Convex quadratic optimization based on generator matrix in credit risk transfer process

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Abstract. In this paper, the generator matrix is calculated based on the grade transfer matrix over a period of time in the quantitative analysis of credit risk, which has become one of the hot issues that many scholars pay attention to and study.From the mathematical model point of view combined with the actual situation, the calculation of the generated matrix can be regarded as optimization problems of the matrix logarithm. This article is based on Prof Kreinin and Prof Sidelnikova's classical optimization model, and we use the combination model of Markov chain and the effective set method of convex quadratic optimization to solve the problem. We mainly make the feasible domain satisfy the applicable conditions by reducing dimensions and correcting variables and theoretically verify the convergence and feasibility of the optimization system. Meanwhile, we verify the data published by Standard and Poor's Credit Review by Matlab.

Keywords. credit risk, convex quadratic optimization, effective set method, generator matrix, Markov chain

1. Introduction

The JLT model is an important model in the quantitative analysis of credit risk. It constructs a non-timed continuous chain to portray the process of change of the credit rating in the risk neutrality measurement space by some market data, such as the country Debt, corporate bond and other prices. An important step in implementing the JLT model is to assume that the credit-level transfer process is a continuous Markov chain in the real-world measurement space. And how to calculate a generator matrix based on the credit-level transfer matrix published by the credit rating agency for a period of time? Expressed in mathematical language, given the transition matrix $P(0,T)$ for a given time period [0,T] to calculate a matrix Q that satisfies the line sum as 0, non-diagonal as non-negative (the nature of the generator matrix), so that e^{QT} is as close as possible to $P(0,T)$. [2] and [5] proposed a method to solve this problem: first calculate the matrix $\log P(0,T)$, and then modify $\log P(0,T)/T$ to a matrix Q that satisfies the properties of the generator matrix. This article has made two improvements to the methods in [2] and [5]:

First, when T is small, $\log P(0,T)$ has a strong diagonal dominance, then $\|\log P(0,T)\|$ is small. We use the inverse scaling and squaring algorithm with specified relative precision to calculate the logarithmic matrix, so that the calculated value guarantees a certain relative precision.

Second, it is well known that the results obtained directly through the matrix logarithm operation do not satisfy the nature of the generation matrix of the credit level transfer in the real world. It is pointed out in [10] that more accurate prediction estimates can be obtained by optimization methods. In this paper, we fully consider this point when doing matrix correction. The properties of the generator matrix are taken as constraints, and $\log P(0,T)/T$ is corrected to the matrix Q by solving the linear optimization problem.

2. Preliminary

In this chapter we will introduce the relevant knowledge of continuous time Markov chain and convex quadratic optimization, which will lay the foundation for the following discussion[3,16].

2.1 Continuous time Markov chain

The random process $\{X_n, n \in \mathbb{N}\}$(where N represents the set of indicators of time) is called the Markov process[7,8].

If the conditional probability satisfies the Markov property, that is, $\forall t$, a positive integer n, and an arbitrary sequence $t_0, t_1, \cdots, t_n$ that satisfies $t_0 < t_1 < \cdots < t_n < t$, the following equation holds:

$$P\{X_t \leq x | X_{t_0} = x_0, X_{t_1} = x_1, \cdots, X_{t_n} = x_n\} = P\{X_t \leq x | X_{t_n} = x_n\}.$$

It can be obtained from the above formula that the Markov process is a process in which future state changes are only related to the current state, and are independent of the previous state.

If the state space of a Markov process is discrete, then this Markov process is called the Markov chain.

