# On the Continuous Dependence of a Functional Integral Equation with Parameter

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***Abstract***

*In this work, we study the exitance of at least one and exactly one continuous or integrable solution of a functional integral equation with parameter. The continuous dependence of the unique solution on parameter and the function it self will be studied.*

**MSC:** 34A12; 34A40

**Keywords -** *Functional integral equation, continuous dependence, existence of solution*

## I. INTRODUCTION

Functional integral equation appear in various research areas and have applications in Euclidean and non-Euclidean geometry, Mechanics, Business mathematics, Biology, and probability theory. It can arise in practice in several ways. Some occur in direct physical applications, others are from defining properties of functions, and still others are generalizations of known identities, see [1]-[2]and[3]-[4]-[7]-[8].

In this work, we are concerning with the functional integral equation

$$x(t) = f_1(t, \int_0^t f_2(s, x(s), \mu) ds), \quad t \in [0, T]. \tag{1}$$

The existence of a least one solution will be study. Also we prove the existence of a unique solution in the two spaces $C[0,T]$ and $L^1[0,T]$. The continuous dependence of the solution on the parameter $\mu$ and the function $f_2$ will be also studied.

## II. PRELIMINARIES

Let $L_1(J,R)$ denotes the space of all Lebesgue integrable functions on the interval $J = (0,T]$ with the norm $||u||_1 = \int_0^T |u(t)| dt$.

$$Let\ C(J,R) = \{u: u(t)\ is\ continuous\ on\ J: ||u|| = \max_{t\in J}|u(t)|\},$$

The following theorems will be needed.

**Theorem 2.1.** (Schauder fixed point theorem [5])

"Let $E$ be a Banach space and $Q$ be a closed, convex subset of $E$, and $F: Q \to Q$ is continuous, compact operator, then $F$ has at least one fixed point in $Q$".

**Theorem 2.2.** (Kolmogorov compactness criterion [5]) Let $\Omega \subseteq L^p(J,R), 1 \le p < \infty$. If

(i) $\Omega$ is bounded in $L^p(J,R)$,

(ii) $u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$,

then $\Omega$ is relatively compact in $L_p(J,R)$, where

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s) ds.$$

The superposition operator associated with $f$ is defined as follows.

**Definition 2.3.** Let $f: J \times R \to R$ be a Carathe′odory function. The superposition operator generated by $f$ is the operator $Fx$ which assigns to each real measurable function on $J$ the real function $(Fx)(t) = f(t, x(t)), t \in J$.

We have the following theorem due to Krasnosel'skii [5]

**Theorem 2.3.** The superposition operator $F$ generated by the function $f$ maps the space $L_1(J,R)$ continuously into itself if and only if

$$|f(t,x)| \le |a(t)| + b|x|, for\ \ all\ t \in J\ and\ x \in R,$$

where $a(t)$ is a function in $L_1(J,R)$ and $b$ is a nonnegative constant.

## III. INTEGRABLE SOLUTIONS

### A. Existence of at least one integrable solutions

Consider the functional integral equation (1) under the following assumptions

$f_1: [0,T] \times R \to R$ satisfies Carathe'odory condition, i.e it is measurable in $t \in [0,T]$ for every $x \in R$ and continuous in $x \in R$ for every $t \in [0,T]$ and there exist a function $a_1 \in L_1[0,T]$ and a constant $b_1 > 0$ s.t

$$|f_1(t,x)| \leq a_1(t) + b_1|x|.$$

$f_2: [0,T] \times R \times R \to R$ satisfies Carathe'odory condition, i.e it is measurable in $t \in [0,T]$ for every $x \in R$ and continuous in $x \in R$ for every $t \in [0,T]$ and there exist a function $a_2(t) \in L^1[0,T]$ and a constant $b_2 > 0$ such that

$$|f_2(t,x(s),\mu)| \leq a_2(t) + b_2|x| + |\mu|.$$

$b_1 b_2 T < 1$.

Now for existence of at least one solution of the functional integral equation (1), we have the following theorem.

