3.7) is equivalent to the abstract measure integral equation (in short AMIE)

$$p(E) = \int_E f(x, p(\bar{S}_x))\, d\mu + \int_E g(x, p(\bar{S}_x))\, d\mu \tag{3.8}$$

if $E \in M_z, E \subset \overline{x_0 z}$,

and $$p(E) = q(E), \quad if\ E \in M_0. \tag{3.9}$$

A solution p of the AMDE (3.6)–(3.7) on $\overline{x_0 z}$ will be denoted by $p(\bar{S}_{x_0}, q)$.

Note that our AMDE (3.6)–(3.7) includes the abstract measure differential equation considered in the previous papers as special case. To see this, define $g(x, y) = 0$ for all $x \in \overline{x_0 z}$ and $y \in R$, then AMDE (3.6)–(3.7) reduces to

$$\frac{dp}{d\mu} = f(x, p(\bar{S}_x)) \ a.e.[\mu] \text{ on } \overline{x_0 z}. \tag{3.10}$$

and $$p(E) = q(E), \quad if\ E \in M_0, \tag{3.11}$$

The AMDE (3.10)–(3.11) has been studied in Joshi [3] and Dhage et al. [8] which further includes the abstract measure differential equations studied by S.Leela[10] and Dhage [9] as special cases. Thus our AMDE (3.6)–(3.7) is more general and we claim that it is a new to the literature on measure differential equations. As a result the results of the present study are new and original contribution to the theory of nonlinear measure differential equations. In the following section we shall prove the existence and uniqueness theorems for AMDE (3.6)–(3.7).

Remark 3.2 In this section, we prove an existence and approximation result for the closed and bounded interval $J = [a, b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. The function space $C(J, R)$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, R)$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.12}$$

And $$x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for all } t \in J \tag{3.13}$$

Clearly, $C(J, R)$ is a Banach space with respect to above supremum norm and also partially ordered w. r. t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, R)$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it.

4. MAIN RESULT

Given a vector measure a $p \in ca(X,M)$ with $p << \mu$, consider the initial and periodic boundry value problems of first order nonlinear abstract measure differential equation (in short AMDE),

$$\left.\begin{array}{c}\frac{d}{d\mu}[p(S_x)-f(x,p(S_x))]+\lambda[p(S_x)-f(x,p(S_x))]=g(x,p(S_x)), x\in\overline{x_0 z},\\ p(E)=q(E), E\in M_0\end{array}\right\} \quad (4.1)$$

and

$$\left.\begin{array}{c}\frac{d}{d\mu}[p(S_x)-f(x,p(S_x))]+\lambda[p(S_x)-f(x,p(S_x))]=g(x,p(S_x)), x\in\overline{x_0 z},\\ x(0)=x(M),\end{array}\right\} \quad (4.2)$$

Where q is a given known vector measure , $\lambda \in$ R, $\lambda > 0$ and *f, g*: $S_z \times R \to R$ are continuous functions.

The solution of the AMDE (4.1) or (4.2) we mean a function $p \in$ ca(S_z, M_z) such that

(i) the function $x \longmapsto x - f(x, p(S_x))$ is differentiable for each x and

(ii) x satisfies the equations in (4.1) or (4.2),

where ca(S_z, M_z) is the space of continuous real-valued functions defined on $\overline{x_0 z}$ .The AMDEs (4.1) and (4.2) are linear perturbations of the second type of the nonlinear differential equations

$$\left.\begin{array}{l}\frac{dp}{d\mu}=g(x,p(S_x)), x\in\overline{x_0 z}\\ p(E)=q(E),\ \ E\in M_0\end{array}\right\} \quad (4.3)$$

and

$$\left.\begin{array}{c}\frac{dp}{d\mu}=g(x,p(S_x)), x\in\overline{x_0 z}\\ x(0)=x(M),\end{array}\right\} \quad (4.4)$$

and a sharp classification of different types of perturbations of a differential equation appears in Dhage [2] which can be treated with the hybrid fixed point theory (see Dhage [15] and S.Heikkilaand and Lakshmikantham [7]). The AMDE (4.1) with $\lambda = 0$ has been thoroughly discussed in the literature for different basic aspects of the solutions such as existence theorem, differential inequalities, maximal and minimal solutions, comparison principle under some mixed Lipschitz and compactness type conditions.

