Simplicity of Groups

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Abstract
The aim of this paper is to check whether a group is simple or not. For this we use some test.

Key Words - *Group, set, prime, simple, sylow subgroup.*

I. INTRODUCTION

The theory of simple group plays an important role in Modern Algebra. In this article we discuss about a few theorems that are useful in providing that a group of particular order is a non abelian simple group

Simple Group:A group G is said to be simple if G has no normal subgroup except {e} and G itself.

Sylow p Subgroup of G : Let G be finit group & let p be a prime advisor of O(G). If p^k/O(G) and $p^{k+1}\nmid$o(G) then any subgroup of G of order p^k is called a sylow p subgroup.

Result: If a finite group contains unique sylow p-subgroup then it is normal subgroup.

II. SOME USEFUL TEST TO CHECK SIMPLICITY OF A GROUP

Theorem 1 **Cayley Test:**

Let G be a group.

O(G) = 2n where n is an odd number greater than 1. Then G contains a normal subgroup of order n. Hence G is not simple.

Example:

1. **O(G) = 58**
Then
O(G) = 2 x 29
29>1 & 29 is odd number. Then by clayey test G is not simple.
Hence, any group of order 58 is not simple.

2. **O(G) = 62**
Then
O(G) = 2 x 31
Again G is not simple by cayley test. Hence, any group of order 62 is not simple.

Theorem2 If O(G) = p^h (h>1). Then G contains a normal subgroup of order p. Hence, G is not simple.

Example:

1. **O(G) = 125**
Then
O(G) = 5^3
By Th^m 2. G contains a normal subgroup of order 5. So G is not simple. Hence any group of order 25 is not simple.

2. **O(G) = 64**
Then
O(G) = 2^6
By Th^m 2. G contains a normal subgroup of order 2. So G is not simple. Hence any group of order 64 is not simple.

Theorem3 **Index theorem**

If G is a finite group and H is a proper sub group of G such that O(G) $\nmid$ ([G:H])! Then G has proper normal subgroup and hence G is not simple

Example:1
If O(G) = 12
Then O(G)= 2^2.3
Hence by sylow first Th^m G has a subgroup H of order 2^2. Let it be that subgroup.

Then $[G:H] = \frac{O(G)}{O(H)} = \frac{12}{4} = 3$

As 12 ∤ 3! So O(G) ∤ ([G:H])! Then by index th^m G contains a proper normal subgroup. Hence, G is not simple.

Example:2

If O(G) = 108

Then O(G) = 108 = $2^2 . 3^3$

Then by sylow first Th^m G has a subgroup of order 3^3. Let it be that subgroup

Then$[G:H] = \frac{O(G)}{O(H)} = \frac{108}{27}$=4

As 108 ∤4! So O(G) ∤ ([G:H])! So by index Thm G has a proper normal subgroup.

Hence G is not a simple.

Sylow Test for Non Simplicity:-

If G is a group of order p.m where p is prime integer and (p.m) = 1, then n_p = 1 + kp ,k≥o and n_p /O(G) where n_p denotes the number of sylow p subgroup of G.

If n_p=1 then G has unique p sylow subgroup which is normal.

Hence G is simple.

Example:- O(G) = 40 =5.8= p^r.q

P = 5 is prime and (5, 8) = 1

Hence G has a 5 sylow subgroup of order 5.

n_5= 1+5k

If	K=0	n_5=1+5.0=1/O(G)				
	K=1	n_5	=	1+5.1	=	6 ∤ O(G)
	K=2	n_5	=	1+5.2	=	11 ∤ O(G)
	K=3	n_5	=	1+5.3	=	16 ∤ O(G)

Hence only one sylow 5-subgroup and it must be normal.

Hence G is not simple

Example: O(G) =30=3.5.2

So	n_3	=	1+3K
	n_3	=	1 or 10
	n_5	=	1+5K
	n_5	=	1 or 6
	n_2	=	1+2K
	n_2	=	1, 3,5,15

.Case1 If $n_3 = 1$ and $n_5 = 1$ and $n_2 is\ anything$

G has a unique slyow 3-subgroup and slyow 5-subgroup, then they must be normal ,hence G is not simple.

Case2 If $n_3 = 1$ and $n_5 = 6$ and $n_2 is\ anything$

G has a unique slyow 3-subgroup and it must be simple , hence G is not simple

Case3 If $n_3 = 10$ and $n_5 = 1$ and $n_2 is\ anything$

G has a unique slyow 5-subgroup and it must be simple , hence G is not simple

Case4 If $n_3 = 10$ and $n_5 = 6$ and $n_2 is\ anything$

Let G_1 G_2 G_3 . . . . G_{10} be 10 distinct slyow 3-subgroup of G

Also $G_i \cap G_j \subseteq G_i$ $j \leq 10, i \geq 1$

$G_i \cap G_j$ is a subgroup of G_i

O($G_i \cap G_j$) /(G_i) i.e. O($G_i \cap G_j$) /3

Also $G_i \neq G_j$ $G_i \neq G_i \cap G_j$

O($G_i \cap G_j$)= 1 $or\ G_i \cap G_j = \{e\}$

Morever if e≠ $a \in G_i$ $a^3 = e$

This shows that O(a) $= 3\ for\ all\ a(\neq e) \in G_i$

Which shows that there are $10\times(3\text{-}1)$ $=20$ $non\ identity\ elements\ of\ order\ 3$
Similarly ther are $6\times(5-1)$ $=24$ elements of order 5
G has atleast (20+24)=44 elements which is a contradiction.
Hence case 4 is not possible
Hence G is not simple.

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