**Theorem 1** *Let the assumption* $(i)-(iii)$ *be satisfied then the functional integral equation (1) has at least one integrable solution* $x \in L_1[0,T]$.

*Proof.* Define the operator F by

$$Fx(t) = f_1(t, \int_0^t f_2(s,x(s),\mu)ds), \quad t \in [0,T].$$

Define the set

$$Q_r = \{x \in R: |x| \leq r\} \quad where \quad r = \frac{\|a_1\| + b_1 T \|a_2\| + \frac{1}{2} b_1 b_2 T^2 |\mu|}{1 - b_1 b_2 T}.$$

Let $x \in Q_r$, then

$$\begin{aligned}
|Fx(t)| &= |f_1(t, \int_0^t f_2(s,x(s),\mu)ds)| \\
&\leq a_1(t) + b_1 |\int_0^t f_2(s,x(s),\mu)ds| \\
&\leq a_1(t) + b_1 \int_0^t |f_2(s,x(s),\mu)|ds \\
&\leq a_1(t) + b_1 \int_0^t [a_2(s) + b_2|x(s)| + |\mu|]ds \\
&\leq a_1(t) + b_1 \int_0^t a_2(s)ds + b_1 b_2 \int_0^t |x(s)|ds + b_1 b_2 \int_0^t |\mu|ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\| Fx \|_1 = \int_0^T |Fx(t)|dt \leq \int_0^T a_1(t)dt + b_1 \int_0^T \int_0^t a_2(s)dsdt \\
&+ b_1 b_2 \int_0^T \int_0^t |x(s)|dsdt + b_1 \int_0^T \int_0^t |\mu|dsdt, \\
&\leq \| a_1 \|_1 + b_1 T \| a_2 \|_1 + b_1 T b_2 r + \frac{1}{2} b_1 T^2 |\mu| = r.
\end{aligned}$$

This proves that $F: Q_r \to Q_r$ and the class of functions $\{Fx\}$ is bounded in $Q_r$.

Let $x \in Q_r$, then

$$\begin{aligned}
&\| (Fx)_h - (Fx) \|_1 = \int_0^T |(Fx(s))_h - (Fx(s))|ds \\
&= \int_0^T \frac{1}{h} |\int_t^{t+h} (Fx(\theta))d\theta - (Fx(s))|ds \\
&\leq \int_0^T \frac{1}{h} \int_t^{t+h} |(Fx(\theta)) - (Fx(s))|d\theta ds \\
&\leq \int_0^T \frac{1}{h} \int_t^{t+h} |f_1(\theta, \int_0^t f_2(\theta, x(s), \mu)) \\
&- f_1(t, \int_0^T f_2(s, x(s), \mu))|d\theta ds
\end{aligned}$$

since $f_1 \in L_1[0,T]$ it follows that

$$\frac{1}{h} \int_t^{t+h} |f_1(\theta, \int_0^t f_2(\theta, x(s), \mu)d\theta) - f_1(t, \int_0^t f_2(s,x(s),\mu)ds)|ds \to 0, \quad as \quad h \to 0.$$

Hence $Fx(t)_h \to (Fx)$ uniformly in $L_1[0,T]$. Thus the class $\{Fx, x \in Q_r\}$ is relatively compact. Hence F is compact operator.

Let $x_n \subset Q_r, x_n \to x$, then

$$Fx_n(t) = f_1(t, \int_0^t f_2(s, x_n(s), \mu)ds)$$

take the limit as $n \to \infty$, then

$$\lim_{n\to\infty} Fx_n(t) = \lim_{n\to\infty} f_1(t, \int_0^t f_2(s, x_n(s), \mu)ds)$$

since $f_1, f_2$ are continuous in $x$, then

$$\lim_{n\to\infty} Fx_n(t) = f_1(t, \lim_{n\to\infty} \int_0^t f_2(s, x_n(s), \mu)ds)$$

Now, from assumptions $(i), (ii)$ and Lebesgue dominated convergence the equation (1) , we have

$$\lim_{n\to\infty} Fx_n(t) = f_1(t, \int_0^t \lim_{n\to\infty} f_2(s, x_n(s), \mu)ds)$$
$$= f_1(t, \int_0^t f_2(s, x(s), \mu)ds)$$
$$= Fx(t).$$

This means that $Fx(t)_h \to (Fx)$. Hence the operator $F$ is continuous. Now all the conditions of Schouder fixed point Theorem[5] are satisfied. Then the functional integral equation (1) has at lest one solution $x \in L_1[0, T]$.