If the indicator set T is continuous, it is called a continuous time Markov chain. In this case, the Markov property is expressed as: $\forall n$ and an arbitrary sequence $t_0, t_1, \cdots, t_n, t_{n+1}$ that satisfies $t_0 < t_1 < \cdots < t_n < t_{n+1}$, The following equation holds:

$$P\{X_{t_{n+1}} = x_{n+1} | X_{t_0} = x_0, X_{t_1} = x_1, \cdots, X_{t_n} = x_n\} = P\{X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n\}.$$

In the continuous Markov chain, $p_{ij}(s,t) = P\{X_t = j | X_s = i\}$ represents the probability of the time point t is in the state j under the condition of the time point s is in the state i, which is called the transition probability equation of the state i to j. Transfer matrix[10] $P(s,t) = (p_{ij}(s,t))$ satisfies:

$$P(s,t) \geq 0; \quad P(s,t)\mathbf{e} = \mathbf{e}, \tag{2.1}$$

where $\mathbf{e}$ represents unit vector, that is $\mathbf{e} = [1, 1, \cdots, 1]^T$.

Unless otherwise stated, the following $\mathbf{e}$ in this paper represents this vector.We know $P(t,t) = I$.

In a continuous Markov chain, $p_{ij}(s,t)$ not only depends on the time s in the state i, but also depends on the time span $t - s$. In order to discuss the probability of moving from one state to another in a sufficiently small time span (assuming that only one transition can occur within this time span) .We define the transfer rate $q_{ij}(t)$ as follows:

$$q_{ij}(t) = \lim_{\triangle t \to 0} \frac{p_{ij}(t, t + \triangle t)}{\triangle t}, \quad j \neq i,$$

$$q_{ii} \equiv \lim_{\triangle t \to 0} \frac{p_{ii}(t, t + \triangle t) - 1}{\triangle t} = -\sum_{j \neq i} q_{ij}(t).$$

We call it $Q(t) = (q_{ij}(t))$ the transfer rate matrix or generator matrix which satisfies:

$$Q(t) = \lim_{\triangle t \to 0} \frac{P(t, t + \triangle t) - I}{\triangle t}.$$

According to the nature (2.1) of the transfer matrix, the generator matrix $Q(t)$ satisfies:

$$Q(t)\mathbf{e} = 0; \quad q_{ij}(t) \geq 0 \;\; \forall i \neq j. \tag{2.2}$$

The continuous Markov chain satisfies the Chapman-Kolmogorov compatible equation:

$$\frac{\partial P(t,T)}{\partial T} = P(t,T)Q(T), \quad \frac{\partial P(t,T)}{\partial t} = -Q(T)P(t,T). \tag{2.3}$$

If the continuous Markov chain is time homogeneity, it indicates that the state change of the Markov chain is independent of the specific time and is only related to the time span. Then the generator matrix $Q(t)$ is a constant matrix. By C-K equation (2.3), we knows that,

$$P(t,T) = e^{Q(T-t)}.$$

It can be seen that there is a close relationship between the generator matrix and the logarithm of the computation matrix when we have known transition matrix of the continuous Markov chain in the time span.

2.2 Effective set for convex quadratic programming

Quadratic programming is one of the special cases in nonlinear optimization. Its objective function is a quadratic real function, and the constraint function is guaranteed to be a linear function. The method of solving quadratic programming has attracted people's attention long ago, making it an important way to solve nonlinear optimization problems. Considering the practicality of our problem, that is, the matrix form of the quadratic function is a positive definite matrix, we call it convex quadratic programming. Here we mainly introduce the effective set method of general constrained convex quadratic programming. For the following optimization problems:

$$\begin{cases} \min & f(x) & = & \frac{1}{2}x^T Hx + c^T x, \\ s.t. & h_i(x) & = & a_i^T x^* - b_i = 0, \ i \in I = 1, \cdots, m, \\ & g_i(x) & = & a_i^T x^* - b_i \geq 0, \ i \in E = m+1, \cdots, n, \end{cases} \tag{2.4}$$

where H is an n order real symmetric matrix, which denoted as $I(x) = \{i | \ a_i^T x^* - b_i = 0, i \in I\}$.

For the above general inequality constraint problem, we mark the following: the feasible domain is $\mathbb{D} = \{x \in \mathbb{R}^n | \ g_i(x) \geq 0, \ i = 1, 2, \cdots, n\}$, and the index set is $I = \{1, \cdots, m\}$. For the convenience of research, we deal with the Lagrangian function of the problem (2.4):

$$L(x, \lambda) = f(x) - \sum_{i=1}^{l} \lambda_i g_i(x),$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)^T$, we call it the multiplier vector.