Definition 4.1. A mapping τ : E $\to$ E is called partially continuous at a point $a \in$ E if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\tau x - \tau a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. τ called partially continuous on E if it is partially continuous at every point of it. It is clear that if τ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 4.2. A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. A nondecreasing mapping τ : E $\to$ E is called partially compact if τ (C) is a relatively compact subset of E for all totally ordered sets or chains C in E. τ is called uniformly partially compact if τ is a uniformly partially bounded and partially compact operator on E. τ is called partially totally bounded if for any totally ordered and bounded subset C of E, τ (C) is a relatively compact subset of E. If τ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Definition 4.3. An upper semi-continuous and monotone nondecreasing function $\psi : R_+ \to R_+$ is called a D-function provided $\psi(0) = 0$. An operator τ : E $\to$ E is called partially nonlinear D-contraction if there exists a D-function ψ such that

$$\|\tau_x - \tau_y\| \le \psi(\|x - y\|)$$

for all comparable elements $x, y \in$ E, where $0 < \psi(r) < r$ for $r > 0$. In particular,

if $\psi(r) = kr$, $k > 0$, τ is called a partial Lipschitz operator with a Lischitz constant k and moreover, if $0 < k < 1$, τ is called a partial linear contraction on E with a contraction constant k.

The Dhage iteration method developed in Dhage [8], S.S.Bellale [18], may be described as "the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution as its first or initial approximation converges monotonically to the solution."

Theorem 4.1. Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such every compact chain C of E. Let A, B : E $\longrightarrow$ E be two nondecreasing operators such that

(a) A is partially bounded and partially nonlinear D-contraction,

(b) B is partially continuous and partially compact, and

(c) there exists an element $x_0 \in$ E such that $x_0 \preceq Ax_0 + Bx_0$ or $x_0 \succeq Ax_0 + Bx_0$.

Then the operator equation Ax + Bx = x has a solution x^* in E and the sequence $\{p_n\}$ of successive iterations defined by $x_{n+1} = Ax_n + Bx_n$, $n = 0,1,\ldots$, converges monotonically to x^*.

Remark 4.1. We consider the following basic hypothesis is as follows.

(H$_1$) The mapping $x \mapsto x - f(x, p(S_x))$ is increasing in R for each $x \in \overline{x_0 z}$.

(H$_2$) There exists a D-function such that

$$0 \le f(x, p(Sx)) - f(x, p(S_y)) \le \psi(x-y)$$

For $x \in \overline{x_0 z}$ all and $x, y \in$ R with $x \ge y$.

Moreover, $0 < \psi(r) < r$ for $r > 0$.

(H$_3$) There exists a constant $M_f > 0$ such that

$|f(x, p(S_x))| \le M_f$ for all $x \in \overline{x_0 z}$ and $x \in$ R.

(H$_4$) There exists a $M_g > 0$ such that $|g(x, p(S_x))| \le M_g$, for all $x \in \overline{x_0 z}$ and $x \in$ R.

(H$_5$) $g(x, p(S_x))$ is nondecreasing in x for each $x \in \overline{x_0 z}$.

Remark 4.2. If the hypothesis (H$_1$) holds, then the function $x \mapsto x - f(x, p(S_x))$ is injective in R for each $x \in \overline{x_0 z}$

Remark 4.3. The hypotheses (H$_2$) and (H$_3$) hold, in particular if f satisfies the inequality

$$0 \le f(x, p(S_x)) - f(x, p(S_y)) \le \frac{L(x-y)}{K + (x-y)}$$

for all $x, y \in$ R with $x \ge y$, where L > 0 and K > 0 are constants satisfying L $\le$ K.

