On the Plane Motion of Incompressible Variable Viscosity fluids with Intermediate Peclet Number via Von-Mises Cooordinates

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Abstract

This paper is to present a class of new exact solutions of the system of partial differential equations governing the plane steady motion with intermediate Peclet number of incompressible fluid with variable viscosity in von-Mises coordinates. The class is characterized by an equation relating a differentiable function $f(x)$ and a function of stream function ψ satisfying a specific relation. The exact solutions for intermediate Peclet number for two values of $f(x)$ are determined. For both values the viscosity function and temperature distribution for intermediate Peclet number are determined. The streamlines, the velocity components, generalized energy function are also obtained. Computer algebra system is used to find solution of two variable coefficient differential equations in terms of special functions.

Keywords - *Variable viscosity fluids, Navier-Stokes equations with body force, Exact solutions in the presence of body force, Martin's system, von-Mises coordinates, Intermediate Peclet number*

I.INTRODUCTION

The fundamental system of partial differential equations (PDE) for the motion of an incompressible fluid of variable viscosity in tensor form is following

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{1}$$

$$\left(v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \frac{1}{R_e}\frac{\partial}{\partial x_j}\left\{ \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} \tag{2}$$

$$\left(v_j \frac{\partial T}{\partial x_j} \right) = \frac{1}{R_e P_r}\frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i} \right) + \frac{\mu E_c}{R_e}\frac{\partial v_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{3}$$

where $\mathbf{F} = \left(F_1(x_i), F_2(x_i), F_3(x_i)\right)$ is the body force per unit mass, $\mathbf{v} = \left(v_1(x_i), v_2(x_i), v_3(x_i)\right)$ the fluid velocity, $p = p(x_i)$ is *pressure*, the coefficients of viscosity $\mu > 0$, the space coordinates x_i and $i, j \in \{1, 2, 3\}$. The dimensionless quantities R_e , P_r and E_c are the *Reynolds number*, the *Prandtl number* and the *Eckert number* respectively. The product of R_e and P_r is the Peclet number $P_{e'}$. Equation (1) is equation of continuity, equation (2) is Navier-Stokes equations (NSE) and equation (3) is the energy equation. The non-dimensional parameters used in equations (2-3) are following

$$x^* = \frac{x}{L_0} \qquad y^* = \frac{y}{L_0} \qquad u^* = \frac{u}{U_0} \qquad v^* = \frac{v}{U_0}$$

$$\mu^* = \frac{\mu}{\mu_0} \qquad p^* = \frac{p}{p_0} \qquad F_1^* = \frac{F_1}{F_0} \qquad F_2^* = \frac{F_2}{F_0}$$

where the thermal conductivity $k = k_0 = Const$., density $\rho = \rho_0 = Const$., the specific heat at constant volume c_v and at constant pressure c_p are such that $c_v = c_p = Const$.

Equations (1-3) for the plane Cartesian space taking $i, j \in \{1, 2\}$, $x_1 = x$, $x_2 = y$, $v_1 = u(x, y)$, $v_2 = v(x, y)$, $F_1 = F_1(x, y)$, $F_2 = F_2(x, y)$ reduces to following

$$u_x + v_y = 0 \tag{4}$$

$$u\, u_x + v\, u_y = -\, p_x + \frac{1}{\mathrm{R_e}} [(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{5}$$

$$u\, v_x + v\, v_y = -\, p_y + \frac{1}{\mathrm{R_e}} [(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{6}$$

$$u\, T_x + v\, T_y = \frac{1}{P_{e'}} (T_{xx} + T_{yy}) + \frac{\mathrm{E_c}}{\mathrm{R_e}} [2\mu(u_x^{\,2} + v_y^{\,2}) + \mu(u_y + v_x)^2] \tag{7}$$

A function $\psi(x, y)$ such that $\frac{\partial^2 \psi}{\partial y\, \partial x} = \frac{\partial^2 \psi}{\partial x\, \partial y}$ and

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{8}$$

satisfies equation (4).

In order to handle the nonlinear terms in equations (5-6) please refer to [1-9] for some coordinates transformation techniques and dimension analysis methods and the references therein. The solution of equation (7) could be obtained for high or very low Peclet number while the solutions at intermediate Peclet number are fascinating [10-13]. The successive transformation technique of this discourse transforms equations (5-7) into curvilinear coordinates (φ, ψ) with Martin's definition then to von-Mises coordinates. Martin [14] defined the curvilinear coordinate lines $\psi = \text{const.}$ as streamlines and left the curvilinear coordinate lines $\varphi = \text{const.}$ arbitrary. The coordinates system (φ, ψ) is referred as Martin's system. The arbitrary coordinate φ in Martin's system may be taken along the $x-axis$. According to definition of von-Mises coordinates (x, ψ) in [15] the function $\varphi = x$ and stream function ψ of Martin's system are independent variables instead of y and x. In addition of selecting the von-Mises coordinate, the streamlines of the class of flows under consideration is characterized by

$$y - f(x) = \text{const.} \tag{9}$$

The function $f(x)$ is a continuously differentiable function. As $\psi = \text{const.}$ are the streamlines therefore, without loss of generality it is reasonable to consider

$$y = f(x) + \nu(\psi) \tag{10}$$

with ν a function of ψ such that $\nu''(\psi) = \nu'^2(\psi)$. The overhead prime represents derivative with respect to ψ.

The paper is organized as follow: Section (2), transforms the fundamental equations into Martin's system (φ, ψ). Section (3) retransforms them into von-Mises coordinates before finding exact solution. The last section presents conclusion.