### *B. Existence of a unique integrable solution*

Consider following assumptions

$f_1: [0, T] \times R \to R$ is measurable in $t \in [0, T]$ and satisfies the Lipschitz condition

$$|f_1(t, x) - f_1(t, y)| \le b_1|x - y|.$$

$f_2: [0, T] \times R \times R \to R$ is measurable in $t \in [0, T]$ and satisfies the Lipschitz condition

$$|f_2(t, x, \mu) - f_2(t, y, \mu^*)| \le b_2|x - y| + |\mu - \mu^*|.$$

$$f_1(t, 0), f_2(s, 0,0) \in L_1[0, T]$$
$$\int_0^t f_1(t, 0)dt \le M, b_1 b_2 \int_0^T \int_0^t f_2(s, 0,0)dsdt \le N.$$

$b_1 b_2 T < 1.$

Now for existence of a unique integrable solution of function integral equation (1) we have the following theorem.

**Theorem 2** *Let the assumptions* $(i^*)$–$(V^*)$ *be satisfied, then the functional integral equation (1), has a unique integrable solution* $x \in L_1[0, T]$.

*Proof.* From assumption $(i^*)$, we obtain

$$|f(t.x)| - |f(t, 0)| \le |f_1(t, x) - f_1(t, 0)| \le b_1|x|.$$

This implant

$$|f(t, x)| \le b_1|x| + |f_1(t, 0)| = b_1|x| + a_1(t), a_1(t) = |f_1(t, 0)|.$$

Similarly

$$|f_2(t, x, \mu)| - |f_2(t, 0,0)| \le |f_2(t, x, \mu) - f_2(t, 0,0)| \le b_2|x| + |\mu|.$$

This implant

$$|f_2(t, x, \mu)| \le b_2|x| + |\mu| + a_2(t), a_2(t) = |f_2(t, 0,0)|.$$

Then the assumptions of Theorem 1 are satisfied. Then the functional integral equation (1) has at lest one solution. Let $x, y$ be the two solution of the functional integral equation (1) then

$$|x(t) - y(t)| = |f_1(t, \int_0^t f_2(s, x(s), \mu)ds) - f_1(t, \int_0^t f_2(s, y(s), \mu)ds)|$$
$$\le b_1 | \int_0^t f_2(s, x(s), \mu)ds - \int_0^t f_2(s, y(s), \mu)ds|$$
$$\le b_1 \int_0^t |f_2(s, x(s), \mu) - f_2(s, y(s), \mu)|ds$$
$$\le b_1 b_2 \int_0^t |x(s) - y(s)|ds,$$

and, we obtain

$$\| x - y \|_1 \le b_1 b_2 \int_0^T \int_0^t |x(s) - y(s)|dsdt$$
$$\le b_1 b_2 T \| x - y \|_1,$$

since $b_1 b_2 T < 1$, $(1 - b_1 b_2 T) \| x - y \|_1 \le 0$ this implant the $x = y \in L_1[0, T]$. i.e the solution of the functional integral equation (1) is unique.

