Idempotents of M_2 (Z_{15}[x])

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Abstract

In this article we will discuss the idempotents of M_2 (Z_{15} [x])

Key Words - *Idempotents, polynomials, trace etc.*

I. INTRODUCTION

Idempotents in ring theory plays a vital role in the ring theory. Here, we will find the idempotents in matrix ring M_2 (Z_{15} [x]) where Z_{15} [x] is the poly nominal ring over the ring Z_{15}. For any ring R, I (R) will denote for set of all idempotents in R. For any positive integer n, M_n (R) will denote the ring of n×n matrices over a ring R.

For a 2×2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over a commutative ring R, determinate of A is ad-bc and trace of A is a+d.

Definition: Let R be a ring. An element a ϵ R is said to be idempotent in R if a^2 = a

Theorem 1 If R is a commutative ring then I(R[x]) = I (R)

Theorem 2 Any non=trivial idempotent is M_2 (Z_{15} [x]) is one of the following form

1. $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$
2. $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$, where a (x)(1-a(x))– b(x) c (x) = 0
3. $\begin{bmatrix} 6a(x) & 6b(x) \\ 6c(x) & 6(1-a(x)) \end{bmatrix}$, where a (x)(1-a(x)) – b(x) c (x) = 5f(x)
4. $\begin{bmatrix} 5a(x) & 5b(x) \\ 5c(x) & 5(5-a(x)) \end{bmatrix}$, where a (x)(5-a(x)) – b(x) c(x) = 3g(x)
5. $\begin{bmatrix} 2+5a(x) & 5b(x) \\ 5c(x) & 2-5a(x) \end{bmatrix}$, where (2+5a(x))(2-5a(x))-25b(x) c(x) = 6
6. $\begin{bmatrix} 1+5a(x) & 5b(x) \\ 5c(x) & 6-5a(x) \end{bmatrix}$, where (1+5a(x))(6-5a(x)) – 25 b(x) c (x) = 6

Where a(x), b(x),c(x), f(x), g(x) are polynomial in Z_{15}[x].

Proof: As the idempotents in Z_{15}[x] are the idempotents in Z_{15}. Therefore the idempotents in Z_{15}[x] are 0, 1, 6, 10 let A=$\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$ be a non trivial idempotent of M_2(Z_{15}[x]). For our convenience, we will take a(x) = a, b(x) = b, c(x) = c, d(x)=d

Now a is idempotent, so a^2+bc=a, b(a+d)=b, c(a+d)=c and bc+d^2=d. Also determinant of A is an idempotent in Z_{15} so the determinant of A is 0 or 1 or 6 or 10.

If determinant of A is 1 Then A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, a trivial idempotent in M_2(Z_{15}[x]). Hence the determinant of A is 0 or 6 or 10. Also, trace of A is in Z_{15} i.e. a+d ϵ $Z_{15.}$

Case 1 : Determinant of A is 0 i.e. ad-bc = 0

Since A is an idempotent

Therefore, a^2+bc+bc+d^2 = a^2+2bc+d^2 = a^2+2ad+d^2 = (a+d)

Thus (a+d) is an idempotent in Z_{15}[x]. Thus (a+d) is either 0 or 1 or 6 of 10.

If a+d=0,then we get A to be a zero matrix, which is trivial idempotent in M_2(Z_{15}[x]).

If a+d=1 then d=1-a and hence ad-bc=0 gives a^2+bc=a, (a+d) b=b. Also (a+d) c=c and bc+d^2=1-a. Thus A^2 =$\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$. Thus in this case, matrix A= $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$, where a(x), b(x), c(x) ϵZ_{15}[x] such that a(x){1-a(x)} = b(x) c(x).

If a+d=6 gives d=6-a and hence ad-bc = 0 gives a^2+bc=6a and so, 5a=0. Also (a+d) b=b implies 5b=0 and (a+d)c=c implies 5c=0

Therefore, a=6a' (x), b=6b'(x) and c = 6c'(x), where a' (x), b' (x), c'(x) are polynomial in Z_{15}[^{x}].