II. FUNDAMENTAL FLOW EQUATIONS IN MARTIN'S SYSTEM

Introducing the vorticity function w and the total energy function L defined by

$$w = v_x - u_y \tag{11}$$

$$L = p + \frac{1}{2}(u^2 + v^2) - \frac{1}{\mathrm{R_e}}(2\mu u_x) \tag{12}$$

the basic system of equations (5-7) is written into a convenient form as follow

$$-v\ w = -L_x + \frac{1}{\mathrm{R_e}} A_y \tag{13}$$

$$u\ w = -L_y - \frac{1}{R_e} B_y + \frac{1}{\mathrm{R_e}} A_x \tag{14}$$

$$u\ T_x + v\ T_y = \frac{1}{P_{e'}} (T_{xx} + T_{yy}) + \frac{\mathrm{E_c}}{\mathrm{R_e}} \frac{1}{4\mu}\left(B^2 + 4A^2\right) \tag{15}$$

where

$$A = \mu(u_y + v_x) \qquad B = 4\mu\ u_x \tag{16}$$

Consider the allowable change of coordinate

$$x = x(\varphi, \psi), \qquad y = y(\varphi, \psi) \tag{17}$$

such that the Jacobian $J = \dfrac{\partial(x, y)}{\partial(\varphi, \psi)}$ of the transformation is non-zero and finite. At a common point $P(x, y)$ let θ be the angle between the streamlines lines $\psi = const$. and the curves $\varphi = const$. then

$$\tan(\theta) = \frac{y_\varphi}{x_\varphi} \tag{18}$$

The basic equations (13-15) reduces to Martin's system as follow

$$\begin{aligned} -R_e\, w\, J\, E = R_e\, J\, E\, L_\psi &+ A_\varphi\left((F^2 - J^2)\cos 2\theta - 2FJ\sin 2\theta\right) \\ &+ EA_\psi\left(J\sin 2\theta - F\cos 2\theta)\right) - B_\varphi\left(\frac{1}{2}(F^2 - J^2)\sin 2\theta + FJ\cos 2\theta\right) \\ &+ E\, B_\psi\left(\frac{1}{2}F\sin 2\theta + J\cos^2\theta\right), \end{aligned} \tag{19}$$

$$\begin{aligned} 0 = -R_e J\, L_\varphi + E\, A_\psi \cos 2\theta - A_\varphi\left[F\cos 2\theta - J\sin 2\theta\right] \\ + B_\varphi\left(\frac{1}{2}F\sin 2\theta - J\sin^2\theta\right) - \frac{E\, B_\psi}{2}\sin 2\theta, \end{aligned} \tag{20}$$

and

$$\frac{1}{\mathrm{J\, R_e\, P_r}}\left[\left(\frac{G\, T_\varphi - F\, T_\psi}{J}\right)_\varphi + \left(\frac{E\, T_\psi - F\, T_\varphi}{J}\right)_\psi\right] = -\frac{\mathrm{E_c}}{\mathrm{R_e}}\frac{1}{4\mu}\left(B^2 + 4A^2\right) + \frac{T_\varphi}{J} \tag{21}$$

where

$$E = x_\varphi^2 + y_\varphi^2,\ F = x_\varphi\, x_\psi + y_\varphi\, y_\psi,\ G = (x_\psi)^2 + (y_\psi)^2, \tag{22}$$

and

$$J = \pm\sqrt{E\, G - F^2}, \tag{23}$$

$$\begin{aligned} B(\varphi, \psi) = \frac{4\mu}{EJ^3}[E_\varphi\,(F\sin\theta + J\cos\theta)^2 - 2E(F\sin\theta + J\cos\theta) \\ (F_\varphi\sin\theta + J_\varphi\cos\theta) + E^2(J_\psi\sin 2\theta + G_\varphi\sin^2\theta)], \end{aligned} \tag{24}$$

$$\begin{aligned} A(\varphi, \psi) = \mu\,[-\frac{(F\cos\theta - J\sin\theta)}{4E^2J^5}\{E_\varphi(2EJ^3\cos\theta + F\sqrt{E}\sin\theta) \\ - 4E^2J^2J_\varphi\cos\theta - 2E\sqrt{E}\,F_\varphi\sin\theta + E\sqrt{E}\,E_\psi\sin\theta\} \end{aligned}$$

$$+\frac{\cos\theta}{2J^3}\{E_\psi(F\sin\theta+J\cos\theta)-2EJ_\psi\cos\theta-EG_\varphi\sin\theta\}$$
$$+\frac{(F\sin\theta+J\cos\theta)}{2EJ^3}\{(JE_\varphi-2EJ_\varphi)\sin\theta$$
$$+\cos\theta\,[-FE_\varphi+2EF_\varphi-EE_\psi]\}$$
$$-\frac{\sin\theta}{2J^3}\{(E_\psi(J\sin\theta-F\cos\theta)-2EJ_\psi\sin\theta+EG_\varphi\cos\theta\}], \tag{25}$$

and

$$w=\frac{(F\sin\theta+J\cos\theta)}{2EJ^3}\{(JE_\varphi-2EJ_\varphi)\sin\theta+\cos\theta\,[-FE_\varphi+2EF_\varphi-EE_\psi]\}$$
$$-\frac{\sin\theta}{2J^3}\{E_\psi(J\sin\theta-F\cos\theta)-2EJ_\psi\sin\theta+EG_\varphi\cos\theta\}]$$
$$+\frac{(F\cos\theta-J\sin\theta)}{4E^2J^5}\{E_\varphi(2EJ^3\cos\theta+F\sqrt{E}\sin\theta)$$
$$-4E^2J^2J_\varphi\cos\theta-2E\sqrt{E}F_\varphi\sin\theta+E\sqrt{E}E_\psi\sin\theta\}$$
$$-[\frac{\cos\theta}{2J^3}\{E_\psi(F\sin\theta+J\cos\theta)-2EJ_\psi\cos\theta-EG_\varphi\sin\theta\}], \tag{26}$$

Thus, equations (19-26) are the basic equations in Martin's system.