If there is a feasible point $\bar{x} \in \mathbb{D}$ of question(2.4), such that $g_i(\bar{x}) = 0$, then the inequality constraint $g_i(x) \geq 0$ is a valid constraint of $\bar{x}$. On the other hand, if $g_i(\bar{x}) > 0$, the inequality constraint $g_i(x) \geq 0$ is a non-effective constraint of $\bar{x}$. The set $I(\bar{x}) = \{i : \ g_i(\bar{x}) = 0\}$ is called the set of effective constraint indicators at $\bar{x}$, which is simply referred to as the effective set (or positive set) at x.

So,the known transition matrix of Markov chains in the credit risk hierarchy is the key step in implementing this model: assuming that the credit level change is a continuous Markov chain. And how to solves the generator matrix by the transition matrix of credit rating over a period of time, become the optimization problem of the logarithmic matrix.

2.3 Symbol Description

In this paper, the matrix norm is the infinite norm without special explanation. For example, for any matrix A, the following formula holds true:

$$norm(A) = ||A|| = ||A||_\infty.$$

3. Calculation of the generator matrix in the credit risk level

In the above chapter, we know that there is a close relationship between the transfer matrix of the Markov chain and the logarithm of the computational matrix. We calculate a logarithmic matrix based on the inverse scaling and squaring algorithm of the logarithm of the specified relative precision in [16]. In this chapter we mainly discuss the optimization of logarithmic matrices in real risk[17,18].

3.1 Application Background and Problem Description of Optimizing Generator Matrix

The credit rating of the company is an indicator of the credit status of the credit rating agency based on the company's operating, financial, capital, debt and other information to evaluate the company's ability to repay debts in the future. Credit ratings are generally divided into AAA[6](representing the highest credit rating), AA(representing the next highest credit rating), A(the first three grades are investment grades, representing higher credibility), BBB, BB, B(these three ratings are for speculative grades, which represent lower credibility, C(the lowest credit rating, indicating significant bankruptcy) and bankruptcy. The company's credit rating affects the price of corporate bonds, loan interest rates and many other factors. Therefore, in the quantitative analysis of credit risk, the company's credit rating and its dynamic changes, especially the possibility of bankruptcy in the future is a very important factor.

The JLT model proposed by Jarrow, Lando, and Turnbull in [1] is an important model in the quantitative analysis of credit risk. Through some market data, such as national debt and corporate bonds, the JLT model constructs a non-timed continuous Markov chain to describe the change process of credit rating in the risk neutrality measurement space. The key step in implementing the JLT model is to assume that the credit level change process is a continuous Markov chain in the real world measurement space, and how to get the generator matrix of this Markov chain by the credit level transfer matrix for a period of time published by the rating agency, such as the annual transfer matrix, semi-annual Transfer matrix, etc. The above questions are expressed in mathematical language as follows.

The state space of the continuous Markov chain $\{X_t\}$ is $\mathbb{N} = \{1, 2, \cdots, K\}$, where $1, 2, \cdots, K-1$ represents each credit level from high to low, and K represents bankruptcy, which is an absorption state, that is, once the transfer process reaches the state, it stays in this state forever. The transfer matrix $P(0, T),\ \ T \leq 1$ is known, and the matrix Q is generated to satisfy the equations $P(0, T) = e^{QT}$ and (2.2). The transfer matrix $P(0, T)$ is generally a strictly diagonally dominant matrix, and the smaller the T, the stronger the diagonal dominance. This is because a company with a credit rating of i has a higher probability of staying at that level within one year(> 0.5), and the shorter the time, the higher the probability.

3.2 Existing calculation method of generator matrix

In Section 3.1, we have elaborated on the calculation of the credit rating matrix and its background. Next, we begin to discuss the calculation method of the generator matrix Q.