Since ad-bc=0, we get 6a'(x)(6-6a'(x)=6b'(x)6c'(x), which is equivalent to a'(x)(a'(x))-b'(x)c'(x)=5f(x) for some polynomial f(x) ϵ Z_{15}[x]

Hence A $=\begin{bmatrix} 6a(x) & 6b(x) \\ 6c(x) & 6(1-a(x)) \end{bmatrix}$,where a(x), b(x), c(x) are polynomial in Z_{15} [x] such that a(x) (1-a(x))-b(x) c(x) = 5f(x) for some f(x) t Z_{15} [x].

If a+d=10 then d=10-a

Now ad-bc = 0 gives a^2+bc=10a. Thus 9a=0

Also (a+d) b=b gives 9b=0 and (a+d) c=c gives 9c=0

Therefore, a=5a'(x), b=5b' (x) and c=5c' (x) and d= 5(5-a' (x), where a' (x), b' (x) and c' (x) are polynominal in Z_{15}[x].

Now since ad-bc = 0, we get 5a'(x) 5 (5-a' (x)) = 5b' (x)-5 c' (x). hence, idempotent matrix is A $=\begin{bmatrix} 5a(x) & 5b(x) \\ 5c(x) & 5(5-a(x)) \end{bmatrix}$, where a(x) (5-a(x)) –b(x)c(x) = 3g(x) for some g(x) ϵ Z_{15}[x].

Case 2 : Determinant of A is 6. This means ad-bc=6.

So, we get a^2+bc+bc+d^2 = a^2+2(ad-6)+d^2 =$(a+d)^2$+3.

Trace of matric A is idempotent if a+d=4 or 7 or 9 of 12.

If a+d=4 then ad-bc=6 implies 3a=6 i.e. a=2 or 7 or 12 in Z_{15}[x]. i.e. a=2+5a(x) for some polynomial a(x) ϵ Z_{15}[x].

Also (a+d) b=b gives 3b=0 and (a+d) c=c gives 3c=0

i.e. b=5b(x) and c=5c(x) for some polynomials b(x) and c(x) in Z_{15}[x].

Hence matrix A $=\begin{bmatrix} 2+5a(x) & 5b(x) \\ 5c(x) & 2-5a(x) \end{bmatrix}$, where a(x), b(x), c(x) are polynomial in Z_{15}[x]. such that (2+5a(x)) (2-5a(x)) – 25b(x) c(x)= 6

If a+d =7 then d=7-a

Now ad-bc=6 gives 6a=6 i.e. a can be 1 or 6 or 11 i.e a=1+5a(x) for some a(x) ϵ Z_{15}[x].

Also b(a+d)=b gives 6b=0 and c(a+d)=c gives 6c=0 a=1+5a(x), b=5b(x), c=5c(x) and d=6-5a(x) for some polynomials a(x), b(x), c(x) in Z_{15}[x].

Hence, matrix A $=\begin{bmatrix} 1+5a(x) & 5b(x) \\ 5c(x) & 6-5a(x) \end{bmatrix}$, where a(x), b(x), c(x) are polynomials in Z_{15}[x] such that (1+5a(x)) (6-5a(x)) – 5b(x) 5c(x) = 6.

If a+d =9 then d=9-a

Now ad-bc = 6 gives a^2+bc = 9a-6 i.e. 4a=3

Thus a = 12. Also b(a+d)=b gives 8b=0 i.e. b=0 and c(a+d)=c gives 8c=0 i.e. c=0

Thus a=12, b=0, c=0 and d=12

Hence matrix A = $\begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$, which is not possible as determinant of A is 9.

If a+d = 12 then d = 12-a

Now ad-bc = 6 gives 11a=6 i.e. a=6

Also b(a+d)=b gives 11b=0 i.e. b=0 and c(a+d)=c gives 11c=0 i.e. c=0

Hence matrix A $=\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

Case 3 : Determinate of A is 10 i.e. ad-bc=0

Since A is idempotent, therefore a^2+bc+bc+d^2 = a^2+2(ad-10)+d^2 = a^2+2ad+d^2-20 =$(a+d)^2$+10

Trace of matrix A is idempotent iff a+d=5

If a+d=5 then d=5-a

Now ad-bc=10 gives 4a=10 i.e. a=10.

Also b(a+d)=b gives 4b=0 i.e. b=0 and c(a+d)=c gives 4c=0 i.e. c=0

Thus a=10, b=0, c=0 and d=10

Hence, matrix A $=\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

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