The lemma given below follows from the theory of calculus and linear differential equations.

Lemma 4.1. Assume that hypothesis (H$_1$) holds. Then for any continuous function $h : S_z \times R \longrightarrow R$, the function $p \in ca(X, M)$ is a solution of the AMDE

$$\left.\begin{array}{c} \frac{d}{d\mu}[p(S_x) - f(x, p(S_x))] + \lambda[p(S_x) - f(x, p(S_x))] = h(x), x \in \overline{x_0 z}, \\ x(0) = M_0, \end{array}\right\} \tag{4.5}$$

if and only if x satisfies the abstract measure integral equation (AMIE)

$$p(x) = ce^{-\lambda x} + f(x, p(S_x)) + e^{-\lambda x}\int_0^x e^{\lambda x} h(x)dx,\ x \in \overline{x_0 z}, \tag{4.6}$$

where $c = M_0 - f(0, M_0)$.

Proof. Let $h \in ca(X, M)$. Assume first that x is a solution of the AMDE (4.5) defined on J. By definition, the function $x \mapsto p(S_x) - f(x, p(S_x))$ is continuous on $\overline{x_0 z}$, and so, differentiable there, whence $\frac{d}{d\mu}[p(S_x) - f(x, p(S_x))]$ is μ integrable on $\overline{x_0 z}$. Applying integration to (4.5) from 0 to x, we obtain the AMIE (4.6) on $\overline{x_0 z}$.

Conversely, assume that x satisfies the AMIE (4.5). Then by direct differentiation we obtain the first equation in (4.6). Again, substituting x = 0 in (4.6) we get

$$x(0) - f(0, x(0)) = M_0 - f(0, M_0).$$

Since the mapping $x \mapsto x - f\left(x, p\left(S_x\right)\right)$ is increasing in R for all $x \in \overline{x_0 z}$ the mapping $x \mapsto x - f(0,x)$ is injective in R, whence $x(0) = M_0$. Hence the proof. We need the following definition in what follows.

Definition 4.4. A function $u \in ca(S_z, M_z)$ is called a lower solution of the AMDE (4.1) on $\overline{x_0 z}$, if the function $x \mapsto x - f\left(x, p\left(S_x\right)\right)$ is differentiable and satisfies the inequalities

$$\left.\begin{aligned} \frac{d}{d\mu}[p(S_x) - f(x, p(S_x))] + \lambda[p(S_x) - f(x, p(S_x))] \le g(x, p(S_x)), x \in \overline{x_0 z}, \\ x(0) = M_{0,} \end{aligned}\right\} \tag{4.7}$$

for all $x \in \overline{x_0 z}$. Similarly, an upper solution of the AMDE(4.1) on ca(S_z,M_z) is defined.

The AMDE (4.1) has a lower solution $u \in ca(S_z, M_z)$.

Now we are in a position to prove the following existence and approximation theorem for the AMDE (4.1) on ca(S_z,M_z).

Theorem 4.2. Assume that the hypotheses (H_1) through (H_3) and (H_4) through (H_6) hold. Then the AMDE (4.1) has a solution *x** defined on ca(S_z,M_z) and the sequence $\{p_n\}_{n=0}^{\infty}$ of successive approximations defined by converges monotonically to *x**, where $c = M_0 - f(0, M_0)$.

Proof. Set E = ca(S_z, M_z). Then, by Lemma 4.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\,.\,\|$ and the order relation $\leq$ so that every compact chain C is in E.