In the literature [1], if the matrix $P(0, T)$ is strictly diagonally dominant, then at most one generator matrix exists. If present, $\log P(0, T)/T$ is the only solution to the generator matrix. However, $\log P(0, T)/T$ generally does not satisfy the properties (2.2) of the generator matrix.

Therefore, there is generally no exact solution to the above problem, we can only calculate the approximate solution. Literature [2] and [5] respectively propose their solutions. Below, we briefly introduce the main ideas of their methods.

3.2.1 Calculation methods of Israel, Rosenthal, and Wei

In [5], Israel, Rosenthal, and Wei proposed a method for approximate calculation of the generator matrix. The calculation method is as follows:

First, we calculate the logarithmic matrix $\log P(0,T)$ using Taylor's expansion and get the calculated value $\tilde{Q}$ of $\log P(0,T)/T$.

Then, the off-diagonal elements less than zero in the matrix are corrected to 0 according to the definition of the generator matrix $\tilde{Q}$, and then the corresponding diagonal guarantee row sum is 0. That is taking

$$q_{ij} = max(\tilde{q_{ij}}, 0), \quad j \neq i \qquad q_{ii} = \tilde{q_{ii}} + \sum_{j \neq i} \min(\tilde{q_{ij}}, 0).$$

Obtaining an approximate generator matrix $Q_{IRW} = (q_{ij})$.

3.2.2 Calculation methods of Kreinin and Sidelnikova

Kreinin and Sidelnikova proposed new calculation methods in the literature:

First, we use the Schur-Parlett method to get $\log P(0,T)$ and get the calculated value $\log P(0,T)/T$ of $\tilde{Q}$.

Second, the matrix $\tilde{Q}$ is corrected line by line, and is corrected to the matrix Q satisfying the formula(2.2). The correction process is to solve the following optimization problem line by line:

For each row $\tilde{q_i}$ of $\tilde{Q} = (\tilde{q_1}^T, \cdots, \tilde{q_K}^T)^T$, find $q_i = (q_{i1}, \cdots, q_{iK})$, and satisfy:

$$\begin{cases} \min \quad dist(\tilde{q_i}, q_i), \quad dist \ indicates \ the \ Euclidean \ distance, \\ s.t. \quad \sum_j q_{ij} = 0, \\ \qquad\quad q_{ij} \geq 0, \qquad \forall j \neq i. \end{cases}$$

We obtain an approximate generator matrix $Q_{AK} = (q_1^T, q_2^T, \cdots, q_K^T)^T$.

The basic idea of the above two algorithms is to first calculate the logarithmic matrix $\log P(0,T)$, and then modify the resulting matrix $\log P(0,T)/T$ into a generator matrix. Obviously the matrix Q_{AK} obtained by the Kreinin and Sidelnikova's algorithm is closer to $\log P(0,T)/T$.

In this paper, we use the effective set algorithm in the modified quadratic programming to solve the problem. Now we briefly introduce the algorithm idea of the effective set method.

3.3 Effective set method for convex quadratic programming

Below we introduce the theoretical basis[4] for studying the optimal solution of the inequality constraint problem.

(Farkas lemma) Let $a_i, b_i \in \mathbb{R}^n$ $(i = 1, \cdots, r)$, then the necessary and sufficient condition that the linear inequality group $b_i^T d \geq 0,\ i = 1, \cdots, r,\ d \in \mathbb{R}^n$ is compatible with the inequality $a^T d \geq 0$ is that there is a non-negative real number $\alpha_1, \cdots, \alpha_r$, so that $a = \sum\limits_{i=1}^{r} \alpha_i b_i$.

Below we give the first order necessary condition for the problem (2.4) to take the minimum value, that is, the KT condition.