III. EXACT SOLUTIONS VON-MISES COORDINATES

The definition of von-Mises coordinates requires setting the coordinate φ of the Martin's system (φ,ψ) along $x-axis$

$$\varphi = x \tag{27}$$

Equation (18), on utilizing equation (10), equations (27), equation (22) and using the trigonometric identities provides

$$\cos\theta=\frac{1}{\sqrt{E}} \tag{28}$$

$$E=1+x^2\,[f'(x)]^2 \tag{29}$$

$$F=J\sqrt{E-1} \tag{30}$$

$$G=x^2\,v'(\psi)^2 \tag{31}$$

$$J=x\,v'(\psi) \tag{32}$$

and the basic equations becomes

$$-R_e\,w=R_e\,L_\psi-JA_x+\sqrt{E-1}\,A_\psi+B_\psi \tag{33}$$

$$0=-R_e\,L_x+\frac{A_\psi(2-E)}{J}+A_x\sqrt{E-1}-\frac{\sqrt{E-1}\,B_\psi}{J} \tag{34}$$

$$JT_{xx}-2\sqrt{E-1}\,T_{vx}v'+\frac{E}{J}T_{vv}(v')^2+\left(J_x-\frac{E_\psi}{2\sqrt{E-1}}-P_{e'}\right)T_x$$
$$+\left(\frac{E_\psi}{J}-\frac{E_x}{2\sqrt{E-1}}-\frac{EJ_\psi}{J^2}+\frac{E}{J}\left(\frac{v''}{v'}\right)\right)T_v v'=-\frac{J\,E_c\,P_r}{4\mu}\left(B^2+4A^2\right) \tag{35}$$

where

$$w = \left(\frac{1}{\nu'(\psi)}\right)\left[\left\{\frac{f'(x)}{x} + f''(x)\right\} + \left\{\frac{1}{x^2} + [f'(x)]^2\right\}\left(\frac{\nu''(\psi)}{\{\nu'(\psi)\}^2}\right)\right] , \tag{36}$$

$$A(x,\psi) = \frac{\mu}{J}\left[\frac{-2J_x\sqrt{E-1}}{J} + \frac{E_x}{2\sqrt{E-1}} - \frac{(2-E)J_\psi}{J^2}\right], \tag{37}$$

$$B(x,\psi) = 4\mu\,\frac{1}{J^3}\,[-J\;J_x + \sqrt{E-1}\;J_\psi] , \tag{38}$$

and

$$q = \frac{\sqrt{E}}{J} \tag{39}$$

where q is the magnitude of velocity vector.

Using the natural integrability condition $L_{x\psi} = L_{\psi x}$, a commonly agreed equation on equations (33-34) is

$$x\nu' A_{xx} - 2xf'A_{x\psi} - \frac{\left[1 - x^2(f')^2\right]}{x\nu'}A_{\psi\psi} + \nu' A_x - A_\psi\left(f' + xf''\right)$$

$$-\left\{B_x - \frac{f'B_\psi}{\nu'}\right\}_\psi = R_e\, w_x \tag{40}$$

Through the solution of equation (40), the function L and temperature distribution T are determined from equations (33-34) and (35), respectively.

In this communication

$$\nu'' = \nu'^2 \tag{41}$$

that is

$$\nu = \ln\left[\frac{-1}{(c_1\psi + c_2)}\right] \tag{42}$$

where $c_1 \neq 0$ and c_2 are constants. Therefore, equations (37-38) on applying equation (41) provide

$$A = \frac{\mu}{x^2\nu'}\left[xM' - 2M - \left(1 - M^2\right)\right] \tag{43}$$

and

$$B = \frac{4\mu}{x^2\nu'}\left(-1 + M\right) \tag{44}$$

where

$$M = xf' \tag{45}$$

Equations (43) and (44) on eliminating μ provides

$$B = Y(x)\;A \tag{46}$$

where $$Y(x) = \frac{4(-1+M)}{xM' - 2M - (1-M^2)}, \qquad M \neq 1 \tag{47}$$

Equation (40), on employing equation (42) and equation (46), become

$$xA_{xx} - (2M + Y)\;A_{\nu x} + \left(\frac{MY - (1-M^2)}{x}\right)A_{\nu\nu} + A_x$$

$$-(M' + Y')\;A_\nu = R_e\left(\frac{e^{-2\nu}}{c_1^2}\right)\left[\frac{M'}{x} + \frac{(1+M^2)}{x^2}\right]' \tag{48}$$

The factor $e^{-2\nu}$ in equation (48) leads to

$$A = C(x,\nu) + e^{-2\nu} P(x) \tag{49}$$

where $C(x,\nu)$ and $P(x)$ are appropriate functions. Equation (49) on substituting equation (49) gives

$$x\,C_{xx} - (2M + Y)\,C_{x\nu} + \left(\frac{MY - (1-M^2)}{x}\right) C_{\nu\nu} + C_x - (M' + Y')\,C_\nu$$

$$+\, e^{-2\nu}\left[x\,P'' + 2(2M + Y)\,P' + 4\left(\frac{MY - (1-M^2)}{x}\right) P + P' + 2(M' + Y')\,P\right]$$

$$= R_e\left(\frac{e^{-2\nu}}{c_1^2}\right)\left[\frac{M'}{x} + \frac{(1+M^2)}{x^2}\right]' \tag{50}$$

Coefficients of $e^{-2\nu}$ on both sides of equation (50) implies

$$x\,C_{xx} - (2M + Y)\,C_{x\nu} + \left(\frac{MY - (1-M^2)}{x}\right) C_{\nu\nu} + C_x - (M' + Y')\,C_\nu = 0 \tag{51}$$

$$x^2 P'' + x\,(4M + 2Y + 1)\,P' + \left[4MY - 4(1-M^2) + 2x(M' + Y')\right] P$$

$$= \left(\frac{R_e}{c_1^2}\right) x \left[\frac{M'}{x} + \frac{(1+M^2)}{x^2}\right]' \tag{52}$$

The coefficients of non-homogeneous equation (52) involve the arbitrary function $f(x)$. The available computer algebra system (CAS) software can solve equation (52) for a given $f(x)$. However, on setting

$$(4M + 2Y + 1) = m_1 \tag{53}$$

and $$4MY - 4(1-M^2) + 2x(M' + Y') = m_2 \tag{54}$$

it reduced to Cauchy equation. The system of equations (47), (53) and (54) on solving provides

$$M = -1, \qquad m_1 = -11, \qquad m_2 = 16 \tag{55}$$

Therefore, the reduced equation (52)

$$x^2 P'' - 11x\,P' + 16P = \left(\frac{R_e}{c_1^2}\right)\left[\frac{-4}{x^2}\right] \tag{56}$$

implies

$$P(x) = p_1\,x^{(6+2\sqrt{5})} + p_2\,x^{(6-2\sqrt{5})} + \left(\frac{-R_e}{11\,c_1^2}\right)\left(\frac{1}{x^2}\right) \tag{57}$$

where p_1 and p_2 are constants.