## IV. CONTINUOUS DEPENDENCE

### *A. Continuous dependence on the parameter $\mu$*

**Definition 1** *The solution $x \in L_1[0,T]$ of the functional integral equation (1) depends continuously on the parameter $\mu$, if*

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon), \quad s.t \quad |\mu - \mu^*| < \delta \Rightarrow \| x - x^* \|_1 < \epsilon$$

where $x^*$ is the unique solution $x^* \in L_1[0,T]$ of the functional integral equation

$$x^*(t) = f_1(t, \int_0^t f_2(s, x^*(s), \mu^*) ds) \quad t \in [0,T]. \tag{2}$$

**Theorem 3** *Let the assumption of Theorem 2 be satisfied, then the solution of the functional integral equation (1) depends continuously on the parameter $\mu$.*

*Proof.* Let $x, x^*$ be the solutions of the functional integral equations (1) and (2), then

$$\begin{aligned}
|x(t) - x^*(t)| &= |f_1(t, \int_0^t f_2(s, x(s), \mu) ds) - f_1(t, \int_0^t f_2(s, x^*(s), \mu^*) ds)| \\
&\leq b_1 | \int_0^t f_2(s, x(s), \mu) ds) - \int_0^t f_2(s, x^*(s), \mu^*) ds)| \\
&\leq b_1 \int_0^t |f_2(s, x(s), \mu) - f_2(s, x^*(s), \mu^*)| ds \\
&\leq b_1 b_2 \int_0^t (|x(s) - x^*(s)| + |\mu - \mu^*|) ds.
\end{aligned}$$

Hence

$$\| x - x^* \|_1 \leq b_1 b_2 \int_0^T (\int_0^t (|x(s) - x^*(s)| + |\mu - \mu^*|) ds) dt$$

$$\| x - x^* \|_1 \leq b_1 b_2 (T \| x - x^* \| + |\mu - \mu^*| (\frac{T^2}{2}))$$

$$\| x - x^* \|_1 \ (1 - b_1 b_2 T) \leq \frac{1}{2} b_1 b_2 T^2 \delta$$

and

$$\| x - x^* \|_1 \leq \frac{\frac{1}{2} b_1 b_2 T^2 \delta}{(1 - b_1 b_2 T)} = \epsilon.$$

This proves the continuous dependence of the solution on the parameter $\mu$.

### *B. Continuous dependence on the function $f_2$*

**Definition 2** *The solution $x \in L_1[0,T]$ of the functional integral equation (1) depends continuously on the function $f_2$, if*

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon), \quad s.t \quad |f_2 - f_2^*| < \delta \Rightarrow \| x - x^{**} \|_1 < \epsilon$$

where $x^{**}$ is the unique solution $x^{**} \in L_1[0,T]$ of functional integral equation.

$$x^{**}(t) = f_1(t, \int_0^t f_2^*(s, x^{**}(s), \mu) ds), \quad t \in [0,T]. \tag{3}$$

**Theorem 4** *Let the assumption of Theorem 2 be satisfied, then the solution of the functional integral equation (1) depends continuously on the function $f_2$.*

*Proof.* Let $x, x^{**}$ be the solutions of the functional integral equations (1) and (3), then

$$\begin{aligned}
|x(t) - x^{**}(t)| &= |f_1(t, \int_0^t f_2(s, x(s), \mu) ds) - f_1(t, \int_0^t f_2^*(s, x^{**}(s), \mu) ds)| \\
&\leq b_1 | \int_0^t f_2(s, x(s), \mu) ds) - \int_0^t f_2^*(s, x^{**}(s), \mu) ds)| \\
&\leq b_1 \int_0^t |(f_2(s, x(s), \mu) - f_2^*(s, x^{**}(s), \mu))| ds \\
&\leq b_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x(s), \mu) + f_2^*(s, x(s), \mu) - f_2^*(s, x^{**}(s), \mu)| ds \\
&\leq b_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x(s), \mu)| ds + b_1 \int_0^t |f_2^*(s, x(s), \mu) - f_2^*(s, x^{**}(s), \mu)| ds \\
&\leq b_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x(s), \mu)| ds + b_1 b_2 \int_0^t |x(s) - x^{**}(s)| ds.
\end{aligned}$$

Hence

$$\| x - x^{**} \|_1 \ (1 - b_1 b_2 T) \leq \frac{1}{2} b_1 T^2 \delta,$$

and

$$\| x - x^{**} \|_{L_1} \le \frac{\frac{1}{2} b_1 T^2 \delta}{(1 - b_1 b_2 T)} = \epsilon.$$

This proves the continuous dependence of the solution on the function $f_2$.