(KT condition) Let x^* be the local minimum of the inequality constraint problem(3.1). Take the effective constraint set $I(x^*) = \{i|\ g_i(x^*) = 0,\ i = 1, \cdots, m\}$,and $f(x)$, and $g_i(x)$ $(1, 2, \cdots, m)$ is divisible at x^*. If vector group $\nabla g_i(x^*)(i \in I(x^*))$ is linearly independent, then vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*)^T$ exists, so that the following formula holds:

$$\begin{cases} \nabla f(x^*) - \sum\limits_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0, \\ g_i(x^*) \geq 0,\ \lambda_i^* \geq 0,\ g_i(x^*) = 0, i = 1, \cdots, m. \end{cases}$$

General constrained quadratic programming has global minima, if and only when H is semi-positive.

x^* is a necessary and sufficient condition for the global minimum point of the convex quadratic programming. When x^* satisfies the KT condition, $\lambda^* \in \mathbb{R}^n$ exists, so that the following formula holds:

$$\begin{cases} Hx^* + c - \sum\limits_{i \in E} \lambda_i^* a_i - \sum\limits_{i \in I} \lambda_i^* a_i = 0, \\ a_i^T x^* - b_i = 0,\ i \in E, \\ a_i^T x^* - b_i \geq 0,\ i \in I, \\ \lambda_i^* \geq 0,\ i \in I;\ \lambda_i^* = 0,\ i \in I \backslash I(x^*). \end{cases}$$

Assume that the matrix H in the convex quadratic programming problem is symmetrically positive. If the matrix $A_k = (a_i)_{i \in S_k}$ in each iteration of the effective set algorithm is full rank and $\alpha_k \neq 0$, then the algorithm can obtain the global minimum of the effective set algorithm in a finite step.

1. Algorithm(Effective set algorithm steps)

Step 1. Select initial value. Give the initial feasible point $x_0 \in \mathbb{R}^n$ and let k:=0.

Step 2. Solve the subproblem. Determine the corresponding efficient set $S_k = E \bigcup I(x_k)$ and solve minimum point d_k and Lagrange multiplier vector λ_k of the subproblem:

$$\begin{cases} min & q_k(d) = \frac{1}{2}d^T H d + g_k^T d, \\ s.t. & a_i^T d = 0, i \in S_k. \end{cases}$$

If $d_k \neq 0$, go to step 4 ; otherwise, go to step 3.

Step 3. Examine the termination criteria. Calculate the Lagrange multiplier $\lambda_k = B_k g_k$, where

$$g_k = Hx_k + c,\ B_k = (A_k H^{-1} A_k^T)^{-1} A_k H^{-1}, A_k = (a_i)_{i \in S_k},$$

and let $(\lambda_k)_t = \min_{i \in I(x_k)} \{(\lambda_k)_i\}$. If $(\lambda_k)_t \geq 0$, x_k is the global minimum and stop. Otherwise, if $(\lambda_k)_t < 0$, let $S_k := S_k \backslash \{t\}$ and go to step 2.

Step 4. Determine the step length α_k. Let $\alpha_k = min\{1, \alpha_k\}$, where $\bar{\alpha}_k = \min_{i \notin S_k} \{\frac{b_i - a_i^T x_k}{a_i^T d_k} | a_i^T d_k < 0\}$. Let $x_{k+1} := x_k + a_k d_k$.

Step 5. Let $k := k + 1$, go to step 2.

3.4 Improved calculation method for modified generator matrix

In this section, we propose a new method for calculating the generated matrix of credit level transfer, which mainly improves the two calculation methods mentioned above from the following two aspects.

On the one hand, because the matrix $P(0, T)$ has strong diagonally dominant property, that is, $\| \log P(0, T) \|$ is small. Compared with the two algorithms mentioned above, we use the improved anti-scaling and squaring algorithm based on relative precision designed in the previous chapter to ensure the relative precision of the results. Then, the improved algorithm is used to obtain logarithm matrix $\log P(0, T)$ and the calculated value $\tilde{Q} = (\tilde{q}_{ij})$ of $logP(0, T)/T$ is also obtained.

On the other hand, in the two algorithms for calculating the generated matrix mentioned in the previous section, only the definition properties of the generator matrix that the matrix Q should satisfy are guaranteed. However, in recent years, scholars have found that in the real world credit risk assessment, more accurate predictions can be obtained through optimization methods. Next, we begin to discuss effective algorithms for solving this optimization problem.