In view of equation (57), the equation (51) becomes

$$x^2 C_{xx} + 6x\,C_{x\nu} + 4C_{\nu\nu} + x\,C_x = 0 \tag{58}$$

Let us search a solution of equation (58) of the form

$$C = C_1(x) + S_1(\nu) + C_2(x)\,S_2(\nu) \tag{59}$$

Substituting equation (59) in equation (58)

$$\{x\,(x\,C_1')'\} + 4S_1'' + \{4C_2\,S_2'' + 6x\,C_2'\,S_2' + x\,(x\,C_2')'\,S_2\} = 0 \tag{60}$$

Differentiation of equation (60) with respect to "x" provides

$$\left[x\,(x\,C_1')'\right]' + 4C_2'\,S_2'' + 6(x\,C_2')'\,S_2' + \left[x\,(x\,C_2')'\right]'\,S_2 = 0 \tag{61}$$

Differentiation of equation (61) with respect to "ν" provides

$$4C_2'\,S_2''' + 6J\,S_2'' + [x\,J]'\,S_2' = 0 \tag{62}$$

where

$$J(x) = \left(x C_2'(x)\right)' \tag{63}$$

Rewriting equation (61) as

$$4C_2'\left(\frac{Z''}{Z}\right) + 6J\left(\frac{Z'}{Z}\right) + [xJ]' = 0 \tag{64}$$

where

$$Z(\nu) = S_2'(\nu) \tag{65}$$

Differentiating equation (64) with respect to "ν" and separation of variables implies

$$\frac{\left(\frac{Z''}{Z}\right)'}{\left(\frac{Z'}{Z}\right)'} = -\frac{3}{2}\frac{J(x)}{C_2'(x)} = d_1 \tag{66}$$

where d_1 is a separation constant. Equation (66) indicates

$$J(x) = -\frac{2}{3} d_1 C_2'(x) \tag{67}$$

$$Z'' = d_1 Z' + d_2 Z \tag{68}$$

where d_2 is a constant of integration.

Integrating equation (64), in view of equations (67-68), implies

$$x C_2'(x) + \frac{2}{3} d_1 C_2(x) = d_3 \tag{69}$$

where d_3 is a constant. The solution of equation (69) is

$$C_2(x) = \frac{3d_3}{2d_1} + d_4 x^{-2d_1/3} \tag{70}$$

where d_4 is constant.

Integration of equation (68) on substituting equation (65) gives

$$S_2''(\nu) - d_1 S_2'(\nu) - d_2 S_2(\nu) = d_5 \tag{71}$$

where d_5 is a constant. Inserting equations (67-68) in equation (62), to obtain

$$C_2'\{S_2''' - d_1 S_2'' + \frac{d_1^2}{9} S_2'\} = 0 \tag{72}$$

As $C_2'(x) \neq 0$, equation (72) gives

$$S_2''' - d_1 S_2'' + \frac{d_1^2}{9} S_2' = 0 \tag{73}$$

or

$$\left(S_2'' - d_1 S_2'\right)' + \frac{d_1^2}{9} S_2' = 0 \tag{74}$$

Consistency of equation (71) and equation (74) implies

$$d_2 = -\frac{d_1^2}{9} \tag{75}$$

Equation (60) on supplying equations (74-75) gives

$$x\left(x C_1'\right)' = -4C_2 d_5 + d_6 \tag{76}$$

Whose solution is

$$C_1(x) = -4d_5 \int \left[\frac{1}{x} \int \frac{C_2(x)}{x} dx \right] dx + d_6 \int \frac{\ln x}{x} dx + d_7 \ln x + d_8 \quad (77)$$

where d_5, d_6, d_7, and d_8 are constants.

Consistency of equation (60), in presence of the equations (74-75) and (77) implies

$$S_1 = -\frac{d_3}{6} \int \left\{ \int \left[9S_2'(\nu) - d_1 S_2(\nu) \right] d\nu \right\} d\nu - \frac{d_6}{8} \nu^2 + d_9 \nu + d_{11} \quad (78)$$

Integration of equation (74) gives

$$S_2(\nu) = d_{12} \, Exp \left[\frac{(3+\sqrt{5})}{6} d_1 \nu \right] + d_{13} \, Exp \left[\frac{(3-\sqrt{5})}{6} d_1 \nu \right] + \frac{9 d_{11}}{d_1^2} \quad (79)$$

where d_9, d_{10}, d_{11}, d_{12} and d_{13} are constants of integration.

On substituting the values $C_1(x)$, $C_2(x)$, $S_1(\nu)$, $S_2(\nu)$ and $P(x)$ in equation (49), we get

$$\begin{aligned} A = & -\frac{3d_3 d_5}{d_1} (\ln x)^2 - \frac{9 d_4 d_5}{d_1^2} x^{-2d_1/3} + d_6 \frac{(\ln x)^2}{2} + d_7 \ln x + d_8 \\ & -\frac{d_3}{6} \int \left\{ \int \left[9S_2'(\nu) - d_1 S_2(\nu) \right] d\nu \right\} d\nu - \frac{d_6}{8} \nu^2 + d_9 \nu + d_{10} \\ & + \left\{ \frac{3d_3}{2d_1} + d_4 x^{-2d_1/3} \right\} \left\{ d_{12} \, Exp \left[\frac{(3+\sqrt{5})}{6} d_1 \nu \right] + d_{13} \, Exp \left[\frac{(3-\sqrt{5})}{6} d_1 \nu \right] + \frac{9 d_{11}}{d_1^2} \right\} \\ & + e^{-2\nu} \left\{ p_1 x^{(6+2\sqrt{5})} + p_2 x^{(6-2\sqrt{5})} + \left(\frac{-R_e}{11 c_1^2} \right) \left(\frac{1}{x^2} \right) \right\} \end{aligned} \quad (80)$$