## V. CONTINUOUS SOLUTIONS

Consider now, the functional integral equation (1) under the following assumptions

1. $f_1 : [0, T] \times R \to R$ is continuous and satisfies the Lipschitz condition
$$|f_1(t, x) - f_1(t, y)| \le K_1 |x - y|.$$

2. $f_2 : [0, T] \times R \times R \to R$ is continuous and satisfies the Lipschitz condition
$$|f_2(t, x, \mu) - f_2(t, y, \mu^*)| \le K_2(|x - y| + |\mu - \mu^*|).$$

3. $K_1 K_2 T < 1$.

Now for existence of a unique continuous solution of the functional integral equation (1), we have the following theorem.

**Theorem 5** *Let the assumptions* $(1^*)$–$(3^*)$ *be satisfied then the functional integral equation (1), has a unique solution* $x \in C[0, T]$.

*Proof.* Define the operator $F$ associated with the functional integral equation (1) by

$$Fx(t) = f_1(t, \int_0^t f_2(s, x(s), \mu) ds) \quad t \in [0, T]. \tag{4}$$

Let $t_1, t_2 \in [0, T]$ $\;$ *and* $\;$ $|t_2 - t_1| \le \delta$ then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |f_1(t_2, \int_0^{t_2} f_2(s, x(s), \mu) ds) - f_1(t_1, \int_0^{t_1} f_2(s, x(s), \mu) ds)| \\
&= |f_1(t_2, \int_0^{t_2} f_2(s, x(s), \mu) ds) - f_1(t_2, \int_0^{t_1} f_2(s, x(s), \mu) ds) \\
&+ f_1(t_2, \int_0^{t_1} f_2(s, x(s), \mu) ds) - f_1(t_1, \int_0^{t_1} f_2(s, x(s), \mu) ds)| \\
&\le |f_1(t_2, \int_0^{t_2} f_2(s, x(s), \mu) ds) - f_1(t_2, \int_0^{t_1} f_2(s, x(s), \mu) ds)| \\
&+ |f_1(t_2, \int_0^{t_1} f_2(s, x(s), \mu) ds) - f_1(t_1, \int_0^{t_1} f_2(s, x(s), \mu) ds)| \\
&\le K_1 \int_{t_1}^{t_2} |f_2(s, x(s), \mu)| ds + \delta_1 \\
&\le K_1 \int_{t_1}^{t_2} (k_2(|x| + |\mu|) + |f_2(s, 0,0)|) ds + \delta_1
\end{aligned}$$

This proves $F : C[0, T] \to C[0, T]$.

Now to prove that $F$ is contraction, we have the following.

Let $x, y \in C[0, T]$, then

$$\begin{aligned}
|Fx(t) - Fy(t)| &= |f_1(t, \int_0^t f_2(s, x(s), \mu) ds) - f_1(t, \int_0^t f_2(s, y(s), \mu) ds)| \\
&\le K_1 | \int_0^t f_2(s, x(s), \mu) ds - \int_0^t f_2(s, y(s), \mu) ds| \\
&\le K_1 \int_0^t |f_2(s, x(s), \mu) - f_2(s, y(s), \mu)| ds \\
&\le K_1 \int_0^t K_2 |x(s) - y(s)| ds \\
&\le K_1 K_2 \int_0^t |x(s) - y(s)| ds \\
&\le K_1 K_2 T \| x - y \|_C.
\end{aligned}$$

since $K_1 K_2 T < 1$, then $F$ is contraction and by using Banach fixed point of Theorem[6], then there exists a unique solution $x \in C[0, T]$ of the functional integral equation (1) .