In the literature [6], the genetic algorithm is used to solve the linear constraint optimization problem, and the global optimization problem is solved by adding the repair operator. In Ma Changfeng's book, *Optimization Method and Its Matlab Program Design[3]*, the methods for solving quadratic problems are optimal conditions, penalty function method, feasible direction method, quadratic programming and sequential quadratic programming. Considering the computational cost of high-dimentional matrices in practical applications and the matrix properties obtained by our improved logarithmic algorithm, we use the effective set method in quadratic programming.

Here, we assume that the transfer matrix published by the company is denoted as P, and the logarithmic matrix of P obtained by the improved algorithm is denoted as $\tilde{Q}$. Considering that the final calculation error $||e^Q - P||$ is as small as possible, due to e, P are constants, we want to convert to $||Q - \tilde{Q}||$ as small as possible. Therefore, the process of modifying the matrix $\tilde{Q}$ can be transformed into solving the following optimization problems:

$$\begin{cases} min & norm(Q - \tilde{Q}), \\ s.t. & \sum_{j=1}^{n} q_{ij} = 0, \\ & q_{ij} \geq 0, i \neq j, \\ & q_{ij} \leq 0, i = j. \end{cases} \tag{3.1}$$

It is easy to know that the problem (3.1) is different in the feasible range of each variable in the system and this does not conform to the expression of the previous effective set quadratic programming. Next we modify the problem (3.1) to the quadratic programming problem of linear inequality constraints which we deal with next.

On the one hand, by observing, in order to make the feasible domain symbols of each variable consistent, we replace the diagonal elements with other linear combinations of non-diagonal elements by the equation transformation.

On the other hand, we choose the closest distance function as the approximation for the objective function. For the complete expansion of quadratic,because it has no effect on the optimization objective function, we can directly discard it. So we can get the optimized form of the following problem (3.1):

$$\begin{cases} \min & (-\sum_{j\neq i} q_{ij} - \tilde{q}_{ii})^2 + \sum_{j\neq i} (q_{ij} - \tilde{q}_{ij})^2, \\ s.t. & -\sum_{j\neq i}^{n} q_{ij} \geq -1, \\ & q_{ij} \geq 0, i \neq j. \end{cases} \tag{3.2}$$

4. Numerical Experiments

In the previous chapter, we systematically analyze how to calculate the generated matrix of the transfer process in the credit risk grade. Now we verify the ***algorithm(Effective set algorithm steps)*** numerically with several practical cases. The matrix P in the following examples is an annual transfer matrix that is processed based on historical migration data published by the rating company.

1. example Take table3 in the literature [1] as matrix P, which is the average annual transfer matrix obtained by processing the annual transfer process published on the Standard and Poor's Credit Review(1993), where $T = 1$.

$$P = \begin{pmatrix} 0.8910 & 0.0963 & 0.0078 & 0.0019 & 0.0030 & 0.0000 & 0.0000 & 0.0000 \\ 0.0086 & 0.9010 & 0.0747 & 0.0099 & 0.0029 & 0.0029 & 0.0000 & 0.0000 \\ 0.0009 & 0.0291 & 0.8894 & 0.0649 & 0.0101 & 0.0045 & 0.0000 & 0.0009 \\ 0.0006 & 0.0043 & 0.0656 & 0.8427 & 0.0644 & 0.0160 & 0.0018 & 0.0045 \\ 0.0004 & 0.0022 & 0.0079 & 0.0719 & 0.7764 & 0.1043 & 0.0127 & 0.0241 \\ 0.0000 & 0.0019 & 0.0031 & 0.0066 & 0.0517 & 0.8246 & 0.0435 & 0.0685 \\ 0.0000 & 0.0000 & 0.0116 & 0.0116 & 0.0203 & 0.0754 & 0.6493 & 0.2319 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