Now, using the hypotheses (H_1) and (H_3), by Lemma 4.1 the AMDE (4.1) is equivalent to the nonlinear AMIE

$$p(x) = ce^{-\lambda x} + f(x, p(S_x)) + e^{-\lambda x} \int_0^x e^{\lambda x} g(x, p(S_x)) dx \tag{4.8}$$

$$x_0 = u,$$

$$p_{n+1}(x) = ce^{-\lambda x} + f(x, p_n(S_x)) + e^{-\lambda x} \int_0^x e^{\lambda x} g(x, p_n(S_x))\, dx \quad x \in \overline{x_0 z}$$

for $x \in \overline{x_0 z}$ Define two operators $A, B : E \to E$ by

$$Ap(x) = f(x, p(S_x)), x \in \overline{x_0 z}, \tag{4.9}$$

and

$$Bp(x) = ce^{\lambda x} + e^{\lambda x} \int_0^x e^{\lambda x} g(x, p(S_x)) dx, x \in \overline{x_0 z}. \tag{4.10}$$

Then, the AMIE (3.6) is transformed into an operator equation as

$$Ap(x) + Bp(x) = p(x), x \in \overline{x_0 z}. \tag{4.11}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 4.1. Firstly, we show that the operators A and B are nondecreasing on E. Let *x*, *y* $\in$ E be such that x $\geq$ y. Then, by hypothesis (H_2),

$$Ap(x) = f(x, p(S_x)) \ge f(x, p(S_y)) = Ap(y)$$

for all $x \in \overline{x_0 z}$. Similarly by hypothesis (H_4),

$$Bp(x) = ce^{-\lambda x} + e^{-\lambda x}\int_0^x e^{\lambda x} g(x, p(S_x))\,dx$$
$$\geq ce^{-\lambda x} + e^{-\lambda x}\int_0^x e^{\lambda x} g(x, p(S_x))\,dx$$
$$= Bp(y)$$

for all $x \in \overline{x_0 z}$. This shows A and B are nondecreasing operators on E into E.
From (H_3) it follows that

$$\| Ap(x) \| \leq \sup_{x \in \overline{x_0 z}} | Ap(S_x) | \leq \sup_{x \in \overline{x_0 z}} | f(x, p(S_x)) | \leq M$$

for all $x \in$ E. As a result, A is bounded and consequently partially bounded on E. Next, we show that A is a partial nonlinear D-contraction on E with a D function ψ. Let $x, y \in$ E be such that $x \geq y$.
Then, by hypothesis (H_2),

$$| Ap(x) - Ap(y) | = | f(x, p(S_x)) - f(x, p(S_y)) |$$
$$= f(x, p(S_x)) - f(x, p(S_y))$$
$$\leq \psi(p(S_x) - p(S_y))$$
$$= \psi(| p(S_x) - p(S_y) |)$$
$$\leq \psi(\| x - y \|)$$

for all $x \in \overline{x_0 z}$. Taking the supremum over x, we obtain

$$\| Ax - Ay \| \leq \psi(\| x - y \|)$$

for all $x, y \in$ E with $x \geq y$. This shows that A is a partial nonlinear D-contraction on E with the D-function .
Next, we show that B is a partially compact and partially continuous operator on E into E. First we show that B is a partially continuous on E. Let $\{p_n\}$ be a sequence in a chain C of E converging to a point $x \in$ C. Then by the dominated convergence theorem for integration, we obtain

$$\lim_{n\to\infty} Bp_n(x) = \lim_{n\to\infty}\left[ce^{-\lambda x} + e^{-\lambda x}\int_0^x e^{\lambda x} g(x, p_n(S_x))\,dx \right]$$
$$= ce^{-\lambda x} + \lim_{n\to\infty} e^{-\lambda x}\int_0^x e^{\lambda x} g(x, p_n(S_x))\,dx$$
$$= ce^{-\lambda x} + e^{-\lambda x}\int_0^x e^{\lambda x}\left[\lim_{n\to\infty} g(x, p_n(S_x))\right]dx$$
$$= ce^{-\lambda x} + e^{-\lambda x}\int_0^x e^{\lambda x}\left[g(x, p_n(S_x))\right]dx$$
$$= Bp_n(x)$$

for all $x \in \overline{x_0 z}$. Moreover, it can be shown as below that $\{Bp_n\}$ is an equicontinuous sequence of functions on E. Now, following the arguments similar to that given in Granas and Dugundji [19], it is proved that B is a a partially continuous operator on E.