The viscosity distribution from equation (43) or (44) on using (46) provides

$$\begin{aligned} \mu = & \left(\frac{-r^2 c_4 e^{\nu}}{2} \right) \left[-\frac{3d_3 d_5}{d_1} (\ln x)^2 - \frac{9 d_4 d_5}{d_1^2} x^{-2d_1/3} + d_6 \frac{(\ln x)^2}{2} + d_7 \ln x + d_8 \right. \\ & -\frac{d_3}{6} \int \left\{ \int \left[9S_2'(\nu) - d_1 S_2(\nu) \right] d\nu \right\} d\nu - \frac{d_6}{8} \nu^2 + d_9 \nu + d_{10} \\ & + \left\{ \frac{3d_3}{2d_1} + d_4 x^{-2d_1/3} \right\} \left\{ d_{12} \, Exp \left[\frac{(3+\sqrt{5})}{6} d_1 \nu \right] + d_{13} \, Exp \left[\frac{(3-\sqrt{5})}{6} d_1 \nu \right] + \frac{9 d_{11}}{d_1^2} \right\} \\ & \left. + e^{-2\nu} \left\{ p_1 x^{(6+2\sqrt{5})} + p_2 x^{(6-2\sqrt{5})} + \left(\frac{-R_e}{11 c_1^2} \right) \left(\frac{1}{x^2} \right) \right\} \right] \end{aligned} \quad (81)$$

The function L from equations (33-34), utilizing equation (57), equation (70) and equations (77-79), is

$$\begin{aligned} R_e L = & -[C_1(x) + S_1(\nu) + C_2(x) S_2(\nu) + e^{-2\nu} D(x)] - 4S_1' \ln x \\ & -4S_2' \int \frac{C_2}{x} dx + 8e^{-2\nu} \int \frac{D}{x} dx + (d_7 + 6d_9)\nu - \frac{3d_6}{4} \nu^2 + p_3 \end{aligned} \quad (82)$$

provided

$$d_3 = 0, \qquad d_5 = d_{11} \quad (83)$$

where p_3 is constant.

Now for this case the equation (35) for T is

$$x^2 T_{xx} + 2x T_{\nu x} + 2T_{\nu\nu} + x \left(1 - \frac{P_{e'}}{c_1} e^{-\nu} \right) T_x$$

$$= \frac{10\ E_c\ P_r}{c_1}\ [e^{-\nu}\ \{d_6\ \frac{(\ln\ x)^2}{2} + d_7\ \ln\ x + d_8\} + e^{-\nu}\ \{-\frac{d_6}{8}\nu^2 + d_9\ \nu + d_{10}\}$$

$$+ e^{-\nu}\ \{d_4\ x^{-2d_1/3}\}\{d_{12}\ Exp\ [\frac{(3+\sqrt{5})}{6}d_1\ \nu] + d_{13}\ Exp\ [\frac{(3-\sqrt{5})}{6}d_1\ \nu]\}$$

$$+ e^{-3\nu}\left(\frac{-R_e}{11\ c_1^2}\right)\left(\frac{1}{x^2}\right) + e^{-3\nu}\left\{p_1\ x^{(6+2\sqrt{5})} + p_2\ x^{(6-2\sqrt{5})}\right\} \tag{84}$$

The non-homogeneous variable coefficients equation (84) is extremely difficult to solve. However, equation (84) on setting

$$T = e^{-\nu}\ T_1(x) + S_3(\nu) + x^b S_4(\nu) \tag{85}$$

and arranging the terms provides

$$e^{-\nu}\left[x^2 T_1''(x) - x T_1'(x) + 2 T_1(x)\right]$$

$$+ x^b \left[b(b-1)S_4(\nu) + 2b S_4'(\nu) + 2 S_4''(\nu) + b\ x^b \left(1 - \frac{P_{e'}}{c_1} e^{-\nu}\right) S_4(\nu) \right]$$

$$+ 2 S_3''(\nu) - \frac{P_{e'}}{c_4} e^{-2\nu}\ x T_1'(x)$$

$$= \frac{10\ E_c\ P_r}{c_1}\ [e^{-\nu}\ \{d_6\ \frac{(\ln\ x)^2}{2} + d_7\ \ln\ x + d_8\} + e^{-\nu}\ \{-\frac{d_6}{8}\nu^2 + d_9\ \nu + d_{10}\}$$

$$+ e^{-\nu}\ \{d_4\ x^{-2d_1/3}\}\{d_{12}\ Exp\ [\frac{(3+\sqrt{5})}{6}d_1\ \nu] + d_{13}\ Exp\ [\frac{(3-\sqrt{5})}{6}d_1\ \nu]\}$$

$$+ e^{-3\nu}\left(\frac{-R_e}{11\ c_1^2}\right)\left(\frac{1}{x^2}\right) + e^{-3\nu}\left\{p_1\ x^{(6+2\sqrt{5})} + p_2\ x^{(6-2\sqrt{5})}\right\} \tag{86}$$

Coefficients of like terms in equation (86) provides

$$T_1(x) = \frac{5 d_8\ E_c\ P_r}{c_1} \tag{87}$$

$$S_3''(\nu) = \frac{5\ E_c\ P_r}{c_1}\ (d_9\ \nu + d_{10})e^{-\nu} \tag{88}$$

and

$$S_4''(\nu) - 2 S_4'(\nu) + \left(2 + \frac{P_{e'}}{c_1} e^{-\nu}\right) S_4(\nu)$$

$$= \frac{5\ E_c\ P_r}{c_1}\ d_4 \{d_{12} e^{\frac{(1+\sqrt{5})}{2}\nu} + d_{13}\ e^{\frac{(1-\sqrt{5})}{2}\nu}\} + e^{-3\nu}\left(\frac{-R_e}{11\ c_1^2}\right) \tag{89}$$

