On the System of Double Equations $x + y^2 = a^2, x - y^2 = b^2$

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Abstract

The paper aims at determining sets of non-zero distinct integer solutions to the system of double equations given by $x + y^2 = a^2, x - y^2 = b^2$. *A few numerical examples are given. A few interesting properties among the solutions are also presented.*

Keywords - *System of double equations, integer solutions.*

Notations:

- Polygonal number of rank n with size m

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$P_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$$

- Star number of rank n

$$S_n = 6n(n-1)+1$$

- Stella Octangular number of rank n

$$SO_n = n(2n^2 - 1)$$

- Pronic number of rank n

$$\Pr_n = n(n+1)$$

- Fourth dimensional Figurate number of rank r whose generating polygon has s-sides

$$F_{4,s}^r = \frac{r(r+1)(r+2)(rs - s + 4)}{4!}$$

I. INTRODUCTION

Systems of indeterminate quadratic equations of the form $ax + c = u^2$, $bx + d = v^2$ where a, b, c, d are non-zero distinct constants, have been investigated for solutions by several authors [1, 2] and with a few possible exceptions, most of the them were primarily concerned with rational solutions. Even those existing works wherein integral solutions have been attempted, deal essentially with specific cases only and do not exhibit methods of finding integral solutions is a general form. In [3], a general form of the integral solutions to the system of equations $ax + c = u^2$, $bx + d = v^2$ where a, b, c, d are non-zero distinct constants is presented when the product ab is a square free integer whereas the product cd may or may not a square integer. For other forms of system of double diophantine equations, one may refer [4-23].

In this paper, yet another system of double equations given by $x + y^2 = a^2$, $x - y^2 = b^2$ is studied for obtaining its non-zero distinct integer solutions. A few numerical examples are given. A few interesting properties among the solutions are also presented.

II. METHOD OF ANALYSIS

The system of equations to be solved is

$$x + y^2 = a^2 \quad (1)$$

$$x - y^2 = b^2 \quad (2)$$

On solving (1) and (2), one obtains

$$x = \frac{a^2 + b^2}{2} \quad (3)$$

and $$y^2 = \frac{a^2 - b^2}{2} \quad (4)$$

As we require *x* and *y* to be in integers, it is seen that *x* and *y* are integers when both *a* and *b* are of the same parity.

Case (1):

Let *a* and *b* be both odd.
That is assume

$$a = 2A + 1 \,,\; b = 2B + 1 \quad (5)$$

From (3) and (5), we have

$$x = 2(A^2 + B^2) + 2(A + B) + 1 \quad (6)$$

Using (5) in (4), and performing a few calculations, we get

$$P^2 = 2y^2 + Q^2 \quad (7)$$

where

$$P = 2A + 1 \,,\; Q = 2B + 1 \quad (8)$$

Note that (7) is satisfied by

$$y(r,S) = 2r(2S + 1) \quad (9)$$

$$\left.\begin{aligned} Q &= 2r^2 - (2S + 1)^2 \\ P &= 2r^2 + (2S + 1)^2 \end{aligned}\right\} \quad (10)$$

Substituting (10) in (8) and in view of (6), we have

$$x(r,S) = 4r^4 + (2S + 1)^4 \quad (11)$$

Thus, (9) and (11) represent the integer solutions to the system of equations (1) and (2)

As the *x*-value is written as the sum of two squares, from (2) note that this *x*-value represents the second order Ramanujan Number.

A few numerical examples are given below in Table 1:

Table 1: Examples

r	S	x	y	$x + y^2$	$x - y^2$
1	3	2405	14	51^2	47^2
2	4	6625	36	89^2	73^2
1	2	629	10	27^2	23^2
2	3	2465	28	57^2	41^2

Properties:

- $x(1,S) - 384\, F^{s}_{4,1} + 32\, SO_s + 152\, \Pr_s - 5 \equiv 0 \pmod{2}$

- $y((S+1)(S+2),S) = 24P_S^3 + 4t_{3,S+1}$
- $y(r, r(r+1)) - 8P_r^5 \equiv 0 \pmod{2}$

It is worth mentioning here that, (7) is also solved by factorization method as follows:
Assume

$$P = \alpha^2 + 2\beta^2 \tag{12}$$

Employing (12) in (7) and applying the method of factorization, define

$$Q + i\sqrt{2}\,y = (\alpha + i\sqrt{2}\beta)^2$$

Equating the rational and irrational parts, note that

$$y = 2\alpha\beta \tag{13}$$

$$Q = \alpha^2 - 2\beta^2 \tag{14}$$

Substituting (12) and (14) in (8), we have

$$\left.\begin{aligned} A &= \frac{\alpha^2 + 2\beta^2 - 1}{2} \\ B &= \frac{\alpha^2 - 2\beta^2 - 1}{2} \end{aligned}\right\} \tag{15}$$

Choosing $\alpha = 2C + 1$ in (15) and (13), we have

$$\left.\begin{aligned} A &= 2C^2 + 2C + \beta^2 \\ B &= 2C^2 + 2C - \beta^2 \end{aligned}\right\} \tag{16}$$

$$y(C,\beta) = 2\beta(2C+1) \tag{17}$$

Using (16) in (6), the value of *x* is given by

$$x(C,\beta) = (2C+1)^4 + 4\beta^4 \tag{18}$$

Thus, (17) and (18) represent the integer solutions to the system of equations (1) and (2).