## VI. CONTINUOUS DEPENDENCE

### *A. Continuous dependence on the parameter* $\mu$

**Definition 3** *The solution* $x \in C[0, T]$ *of the functional integral equation (1) depends continuously on the parameter* $\mu$, *if*

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon), \quad s.t \quad |\mu - \mu^*| < \delta \Rightarrow \| x - x^* \|_c < \epsilon$$

where $x^*$ is the unique solution $x^* \in C[0, T]$ of problem (2).

**Theorem 6** *Let the assumption of Theorem 5 be satisfied, then the solution of the functional integral equation (1) depends continuously on the parameter* $\mu$.

*Proof.* Let $x, x^*$ be the solutions of the function integral equations (1) and (2), then

$$|x(t) - x^*(t)| = |f_1(t, \int_0^t f_2(s, x(s), \mu)ds) - f_1(t, \int_0^t f_2(s, x^*(s), \mu^*)ds)|$$
$$\leq K_1 |\int_0^t f_2(s, x(s), \mu)ds - \int_0^t f_2(s, x^*(s), \mu^*)ds|$$
$$\leq K_1 \int_0^t |f_2(s, x(s), \mu) - f_2(s, x^*(s), \mu^*)|ds$$
$$\leq K_1 K_2 \int_0^t (|x(s) - x^*(s)| + |\mu - \mu^*|)ds$$
$$= K_1 K_2 \int_0^t (|x(s) - x^*(s)|ds + K_1 K_2 \int_0^t |\mu - \mu^*|)ds$$
$$\leq K_1 K_2 T \parallel x - x^* \parallel_C + K_1 K_2 T\delta.$$

Hence

$$\parallel x - x^* \parallel_C (1 - K_1 K_2 T) \leq K_1 K_2 T\delta$$

and

$$\parallel x - x^* \parallel_C \leq \frac{K_1 K_2 T\delta}{(1 - K_1 K_2 T)} = \epsilon.$$

This proves the continuous dependence of the solution on the parameter $\mu$.

### *B. Continuous dependence on the function* $f_2$

**Definition 4** *The solution* $x \in C[0, T]$ *of the functional integral equation (1) depends Continuously on the function* $f_2$, *if*

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon), \quad s.t \quad |f_2 - f_2^*| < \delta \Rightarrow \parallel x - x^{**} \parallel_C < \epsilon$$

where $x^{**}$ is the unique solution $x^{**} \in C[0, T]$ of problem (3).

**Theorem 7** *Let the assumption of Theorem* (5) *be satisfied, then the solution of the functional integral equation (1) depends continuously on the function* $f_2$.

*Proof.* Let $x, x^{**}$ be the solutions of the functional integral equations (1) and (3), then

$$|x(t) - x^{**}(t)| = |f_1(t, \int_0^t f_2(s, x(s), \mu)ds) - f_1(t, \int_0^t f_2^*(s, x^{**}(s), \mu)ds)|$$
$$\leq K_1 |\int_0^t f_2(s, x(s), \mu)ds - \int_0^t f_2^*(s, x^{**}(s), \mu)ds|$$
$$\leq K_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x^{**}(s), \mu)|ds$$
$$\leq K_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x(s), \mu) + f_2^*(s, x(s), \mu) - f_2^*(s, x^{**}(s), \mu)|ds$$
$$\leq K_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x(s), \mu)|ds + K_1 \int_0^t |f_2^*(s, x(s), \mu) - f_2^*(s, x^{**}(s), \mu)|ds$$
$$\leq K_1 \int_0^t |f_2(s, x(s), \mu) - f_2^*(s, x(s), \mu)|ds + K_1 K_2 \int_0^t |x(s) - x^{**}(s)|ds$$
$$\leq K_1 T\delta + K_1 K_2 T \parallel x - x^{**} \parallel_C.$$

Hence

$$\parallel x - x^{**} \parallel_C (1 - K_1 K_2 T) \leq K_1 T\delta$$

and

$$\parallel x - x^{**} \parallel_C \leq \frac{K_1 T\delta}{(1 - K_1 K_2 T)} = \epsilon.$$

This proves the continuous dependence of the solution on the function $f_2$.

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