Provided

$$p_1 = p_2 = d_6 = d_7 = 0, \qquad d_1 = 3, \qquad b = -2, \tag{90}$$

Solutions of equations (88-89) are

$$S_3(\nu) = \frac{5\ E_c\ P_r}{c_1} \iint (d_9\ \nu + d_{10})e^{-\nu}\ d\nu + s_1\nu + s_2 \tag{91}$$

$$S_4(\nu) = \frac{C_1}{A1} e^{\nu}\ \text{Gamma[1} \quad -2\ \text{i] BesselJ[} \quad -2i, 2\sqrt{A1\, e^{-\nu}}\ ]$$

$$+\frac{C_2}{A1}e^{\nu}\ \text{Gamma}[1 \quad + 2\text{ i}]\ \text{BesselJ}[\quad 2i,\ 2\sqrt{A1\,e^{-\nu}}\ ]$$

$$+\frac{i}{2}e^{\nu}\ \text{Gamma}[1 \quad - 2\text{ i}]\ \text{Gamma}[1 \quad + 2\text{ i}]$$

$$\Big\{\text{BesselJ}[\quad 2i,\ 2\sqrt{A1\,e^{-\nu}}\ ]$$

$$\int e^{-4\nu}\left\{A4 + A3e^{(7-\sqrt{5})\nu/2} + A2\,e^{(7+\sqrt{5})\nu/2}\right\}\text{BesselJ}[\quad -2i,\ 2\sqrt{A1\,e^{-\nu}}\ ]\,d\nu$$

$$-\ \text{BesselJ}[\quad -2i,\ 2\sqrt{A1\,e^{-\nu}}\ ]$$

$$\int e^{-4\nu}\left\{A4 + A3e^{(7-\sqrt{5})\nu/2} + A2\,e^{(7+\sqrt{5})\nu/2}\right\}\text{BesselJ}[\quad 2i,\ 2\sqrt{A1\,e^{-\nu}}\ ]\,d\nu\Big\} \qquad (92)$$

where

$$A1 = \frac{P_{e'}}{c_1}, \qquad A2 = \frac{5\,E_c P_r d_4 d_{12}}{c_1},$$

$$A3 = \frac{5\,E_c P_r d_4 d_{13}}{c_1}, \qquad A4 = -\frac{R_e}{11\,c_1^2} \qquad (93)$$

Therefore, temperature distribution is obtained by substituting equation (78) and equations (91-93) in equation (85) for intermediate Peclet number. Now it is easy to find the function μ from equation (81), (u, v) from equation (7), p from equation (9) utilizing equation (82), and streamlines from equation (2) using (10). The CAS software Mathemaitca is used to obtain (92).

Now substitute $M = 1$ in equations (43-44) gives

$$B = 0 \qquad (94)$$

$$A = \frac{-2\mu}{x^2\,\nu'} \qquad (95)$$

and equation (40) implies

$$x^2 A_{xx} - 2x\,A_{\nu r} + x\,A_r = R_e\left(\frac{e^{-2\nu}}{c_1^2}\right)\left(\frac{-4}{x^2}\right) \qquad (96)$$

Similar to the previous case, let us search solution of equation (96) the form

$$A = N(x,\nu) + e^{-2\nu}\,Q(x) \qquad (97)$$

Supplying equation (97) in equation (96) provides

$$\{x^2 N_{xx} - 2x\,N_{\nu x} + x\,N_x\} + e^{-2\nu}\{x^2 Q'' + 5xQ'\} = R_e\left(\frac{e^{-2\nu}}{c_1^2}\right)\left(\frac{-4}{x^2}\right) \qquad (98)$$

The comparison of like terms on both sides of (98) implies

$$x^2 Q'' + 5xQ' = \left(\frac{R_e}{c_1^2}\right)\left(\frac{-4}{x^2}\right) \qquad (99)$$

$$x^2 N_{xx} - 2x\,N_{\nu x} + x\,N_x = 0 \qquad (100)$$

The solution of equation of (99) is

$$Q(x) = \frac{R_e}{c_1^2}\left(\frac{1}{x^2}\right) - \frac{4\,d_{14}}{x^4} + d_{15} \qquad (101)$$

Seeking solution of equation (100) of the form

$$N = C_3(x) + S_5(\nu) + C_4(x)S_6(\nu) \qquad (102)$$

implies

$$\{x^2 C_3'' + xC_3'\} + S_6\{x^2 C_4'' + xC_4'\} - 2xC_4' S_6' = 0 \tag{103}$$

Differentiating equation (103) with respect to ν and separating the variables one finds

$$x C_4''(x) + (1 - d_{16}) C_4'(x) = 0 \tag{104}$$

$$S_6''(\nu) - \frac{d_{16}}{2} S_6'(\nu) = 0 \tag{105}$$

where d_{16} is a separation constant. Solutions of equations (104-105) are

$$C_4(x) = \frac{d_{17}}{d_{16}} x^{d_{16}} + d_{18} \tag{106}$$

$$S_6(\nu) = \frac{-2d_{19}}{d_{16}} + d_{20}\, e^{d_{16}\nu/2} \tag{107}$$

Supplying equation (106-107) in equation (103) gives

$$x^2 C_3'' + xC_3' = 2d_{19}\, d_{17}\, x^{d_{16}} \tag{108}$$

Solution of equation (108) is

$$C_3(x) = \frac{2d_{17}\, d_{19}}{d_{16}^2} x^{d_{16}} + d_{21} \ln x + d_{22} \tag{109}$$

Substituting equations (101-102) in equation (97) , we get

$$\begin{aligned} A = d_{22} - \frac{2d_{18} d_{19}}{d_{16}} + d_{21} \ln x + S_5(\nu) + d_{18} d_{20}\, e^{d_{16}\nu/2} \\ + \frac{2d_{17}\, d_{19}}{d_{16}^2} x^{d_{16}} + \frac{d_{17}}{d_{16}} x^{d_{16}} \left[d_{20}\, e^{d_{16}\nu/2} - \frac{2d_{19}}{d_{16}} \right] \\ + e^{-2\nu} \left\{ \frac{R_e}{c_1^2} \left(\frac{1}{x^2} \right) - \frac{4d_{14}}{x^4} + d_{15} \right\} \end{aligned} \tag{110}$$

The viscosity is obtained from equation (95) by substituting equation (110) and equation (47) which is

$$\begin{aligned} \mu(x,\nu) = \left(\frac{-x^2 e^{-\nu}}{2c_1} \right) \Big[d_{22} - \frac{2d_{18} d_{19}}{d_{16}} + d_{21} \ln x + S_5(\nu) + d_{18} d_{20}\, e^{d_{16}\nu/2} \\ + \frac{2d_{17}\, d_{19}}{d_{16}^2} x^{d_{16}} + \frac{d_{17}}{d_{16}} x^{d_{16}} \left[d_{20}\, e^{d_{16}\nu/2} - \frac{2d_{19}}{d_{16}} \right] \\ + e^{-2\nu} \left\{ \frac{R_e}{c_1^2} \left(\frac{1}{x^2} \right) - \frac{4d_{14}}{x^4} + d_{15} \right\} \Big] \end{aligned} \tag{111}$$