Next, we show that B is a partially compact operator on E. It is enough to show that B(C) is a uniformly bounded and equi-continuous set in E for every chain C in E. Let $x \in$ C be arbitrary. Then by the hypothesis (H_3),

$$| Bp(x) | \leq | ce^{-\lambda x} | + \left| e^{-\lambda x}\int_0^x e^{\lambda x} g(x, p_n(S_x))\,dx \right|$$

$$\leq | M_0 - f(x_0, M_0) | + \int_0^M e^{\lambda x} M_g\,dx$$

$$\leq |M_0 - f(x_0, M_0)| + \frac{(e^{\lambda M} - 1)M_g}{\lambda}$$

for all $x \in \overline{x_0 z}$. Taking the supremum over *x*, we obtain

$$\| Bp(x) \| \leq |M_0 - f(x_0, M_0)| + \frac{(e^{\lambda M} - 1)M_g}{\lambda}$$

for all $x \in C$. This shows that B is uniformly bounded on C.

Again, let $x_1, x_2 \in \overline{x_0 z}$ be arbitrary. Then for any $x \in C$, one has

$$|Bp(x_1) - Bp(x_2)| = |c|\, |e^{-\lambda x_1} - e^{-\lambda x_2}|$$

$$+ \left| e^{-\lambda x_1} \int_0^{x_1} e^{\lambda x} g(x, p(S_x))dx - e^{-\lambda x_2} \int_0^{x_2} e^{\lambda x} g(x, p(S_x))\, dx \right|$$

$$\leq |c|\, |e^{-\lambda x_1} - e^{-\lambda x_2}|$$

$$+ |e^{-\lambda x_1} - e^{-\lambda x_2}| \left| \int_0^{x_1} e^{\lambda x} g(x, p(S_x)dx \right|$$

$$+ |e^{-\lambda x_2}| \left| \int_0^{x_1} e^{\lambda x} g(x, p(S_x))dx - \int_0^{x_2} e^{\lambda x} g(x, p(S_x))\, dx \right|$$

$$\leq |c|\, |e^{-\lambda x_1} - e^{-\lambda x_2}| + \left[\frac{(e^{\lambda M} - 1)M_g}{\lambda} \right] |e^{-\lambda x_1} - e^{-\lambda x_2}|$$

$$+ \left| \int_{x_2}^{x_1} e^{\lambda x} |g(x, p(S_x))|\, dx \right|$$

$$\leq \left[|c| + \frac{(e^{\lambda M} - 1)M_g}{\lambda} \right] |e^{-\lambda x_1} - e^{\lambda x_2}| + |p(x_1) - p(x_2)|$$

where, $p(x) = \int_0^x e^{\lambda x} M_g dx$

Since the functions $x \mapsto e^{-\lambda x}$ and $x \mapsto p(x)$ are continuous on compact, ca(S_z,M_z) they are uniformly continuous there. Hence, for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x_1 - x_2| < \delta \Rightarrow |Bp(x_1) - Bp(x_2)| < \varepsilon$$

uniformly for all $x_1, x_2 \in \overline{x_0 z}$ and for all $x \in S$. This shows that B(C) is an equi-continuous set in E. Now the set B(C) is uniformly bounded and equicontinuous in E, so it is compact by Arzela-Ascoli theorem. As a result, B is a partially continuous and partially compact operator on E.

Thus, all the conditions of theorem 4.1 are satisfied and hence the operator equation $Ax + Bx = x$ has a solution x^* in E and the sequence of successive approximations $\{p_n\}$ defined by $p_n = Ap_{n-1} + Bp_{n-1}$ converges monotonically to x^*. As a result, the AMDE (4.1) has a solution x^* defined on ca(S_z,M_z) and the sequence of successive approximations $\{p_n\}$ defined by (4.3) converges monotonically to x^*. This completes the proof.