A few numerical examples are given below in Table 2:

Table 2: Examples

β	C	x	y	$x+y^2$	$x-y^2$
1	3	2405	14	51^2	47^2
2	4	6625	36	89^2	73^2
1	2	629	10	27^2	23^2
2	3	2465	28	57^2	41^2

Remark:
It is worth mentioning that the values of x and y given by (9), (11) and (17), (18) are the same but the approach is different.

Case (2):

Let *a* and *b* be both even.
(i.e) Take

$$a = 2A,\ b = 2B \tag{19}$$

From (3) and (4), we have

$$x = 2(A^2 + B^2) \tag{20}$$

$$y^2 = 2(A^2 - B^2) \tag{21}$$

Now, (21) is written in the form of ratio as

$$\frac{y}{A-B} = \frac{2(A+B)}{y} = \frac{P}{Q}, \quad Q > 0$$

which is equivalent to the system of double equations

$$PA - PB - Qy = 0$$

$$2QA + 2QB - Py = 0$$

Employing the method of cross multiplication, we have

$$y(P,Q) = 4PQ \tag{22}$$

$$\left.\begin{array}{l} A = P^2 + 2Q^2 \\ B = P^2 - 2Q^2 \end{array}\right\} \tag{23}$$

Using (23) in (20), the value of x is given by

$$x(P,Q) = 4(P^4 + 4Q^4) \tag{24}$$

Then, (22) and (24) represent the integer solutions to the system of equations (1) and (2).

A few numerical examples are given below in Table 3:

Table 3: Examples

P	Q	x	y	$x + y^2$	$x - y^2$
1	3	1300	12	38^2	34^2
2	4	4160	32	72^2	56^2
1	2	260	8	18^2	14^2
2	3	1360	24	44^2	28^2

Properties:

- $x(P, P+1) - 480\, F^{P}_{4,1} + 28\, SO_P + 124\, \Pr_P - 16 \equiv 0 \pmod{2}$
- $y(P, P+1) - 4\Pr_P = 0$
- $x(Q+1, Q) + y(Q+1, Q) - 20(t_{4,Q})^2 - 32\, P^5_Q - 16\, t_{3,Q} - 4\Pr_Q - 4 \equiv 0 \pmod{4}$

Also, (21) is written as

$$2B^2 = 2A^2 - y^2 \tag{25}$$

Assume

$$B = 2\alpha^2 - \beta^2 \tag{26}$$

Write 2 as

$$2 = (3\sqrt{2} + 4)(3\sqrt{2} - 4) \tag{27}$$

Substituting (26) and (27) in (25) and applying the method of factorization, define

$$\sqrt{2}A + y = (3\sqrt{2} + 4)(\sqrt{2}\alpha + \beta)^2$$

Equating the rational and irrational parts, we get

$$y(\alpha, \beta) = 8\alpha^2 + 4\beta^2 + 12\alpha\beta \tag{28}$$

$$A = 6\alpha^2 + 3\beta^2 + 8\alpha\beta \tag{29}$$

In view of (20), we have

$$x(\alpha,\beta) = 2[(6\alpha^2 + 3\beta^2 + 8\alpha\beta)^2 + (2\alpha^2 - \beta^2)^2] \tag{30}$$

Thus (30) and (28) represent the integer solutions to the system of equations (1) and (2).

A few numerical examples are given below in Table 4:

Table 4: Examples

α	β	x	y	$x+y^2$	$x-y^2$
1	3	6596	80	114^2	14^2
2	4	37120	192	272^2	16^2
1	2	2320	48	68^2	4^2
2	3	19604	140	198^2	2^2

Properties:

- $x(\alpha,1) - 80(t_{4,\alpha})^2 - 384P_\alpha^5 - 20 \equiv 0 \pmod 2$
- $y(1,\beta) - S_\beta + 4t_{3,\beta} - 7 \equiv 0 \pmod 2$
- $y(\alpha^2,\alpha+1) - 8(t_{4,\alpha})^2 - 24P_\alpha^5 - 4\Pr_\alpha - 4 \equiv 0 \pmod 4$

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