The solution of equations (33-34) , utilizing equations (110) and equations (94-95) give

$$\begin{aligned} R_e L = R_e \left(\frac{1}{x^2 c_1^2} e^{-2\nu} \right) + x\, [C_3'(x)\, \nu + C_4'(x) \int S_6(\nu)\, d\nu \\ + \left(\frac{e^{-2\nu}}{-2} \right) Q'(x)] - A + 2d_{21} \ln x + 2d_{22} - \frac{4d_{18} d_{19}}{d_{16}} + p_3 \end{aligned} \tag{112}$$

where p_3 is constant.

The energy equation on substituting equations (93) and equation (110) becomes

$$x^2 T_{xx} - 2x T_{\nu x} + 2T_{\nu\nu} + x\left(1 - \frac{P_{e'}}{c_4} e^{-\nu} \right) T_x$$

$$= \frac{2E_c P_r d_{21}}{c_1} \ln x\, e^{-\nu} + \frac{2E_c P_r d_{22}}{c_1} e^{-\nu} + \frac{2E_c P_r}{c_1} S_5(\nu)\, e^{-\nu}$$

$$- \frac{4E_c P_r d_{19} d_{18}}{c_1 d_{16}} e^{-\nu} + \frac{2E_c P_r d_{18} d_{20}}{c_1} e^{\left(\frac{d_{16}}{2}-1\right)\nu} + \frac{2E_c P_r d_{17} d_{20}}{c_1 d_{16}} x^{d_{16}} e^{\left(\frac{d_{16}}{2}-1\right)\nu}$$

$$+ \frac{2E_c P_r}{c_1} e^{-3\nu} \left\{ \frac{R_e}{c_1^2}\left(\frac{1}{x^2}\right) - \frac{4d_{14}}{x^4} + d_{15} \right\} \tag{113}$$

Searching solution of equation (113) of the form

$$T = e^{-\nu} T_2(x) + S_7(\nu) + x^b S_8(\nu) \tag{114}$$

implies

$$e^{-\nu}\left[r^2 T_2''(x) + 3x T_2'(x) + 2T_2(x)\right]$$

$$+ x^b \left[2S_8''(\nu) - 2bS_8'(\nu) + b^2 S_8(\nu) - \frac{P_{e'}}{c_4} e^{-\nu} bS_8(\nu) \right]$$

$$+ 2S_7''(\nu) - \frac{P_{e'}}{c_1} e^{-2\nu} x T_2'(x) = e^{-\nu}\left[\frac{2E_c P_r d_{21}}{c_1} \ln x \right]$$

$$+ e^{-\nu}\left[\frac{2E_c P_r}{c_1}\left(d_{22} - \frac{2d_{19} d_{18}}{d_{16}}\right) + \frac{2E_c P_r}{c_1} S_5(\nu) + \frac{2E_c P_r d_{18} d_{20}}{c_1} e^{\left(\frac{d_{16}}{2}-1\right)\nu} \right]$$

$$+ \frac{2E_c P_r d_{17} d_{20}}{c_1 d_{16}} x^{d_{16}} e^{\left(\frac{d_{16}}{2}-1\right)\nu} + \frac{2E_c P_r}{c_1} e^{-3\nu}\left[\frac{R_e}{c_1^2}\left(\frac{1}{x^2}\right) \right]$$

$$+ \frac{2E_c P_r}{c_1} e^{-3\nu}\left[-\frac{4d_{14}}{x^4} + d_{15} \right] \tag{115}$$

Coefficients of like terms of equation (115) give

$$T_2(x) = \frac{E_c d_{17} d_{20}}{R_e}\left(\frac{x^{-2}}{-2}\right) \tag{116}$$

$$2S_7''(\nu) = \frac{2E_c P_r d_{22}}{c_1} e^{-\nu} + \frac{2E_c P_r}{c_1} e^{-\nu} S_5(\nu) - \frac{4E_c P_r d_{19} d_{18}}{c_1 d_{16}} e^{-\nu}$$

$$+ \frac{2E_c P_r d_{18} d_{20}}{c_1} e^{\left(\frac{d_{16}}{2}-1\right)\nu} \tag{117}$$

and

$$S_8''(\nu) + 2S_8'(\nu) + \left(2 + \frac{P_{e'}}{c_1} e^{-\nu}\right) S_8(\nu) = \frac{E_c d_{17} d_{20}}{2R_e} e^{-\nu} + \frac{E_c P_{e'}}{c_1^3} e^{-3\nu} \tag{118}$$

provided

$$d_{14} = 0, \qquad d_{15} = 0 \qquad d_{16} = -2 \qquad d_{21} = 0, \qquad b = -2 \tag{119}$$

Solution of equations (117-118) are

$$S_7(\nu) = \left[\frac{E_c P_r d_{22}}{c_1} - \frac{2E_c P_r d_{19} d_{18}}{c_1 d_{16}} \right] e^{-\nu} + \frac{E_c P_r}{c_1} \iint e^{-\nu} S_5(\nu)\, d\nu$$