Remark 4.4. We remark that Theorem 4.2 also remains true if we replace the hypothesis (H_6) with the following one:
(H_7) The AMDE (4.1) has an upper solution $v \in ca(X, M)$.

Remark 4.5. We note that if the AMDE (4.1) has a lower solution u as well as an upper solution v such that $u \le v$, then under the given conditions of Theorem 4.2 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \le x^*$. Hence they are the minimal and maximal solutions of the AMDE (4.1) in the vector segment [u, v] of the Banach space E = ca(X, M), where the vector segment $[u,v]$ is a set of elements in ca(X,M) defined by

$$[u,v] = \{x \in C(X,M) \mid u \le x \le v\}$$

This is because the order relation $\le$ defined by (3.2) is equivalent to the order relation defined by the order cone $K = \{p \in ca(S_z, M_z) \mid p(E) \ge 0, E \in M_z\}$ which is a closed set in ca(Sz, M_z).

Example 4.1. consider the AMDE

$$\left.\begin{array}{c} \frac{d}{d\mu}[p(S_x) - \tan^{-1} p(S_x)] + [p(S_x) - \tan^{-1} p(S_x)] = \tanh p(S_x), x \in \overline{x_0 z}, \\ x(0) = 1 \end{array}\right\}$$

Here, $\lambda = 1$ and the functions f and g are given by

$$f(x, p(S_x)) = \tan^{-1} x \text{ and } g(x, p(S_x)) = \tanh x$$

for all $x \in \overline{x_0 z}$ and $x \in R$. We show that the functions f and g satisfy all the hypotheses of theorem 4.3. First we show that f satisfies the hypotheses (H_1)-(H_3). Now,

$$\frac{\partial}{\partial x}[x - f(x, p(S_x))] = \frac{d}{dx}[x - \tan^{-1} x] = 1 - \frac{1}{1+x^2} > 0,$$

for all $x \in R$ and $x \in \overline{x_0 z}$, so that the function $x \mapsto x - f(x, p(S_x))$ is increasing in R for each $x \in \overline{x_0 z}$. Therefore, hypothesis (H_1) holds. Next, let $x, y \in$ R be such that $x \ge$ y. Then,

$$0 \le f(x, p(S_x)) - f(x, p(S_y)) \le \tan^{-1} x - \tan^{-1} y = \frac{1}{1+\varepsilon^2}(x - y)$$

for all $x > \varepsilon > y$, showing that f satisfies the hypothesis (H_2) with D-function ψ given by

$$\psi(r) = \frac{r}{1+\varepsilon^2} < r, r > 0,$$

Where $\varepsilon \ne 0$ Again

$$|f(x, p(S_x))| = |\tan^{-1} x| \le \frac{\pi}{2},$$

for all $x \in \overline{x_0 z}$ and $x \in R$. This shows that f satisfies hypothesis (H_3) with $M_f = \frac{\pi}{2}$.

Furthermore,

$$|g(x, p(S_x))| = |\tanh x| \le 1,$$

for all $x \in \overline{x_0 z}$ and $x \in R$, so that the hypothesis (H_4) holds with M_g = 1. Again, since the function $x \mapsto \tanh x$ is nondecreasing in R and so the hypothesis (H_5) is satisfied. Finally, the function $u(x)$ = - (x + 3) is a lower solution of the AMDE defined on ca(S_z,M_z) .Thus the functions f and g satisfy all the conditions of Theorem 4.2. Hence we apply and conclude that the AMDE (3.10) has a solution x^* defined on ca(S_z,M_z) and the sequence $\{p_n\}$ of successive approximations defined by

$$x_0 = u,$$

$$p_{n+1}(x) = 1 - \frac{\pi}{2} + \tan^{-1} p_n(x) + e^{-x}\int_0^x e^x \tanh p_n(x)dx$$

for each $x \in \overline{x_0 z}$, converges monotonically to x^*. A similar conclusion also remains true if we replace the lower solution u with the upper solution $v(x) = x + 3$, ca(S_z,M_z).