We first transfer the matrix P for the improved algorithm logarithm calculation, and then the matrix calculated by Matlab is optimized to the form of the problem 3.1. In the end, the effective matrix method is used to solve the problem to obtain the modified matrix Q. Obviously, $P \in R^{8\times 8}$, here take $n = 8$ to determine the quadratic matrix H and vector c of the objective function respectively(for the convenience of the mark, reduce the dimension first and take line i as an example):

$$H = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}_{7\times 7}, c = \begin{bmatrix} \vdots \\ -2(\tilde{q}_{i,j-1} + \tilde{q}_{ii}) \\ -2(\tilde{q}_{i,j+1} + \tilde{q}_{ii}) \\ \vdots \end{bmatrix}, Ae = [], be = [],$$

$$Ai = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, bi = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Take the initial feasible point $q_{i0} = (0, \cdots, 0)^T$. If $norm(A)$ represents the sum of the absolute values of the elements of matrix A,we can get $norm(e^Q - P) = 0.0681730$ calculated by Matlab when $tol = 10^{-10}$.

In the literature [5], the author considers the correction factor, but the Q_{IRW} obtained without optimization processing is compared with the approximate generated matrix Q_{JLT} obtained without correction processing in the literature [1], The error results are shown in the table below:

Table 4-1: Comparison of error values in the first example

$norm(e^Q - P)$	$norm(e^{Q_{IRW}} - P)$	$norm(e^{Q_{JLT}} - P)$
0.0681730	0.002736	0.116900

In this way, we can find that the generated matrix obtained by using the improved algorithm of computing logarithm first, and then the modified optimization into inequality constraints in the form of quadratic programming. Finally, the calculation accuracy of the modified matrix we finally get is one order of magnitude worse than Q_{IRW}, but one order of magnitude better than Q_{JLT}.

2. example Take matrix P in the literature [1], which is the average annual transfer matrix obtained by processing the annual transfer process published on the Standard and Poor's

Credit Review(1999), where $T = 1$.

$$P = \begin{pmatrix} 0.919347 & 0.074592 & 0.004829 & 0.000822 & 0.000411 & 0.000000 & 0.000000 & 0.000000 \\ 0.006396 & 0.918085 & 0.067575 & 0.005984 & 0.000619 & 0.001135 & 0.000310 & 0.000000 \\ 0.000730 & 0.022725 & 0.916814 & 0.051183 & 0.005629 & 0.002502 & 0.000104 & 0.000417 \\ 0.000425 & 0.002658 & 0.055609 & 0.878894 & 0.048272 & 0.010207 & 0.001701 & 0.002339 \\ 0.000441 & 0.000991 & 0.006058 & 0.077542 & 0.814847 & 0.078973 & 0.011125 & 0.010133 \\ 0.000000 & 0.001019 & 0.002830 & 0.004641 & 0.069496 & 0.827957 & 0.039615 & 0.054556 \\ 0.001860 & 0.000000 & 0.003720 & 0.007439 & 0.024294 & 0.121237 & 0.604557 & 0.237010 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{pmatrix}$$

In a similar way,$norm(e^Q - P) = 0.04257830$ is calculated by Matlab, when $tol = 10^{-5}$. The error results are shown in the table below:

Table 4-2: Comparison of error values in the first example

$norm(e^Q - P)$	$norm(e^{Q_{IRW}} - P)$	$norm(e^{Q_{JLT}} - P)$
0.04257830	0.001096	0.103516

And we can also find that the calculation accuracy of the modified matrix, which we finally get is one order of magnitude worse than Q_{IRW}, but one order of magnitude better than Q_{JLT}.

5. Conclusions

Firstly, the logarithmic generated matrix to be optimized is obtained from the annual transfer matrix of credit rating risk by the improved inverse scaling and squaring square method.

Secondly, under the original K-S optimization model, the newly obtained logarithmic matrix is modified to the Convex optimization problem of constraint of linear inequality by dimensionality reduction, variable substitution and modification. The efficient set method is used to optimize the Matlab calculation to satisfy the definition properties of the generating matrix. At the same time, the method has complete convergence and stability, and the calculation accuracy of the final results is also higher than that of the original model.

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