$$+ \frac{E_c P_r d_{18} d_{20}}{2c_1} e^{-2\nu} + s_3 \nu + s_4 \tag{120}$$

and

$$S_8(v) = C_3\ A5\ e^{-v}\ \text{BesselJ}[\ -2i,\ 2\sqrt{A5\,e^{-v}}\ ]\ \text{Gamma}[1\ -2\text{ i}]$$
$$+\ C_4\ A5\,e^{-v}\ \text{BesselJ}[\ 2i,\ 2\sqrt{A5\,e^{-v}}\ ]\ \text{Gamma}[1\ +2\text{ i}]$$
$$+\frac{i}{2}e^{-v}\ \text{Gamma}[1\ -2\text{ i}]\ \text{Gamma}[1\ +2\text{ i}]$$
$$\Big\{\text{BesselJ}[\ 2i,\ 2\sqrt{A5\,e^{-v}}\ ]\int e^{-v}(A7 + A6\,e^{2v})\,\text{BesselJ}[\ -2i,\ 2\sqrt{A5\,e^{-v}}\ ]\,dv$$
$$-\ \text{BesselJ}[\ -2i,\ 2\sqrt{A5\,e^{-v}}\ ]\int e^{-v}(A7 + A6\,e^{2v})\,\text{BesselJ}[\ 2i,\ 2\sqrt{A5\,e^{-v}}\ ]\,dv\Big\} \qquad (121)$$

where C_3 and C_4 are constant and

$$A5 = \frac{P_{e'}}{c_1}, \qquad A6 = \frac{E_c\, d_{17}\, d_{20}}{2R_e}, \qquad A7 = \frac{E_c\, P_{e'}}{c_1^{\,3}} \qquad (122)$$

Therefore, temperature distribution is obtained by substituting equation (116) and equations (120-121) in equation (114) for intermediate Peclet number.

Now it is easy to find the function μ from equation (111), (u, v) from equation (7), p from equation (9) utilizing equation (112), and streamlines from equation (2) using (10). The CAS software Mathemaitca is used to solve equation (118).

IV. CONCLUSION

This paper finds a class of new exact solutions of the equations governing the two-dimensional steady motion with intermediate Peclet number of incompressible fluid of variable viscosity in presence and absence of body force in von-Mises coordinates. The characteristic equation for the stream function ψ is found as equation $y = \ln x + \ln\left[\frac{-1}{(c_1\psi + c_2)}\right]$ for the first case and for the second case, it is $y = \ln\left(\frac{1}{x}\right) + \ln\left[\frac{-1}{(c_1\psi + c_2)}\right]$ where c_1 and c_2 are constants. The exact solutions for intermediate Peclet number is determined for both the cases when $f(x) = \pm \ln x$ and $v(\psi) = \ln\left[\frac{-1}{(c_4\psi + c_5)}\right]$. In both the cases an infinite set of velocity components, viscosity function, generalized energy function and temperature distribution for intermediate Peclet number can be constructed. CAS software Mathematica is used to solve two variable coefficient differential equations in terms of special functions and graph of streamlines can be drawn to observe the streamline patterns using this software.

REFERENCES

[1] Chandna, O. P., Oku-Ukpong E. O.; Flows for chosen vorticity functions-Exact solutions of the Navier-Stokes Equations: International Journal of Applied Mathematics and Mathematical Sciences, **17(1)** (1994) 155-164.

[2] Naeem, R. K.; Steady plane flows of an incompressible fluid of variable viscosity via Hodograph transformation method: Karachi University Journal of Sciences, 2003, **3(1)**, 73-89.

[3] Naeem, R. K.; On plane flows of an incompressible fluid of variable viscosity: Quarterly Science Vision, 2007, **12(1)**, 125-131.

[4] Naeem, R. K.; Mushtaq A.; A class of exact solutions to the fundamental equations for plane steady incompressible and variable viscosity fluid in the absence of body force: International Journal of Basic and Applied Sciences, 2015, **4(4),** 429-465. www.sciencepubco.com/index.php/IJBAS , doi:10.14419/ijbas.v4i4.5064

[5] Mushtaq A., On Some Thermally Conducting Fluids: Ph. D Thesis, Department of Mathematics, University of Karachi, Pakistan, 2016.

[6] Mushtaq A.; Naeem R.K.; S. Anwer Ali; A class of new exact solutions of Navier-Stokes equations with body force for viscous incompressible fluid,: International Journal of Applied Mathematical Research, 2018, **7(1),** 22-26. www.sciencepubco.com/index.php/IJAMR , doi:10.14419/ijamr.v7i1.8836

[7] Mushtaq Ahmed, Waseem Ahmed Khan : A Class of New Exact Solutions of the System of PDE for the plane motion of viscous incompressible fluids in the presence of body force,: International Journal of Applied Mathematical Research, 2018, 7 **(2)** , 42-48. www.sciencepubco.com/index.php/IJAMR , doi:10.14419 /ijamr.v7i2.9694

[8] Mushtaq Ahmed, Waseem Ahmed Khan , S. M. Shad Ahsen : A Class of Exact Solutions of Equations for Plane Steady Motion of Incompressible Fluids of Variable viscosity in presence of Body Force,: International Journal of Applied Mathematical Research, 2018, **7 (3)** , 77-81. www.sciencepubco.com/index.php/IJAMR, doi:10.14419/ijamr.v7i2.12326

[9] Mushtaq Ahmed, (2018), A Class of New Exact Solution of equations for Motion of Variable Viscosity Fluid In presence of Body Force with Moderate Peclet number , International Journal of Fluid Mexhanics and Thermal Sciences, 4 (4) 429- www.sciencepublishingdroup.com/j/ijfmts doi: 10.11648/j.ijfmts.20180401.12

[10] D.L.R. Oliver & K.J. De Witt, High Peclet number heat transfer from a droplet suspended in an electric field: Interior problem, Int. J. Heat Mass Transfer, vol. 36: 3153-3155, 1993.

[11] B. Abramzon and C. Elata, Numerical analysis of unsteady conjugate heat transfer between a single spherical particle and surrounding flow at intermediate Reynolds and Peclet numbers, 2nd Int. Conf. on numerical methods in Thermal problems, Venice, pp. 1145-1153,1981.

[12] Z.G. Fenz, E.E. Michaelides, Unsteady mass transport from a sphere immersed in a porous medium at finite Peclet numbers, Int. J. Heat Mass Transfer 42: 3529-3531, 1999.

[13] Fayerweather Carl , Heat Transfer From a Droplet at Moderate Peclet Numbers with heat Generation. PhD. Thesis, U of Toledo, May 2007.

[14] Martin, M. H.; The flow of a viscous fluid I: Archive for Rational Mechanics and Analysis, 1971, **41**(4), 266-286.

[15] Daniel Zwillinger; Handbook of differential equations; Academic Press, Inc. (1989)