The following useful lemma is obvious and may be found in Dhage [1] and Nieto [13].

Definition 4.5. A function $u \in ca(S_z, M_z)$ is called a lower solution of the AMDE (4.1) if the function $x \mapsto x - f(x, p(S_x))$ is differentiable and satisfies the inequalities

$$\left.\begin{array}{c} \frac{d}{d\mu}[u(x) - f(x, w(x))] + \lambda[u(x) - f(x, u(x))] \leq g(x, u(x)), \\ u(0) \leq u(M) \end{array}\right\}$$

for all $x \in \overline{x_0 z}$. Similarly, an upper solution $v \in ca(S_z, M_z)$ of the AMDE (4.2) is defined.

We need the following hypotheses in what follows.

(H_6) The function $f(x, p(S_x))$ is periodic in x with measure M for all $x \in R$,

i.e., $f(0, x) = f(M, x)$ for all $x \in R$.

(H_5) The AMDE (4.2) has a lower solution $u \in ca(S_z, M_z)$.

Remark 4.6. We remark that Theorem 4.2 also remains true if we replace the hypothesis (H_8) with the following one.

(H_9) The AMDE (4.2) has an upper solution $v \in ca(S_z, M_z)$.

Remark 4.7. We note that if the AMDE (4.2) has a lower solution u as well as an upper solution v such that $u \leq v$, then under the given conditions of Theorem 4.2 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the AMDE (4.2) in the vector segment $[u, v]$ of the Banach space E = ca(S_z, M_z), where the vector segment $[u, v]$ is a set of elements in C(X, M) defined by

$$[u, v] = \{x \in C(X, M) | u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (4.2) is equivalent to the order relation defined by the order cone $K = \{p \in ca(S_z, M_z) \mid p(E) \geq 0, E \in M_z\}$ which is a closed set in ca(S_z, M_z).

Example 4.2. Given a closed and bounded interval $p \in ca(S_z, M_z)$, consider the AMDE

$$\left.\begin{array}{c} \frac{d}{dx}[p(x) - \tan^{-1} p(x)] + [p(x) - \tan^{-1} p(x)] = \tanh p(x), x \in \overline{x_0 z}, \\ x(0) = x(M). \end{array}\right\} \tag{4.12}$$

Here, $\lambda = 1$ and the functions f and g are given by

$$f(x, p(S_x)) = \tan^{-1} x \text{ and } g(x, p(S_x)) = \tanh x$$

for all $x \in \overline{x_0 z}$ and $x \in R$. Now, it can be shown that the functions f and g satisfy all the hypotheses of Theorem 4.2 with $u(x) = -4e^x, x \in (X, M)$. Hence we conclude that the AMDE (4.12) has a solution x^* defined on ca(S_z,M_z) and the sequence $\{p_n\}$ of successive approximations defined by, $x_0 = u$,

$$p_{n+1}(x) = \tan^{-1} p_n(x) + \int_0^1 G(x, p(S_x)) \tanh p_n(x)\, dx,$$

for each $x \in \overline{x_0 z}$, converges monotonically to x^*, where G(x, $p(S_x)$) is a Green's function associated with the homogeneous PBVP

$$\left.\begin{aligned} \frac{dp}{d\mu} + p(x) = 0,\ x \in \overline{x_0 z}, \\ x(0) = x(M), \end{aligned}\right\} \tag{4.13}$$

given by

$$G(t,s) = \begin{cases} \dfrac{e^{s-t+1}}{e-1}, & if \quad 0 \le s \le t \le 1, \\ \dfrac{e^{s-t}}{e-1}, & if \quad 0 \le t < s \le 1. \end{cases}$$

Again, a similar conclusion holds if we replace the lower solution u with the upper solution $v(x) = 4e^x, x \in (X, M)$.

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